VERSITA

Tatra Mountains
Mathematical Publications
DOI: 10.2478/tmmp-2013-0029
Tatra Mt. Math. Publ. **56** (2013), 109–229

UNSOLVED PROBLEMS

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ABSTRACT. In this paper there are given problems from the Unsolved Problems Section on the homepage of the journal Uniform Distribution Theory `http://www.boku.ac.at/MATH/udt/unsolvedproblems.pdf`

It contains 38 items and 5 overviews collected by the author and by Editors of UDT. They are focused at uniform distribution theory, more accurate, distribution functions of sequences, logarithm of primes, Euler totient function, van der Corput sequence, ratio sequences, set of integers of positive density, exponential sequences, moment problems, Benford's law, Gauss-Kuzmin theorem, Duffin-Schaeffer conjecture, extremes $\int_0^1\int_0^1 F(x,y)\mathrm{d}g(x,y)$ over copulas $g(x,y)$, sum--of-digits sequence, etc. Some of them inspired new research activities. The aim of this printed version is publicity.

Contents

2010 Mathematics Subject Classification: 11K, 11J, 11B, 11L.
Keywords: Sequence, binary, integer, ratio, random, uniform distribution, discrepancy, integration, distribution function, diophantine approximation, continued fraction, moment problem.
The work was supported by the grant APVV-0586-11, the grant VEGA 1/0173/13 and VEGA 2/0206/10.

Introduction

Notations, definitions and basic properties should be consulted by the following monographs:

Kuipers, L.—Niederreiter, H.: *Uniform Distribution of Sequences* published by John Wiley in 1974 (reprint edition published by Dover Publications, Inc. Mineola, New York in 2006) and hereafter referred to as [KN];

Rauzy, G.: *Propriétés statistiques de suites arithmétiques* published by Presses Universitaires de France in 1976;

Hlawka, E.: *Theorie der Gleichverteilung* published in German by Bibliographisches Institut in 1979 and English under the title *The Theory of Uniform Distribution* by A B Academic Publishers in 1984;

Niederreiter, H.: *Random Number Generation and Quasi–Monte Carlo Methods* published by SIAM in 1992 and referred to as [N];

Drmota, M.—Tichy, R. F.: *Sequences, Discrepancies and Applications* published by Springer Verlag in 1997 and referred to as [DT];

Strauch, O.—Porubský, Š.: *Distribution of Sequences: A Sampler*, published by Peter Lang in 2005 and referred to as [SP] (Elektronic revised version published in `http://www.boku.ac.at/MATH/udt/`);

Tezuka, S.: *Uniform Random Numbers. Theory and Practice*, published by Kluwer Academic Publishers in 1995;

Matoušek, J.: *Geometric Discrepancy. An Illustrated Guide, Algorithms and Combinatorics* published by Springer-Verlag in 1999;

Niederreiter, H.: *Quasi–Monte Carlo methods and pseudo–random numbers,* Bull. Amer. Math. Soc. **84** (1978), no. 6, 957–1040 MR 80d:65016;

Hlawka, E.: *Statistik und Gleichverteilung (Statistics and uniform distribution)* (German), Grazer Math. Ber. **335** (1998), ii+206 pp. MR 99g:11093;

Koksma, J. F.: *Diophantische Approximationen (Diophantine Approximations)* (German), published by Springer-Verlag in 1936;

Dick, J.—Pillichshammer, F.: *Digital Nets and Sequences (Discrepancy Theory and Quasi-Monte Carlo Integration)* published by Cambridge University Press in 2010.

Definitions and notations

• A function $g\colon [0,1] \to [0,1]$ will be called **distribution function** (abbreviated as d.f.) if the following two conditions are satisfied:

(i) $g(0) = 0$, $g(1) = 1$,

(ii) g is non–decreasing.

We shall identify any two distribution functions $g, \widetilde{g}$ which coincide at common continuity points, or equivalently, if $g(x) = \widetilde{g}(x)$ a.e.

• Given a sequence x_n of real numbers, a positive integer N and a subset I of the unit interval $[0,1)$, the **counting function** $A(I;N;x_n \bmod 1)$ is defined as the number of terms of x_n with $1 \le n \le N$, and with x_n taken modulo one, belonging to I, i.e.,

$$A(I;N;x_n \bmod 1) = \#\big\{n \le N; \{x_n\} \in I\big\} = \sum_{n=1}^{N} c_I\big(\{x_n\}\big),$$

where $c_I(t)$ is the **characteristic function** of I.

• For a sequence $x_1, \dots, x_N \bmod 1$ we define the **step distribution function** $F_N(x)$ for $x \in [0,1)$ by

$$F_N(x) = \frac{A([0,x);N;x_n \bmod 1)}{N},$$

while $F_N(1) = 1$.

• A d.f. g is called a **distribution function** of the sequence $x_n \bmod 1$ if an increasing sequence of positive integers $N_1, N_2, \dots$ exists such that the equality

$$g(x) = \lim_{k\to\infty} \frac{A([0,x);N_k;x_n \bmod 1)}{N_k} \left(= \lim_{k\to\infty} F_{N_k}(x)\right)$$

holds at every point x, $0 \le x \le 1$, of the continuity of $g(x)$ and thus a.e. on $[0,1]$.

• If there exists a limit $\lim_{N\to\infty} F_N(x) = g(x)$ a.e. on $[0,1]$, then $g(x)$ is called **asymptotic distribution function** (abbreviating a.d.f.) of $x_n \bmod 1$ and if $g(x) = x$, then $x_n \bmod 1$ is called **uniformly distributed** in $[0,1]$ (abbreviating u.d.).

• The **set of all distribution functions** of a sequence $x_n \bmod 1$ will be denoted by $G(x_n \bmod 1)$. We shall identify the notion of the **distribution** of a sequence $x_n \bmod 1$ with the set $G(x_n \bmod 1)$, i.e., the distribution of $x_n \bmod 1$ is known if we know the set $G(x_n \bmod 1)$. The set $G(x_n \bmod 1)$ has the following fundamental properties for every sequence $x_n \bmod 1$:

- $G(x_n \bmod 1)$ is non–empty, and it is either a singleton or has infinitely many elements;
- $G(x_n \bmod 1)$ is closed and connected in the topology of the weak convergence, and these properties are characteristic, i.e.,
- for given a non–empty set H of distribution functions, there exists a sequence x_n in $[0,1)$ such that $G(x_n) = H$ if and only if H is closed and connected.

• Let $x_1, \dots, x_N$ be a given sequence of real numbers from the unit interval $[0,1)$. Then the number

$$D_N = D_N(x_1, \dots, x_N) \;= \sup_{0\le\alpha<\beta\le 1} \left| \frac{A([\alpha,\beta); N; x_n)}{N} - (\beta - \alpha) \right|$$

is called the **(extremal) discrepancy** of this sequence. The number

$$D_N^* = \sup_{x\in[0,1]} \left| \frac{A([0,x); N; x_n)}{N} - x \right|$$

is called **star discrepancy**, and the number

$$D_N^{(2)} = \int_0^1 \left(\frac{A([0,x); N; x_n)}{N} - x \right)^2 \mathrm{d}x$$

is called its L^2 **discrepancy**.

• For multidimensional case see [2.2, p. 196].

• The Riemann-Stiltjes integral $\int_0^1 \int_0^1 f(x,y) \mathrm{d}_x \mathrm{d}_y g(x,y)$ is defined as the limit

$$\sum_{k=1}^{m} \sum_{l=1}^{n} f(\alpha_k, \beta_l) \big(g(x_k, y_l) + g(x_{k+1}, y_{l+1}) - g(x_k, y_{l+1}) - g(x_{k+1}, y_l) \big)$$
$$\to \int_0^1\int_0^1 f(x,y) \mathrm{d}_x \mathrm{d}_y g(x,y)$$

if diameters of $[x_k, x_{k+1}] \times [y_l, y_{l+1}]$ tend to zero for partition $0 = x_0 < x_1 < \cdots < x_m = 1$ of x-axis, $0 = y_0 < y_1 < \cdots < y_n = 1$ of y-axis and for $(\alpha_k, \beta_l) \in [x_k, x_{k+1}] \times [y_l, y_{l+1}]$. This integral exists for continuous $f(x,y)$ and $g(x,y)$ with bounded variation. Let $\Box$ denote the rectangle $[x_k, x_{k+1}] \times [y_l, y_{l+1}]$ and denote

$$\Box g(x,y) = g(x_k, y_l) + g(x_{k+1}, y_{l+1}) - g(x_k, y_{l+1}) - g(x_{k+1}, y_l).$$

If diameter $\Box \to 0$, then we find the differential $\mathrm{d}_x \mathrm{d}_y g(x,y)$ as

$$\mathrm{d}_x \mathrm{d}_y g(x,y) = g(x,y) + g(x + \mathrm{d}x, y + \mathrm{d}y) - g(x, y + \mathrm{d}y) - g(x + \mathrm{d}x, y).$$

In some cases we shorten $\mathrm{d}_x \mathrm{d}_y g(x,y) = \mathrm{d}g(x,y)$.

1. Problems

1.1. Extended van der Corput difference theorem

Prove or disprove: If the sequence

$$k(x_{n+h} - x_n) - h(x_{n+k} - x_n) \bmod 1, \qquad n = 1, 2, \dots,$$

is u.d. for every $k, h = 1, 2, \dots,\ k > h$, then the original sequence

$$x_n \bmod 1, \qquad n = 1, 2, \dots,$$

is also

u.d.

NOTES. This problem was posed by M. H. Huxley at the Conference on Analytic and Elementary Number Theory, Vienna, July 18–20, 1996.

Submitted by O. Strauch.

1.2. Inverse modulo prime

Let $p > 2$ be a prime number. For an integer $0 < n < p$, define n^* by the congruence $nn^* \equiv 1 \pmod p$, $0 < n^* < p$. Is it true that the sequence of blocks

$$\left(\frac{n^*}{p}, \frac{(n+1)^*}{p}\right), \qquad n = 1, 2, \dots, p-2,$$

is u.d. as $p \to \infty$?

NOTES. Tsz Ho Chan (2004) proved that

$$\frac{1}{p}\sum_{n=1}^{p-2}\left|\frac{n^*}{p} - \frac{(n+1)^*}{p}\right| = \frac{1}{3} + \mathcal{O}\left(\frac{(\log p)^3}{\sqrt{p}}\right)$$

for every prime $p > 2$. Moreover, the sequence

$$\left(\frac{n}{p}, \frac{n^*}{p}\right), \qquad n = 1, 2, \dots, p-1,$$

is u.d. as $p \to \infty$,

see [SP, p. 3–25, 3.7.2].

Solution. The s-dimensional sequence

$$\left(\frac{n^*}{p}, \frac{(n+1)^*}{p}, \dots, \frac{(n+s-1)^*}{p}\right), \qquad n = 1, 2, \dots, p,$$

is u.d. as $p \to \infty$,

and the discrepancy bound is

$$D_p^* = \mathcal{O}\left(\frac{(\log p)^s}{\sqrt{p}}\right), \tag{1}$$

for all $s \geq 2$, and this estimate is essentially best possible up to the logarithmic factor.

NOTES. A. W i n t e r h o f sent to us that (1) was proved by H. N i e d e r r e i t e r (1994). A generalization is given in H. N i e d e r r e i t e r and A. W i n t e r h o f (2000).

Proposed by O. Strauch.

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1.3. Logarithm of primes

See [SP, p. 2–175, 2.19.8]. Let p_n be the nth prime. Find the set of all distribution functions $G(x_n)$ of the sequence

$$x_n = \log p_n \bmod 1, \qquad n = 1, 2, \dots$$

NOTES. (I) A. W i n t n e r (1935) has shown that $\log p_n \bmod 1$ is not u.d. A proof can be found in D. P. P a r e n t [1984, pp. 282–283, Solut. 5.19].

(II) S. A k i y a m a (1996, 1998) proved: Let c_i, $i = 0, 1, 2, \dots, k-1$, be real numbers with $\sum_{i=0}^{k-1} c_i \neq 0$. Then the sequence $x_n = \sum_{i=0}^{k-1} c_i \log p_{n+i} \bmod 1$, $n = 1, 2, \dots$ is not almost u.d., i.e., $x \notin G(x_n)$.

(III) R. E. W h i t n e y (1972) proved that $\log p_n \bmod 1$ is u.d. with respect to the logarithmic weighted means, i.e.,

$$\lim_{N\to\infty} \left(\sum_{n=1}^{N} \frac{1}{n}\right)^{-1} \sum_{n=1}^{N} \frac{c_{[0,x)}(\{\log p_n\})}{n} = x$$

for all $x \in [0, 1]$.

(IV) D. I. A. C o h e n and T. M. K a t z (1984) have shown the u.d. of $\log p_n \bmod 1$ with respect to the zeta distribution, i.e.,

$$\lim_{\alpha\to 1^+} \frac{1}{\zeta(\alpha)} \sum_{n=1}^{\infty} \frac{c_{[0,x)}(\{\log p_n\})}{n^\alpha} = x$$

for all $x \in [0, 1]$.

Solution. Y. O h k u b o (2011) proved the following results (i)–(ix):

(i) Two sequences $\log p_n \bmod 1$ and $\log n \bmod 1$ have the same d.f.s, i.e.,

$$G(\log p_n \bmod 1) = G(\log n \bmod 1).$$

(ii) Every u.d. sequence $x_n \bmod 1$ is statistically independent of $\log p_n \bmod 1$, i.e., $x_n \bmod 1$ and $(x_n + \log p_n) \bmod 1$ are u.d. simultaneously.

(iii) The result (ii) follows from that every u.d. sequence $x_n \bmod 1$ is statistically independent with $\log(n \log n) \bmod 1$ and

$$\lim_{n\to\infty} \bigl(\log p_n - \log(n \log n)\bigr) = 0.$$

(iv) The result (ii) implies that, for every irrational θ the sequence $p_n\theta + \log p_n$ is u.d. mod 1.

(v) Also, every u.d. sequence $x_n \bmod 1$ is statistically independent with $\frac{p_n}{n} \bmod 1$. It follows from the limit

$$\lim_{n\to\infty} \left(\frac{p_n}{n} - \log(n \log n)\right) = -1.$$

(vi) The result (v) implies that $p_n\theta + \frac{p_n}{n}$ is u.d. mod 1.

(vii) Theorem: Let the real-valued function $f(x)$ be strictly increasing for $x \ge 1$ and let $f^{-1}(x)$ be the inverse function of $f(x)$. Suppose that

- $\lim_{k\to\infty} f^{-1}(k+1) - f^{-1}(k) = \infty$,
- $\lim_{k\to\infty} \frac{f^{-1}(k+w_k)}{f^{-1}(k)} = \psi(u)$ for every sequence $w_k \in [0,1]$ for which $\lim_{k\to\infty} w_k = u$, where this limit defines the function $\psi(u)$ on $[0,1]$,
- $\psi(1) > 1$. Then

$$\begin{aligned} &G(f(p_n) \bmod 1) \\ &= \left\{ \tilde{g}_u(x) = \frac{\min(\psi(x), \psi(u)) - 1}{\psi(u)} + \frac{1}{\psi(u)} \cdot \frac{\psi(x) - 1}{\psi(1) - 1} : u \in [0,1] \right\}. \end{aligned}$$

(viii) The result (vii) implies that $\log p_n$ and the sequences $\log(p_n \log^{(i)} p_n)$, $i = 1, 2, \dots$ have the same distribution as the sequence $\log n$.

(ix) S. Akiyama (1998) proved

$$\lim_{n\to\infty} \left(\sum_{i=0}^{\ell-1} c_i \log p_{n+i} - \left(\sum_{i=0}^{\ell-1} c_i\right) \log p_n \right) = 0.$$

Then

$$G\left(\sum_{i=0}^{\ell-1} c_i \log p_{n+i} \bmod 1\right) = G\left(\left(\sum_{i=0}^{\ell-1} c_i\right) \log p_n \bmod 1\right).$$

and by (vii) we have

$$G\,(c\log p_n \bmod 1)$$
$$= \left\{ \frac{e^{x/c}-1}{e^{1/c}-1}e^{-u/c} + (e^{\min(x/c,u/c)}-1)e^{-u/c} : u\in[0,1] \right\},$$

where $c=\sum_{i=0}^{\ell-1} c_i$.

Submitted by O. Strauch.

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1.4. Fractional part of $n\alpha$

See [SP, p. 2–86, 2.8.12]. Characterize the set $G(x_n)$ of all d.f.'s of the sequence

$$x_n = \begin{cases} \{n\alpha\}\alpha & \text{if } \{n\alpha\} < 1-\alpha, \\ \big(1-\{n\alpha\}\big)(1-\alpha) & \text{if } \{n\alpha\} \geq 1-\alpha, \end{cases}$$

for $0<\alpha<1$.

Notes. A. F. Timan (1987) proved that the series $\sum_{n=1}^{\infty}\frac{x_n}{n^r}$ converges for all $\alpha\in(0,1)$ if and only if $r>1$.

Solution. S. Steinerberger: For irrational $0<\alpha<1$ we have $x_n = f\big(\{n\alpha\}\big)$, where

$$f(x) = \begin{cases} x\alpha & \text{if } x\in[0,1-\alpha], \\ (1-x)(1-\alpha) & \text{if } x\in[1-\alpha,1]. \end{cases}$$

Then a.d.f. $g(x)$ of x_n is

$$g(x) = \left|f^{-1}\big([0,x)\big)\right| = \begin{cases} 1 & \text{if } x \in [\alpha(1-\alpha),1], \\ \frac{x}{\alpha(1-\alpha)}, & \text{others}\ . \end{cases}$$

Proposed by O. Strauch.

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1.5. Strange recurring sequence

[SP, p. 2–243, 2.24.10]: Characterize the $G(x_n)$ of the so–called strange recurring sequences of the form

(i) $x_n = x_{n-[x_{n-1}]} + x_{n-[x_{n-2}]}$,
(ii) $x_n = x_{n-[x_{n-1}]} + x_{[x_{n-1}]}$,
(iii) $x_n = x_{[x_{n-2}]} + x_{n-[x_{n-2}]}$

with real initial values x_1, x_2.

Notes. If $x_1 = x_2 = 1$, the sequence (i) was defined by D. R. Hofstadter (1979), (ii) was defined by J. H. Conway (1988) during one of his lectures and C. L. Mallows (1991) established the regular structure of (ii) and introduced the monotone sequence (iii).

Proposed by O. Strauch.

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1.6. Function $\pi(n)$

[SP, p. 2–193]: Riemann hypothesis implies that the sequence

$$\frac{n}{\pi(n)} \bmod 1, \qquad n = 1, 2, \dots,$$

is not u.d. Find all its d.f.'s.

Notes. Under the Riemann hypothesis

$$\pi(x) = \mathrm{li}(x) + \mathcal{O}\left(\sqrt{x}\log x\right)$$

which implies $\lim_{n\to\infty}\big(n/\pi(n)\big) - \big(n/\mathrm{li}(n)\big) = 0$ the sequences

$$n/\pi(n) \pmod 1 \quad \text{and} \quad n/\mathrm{li}(n) \pmod 1$$

have the same d.f.s if we prove the continuity of all d.f.'s of $n/\mathrm{li}(n) \bmod 1$ at 0 and 1, cf. [SP, p. 2–24, 2.3.3]. Niederreiter's theorem: If x_n, $n = 1, 2, \dots,$ is a monotone sequence that is u.d. mod 1, then

$$\lim_{n\to\infty} \frac{|x_n|}{\log n} = \infty.$$

implies that the sequence $n/\pi(n) \bmod 1$ is not u.d. (probably without the Riemann hypothesis).

Solution. F. Luca: without the Riemann hypothesis sequences $n/\pi(n)$ and $\log n$ have the same d.f.'s mod 1.

This follows from

$$\left|\frac{n}{\pi(n)} - \frac{n}{\mathrm{li(n)}}\right| = \mathcal{O}\left((\log n)^2 \exp(-c\sqrt{\log n})\right) = o(1);$$

$$\left|\frac{n}{\mathrm{li(n)}} - \frac{n}{f(n)}\right| = \mathcal{O}\left((\log n)^{-1}\right) = o(1), \quad \text{where} \quad f(n) = \frac{n}{\log n} + \frac{n}{(\log n)^2};$$

$$\frac{n}{f(n)} = \log(n) - 1 + o(1);$$

immediately.

Proposed by O. Strauch.

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1.7. Glasner sets

A strictly increasing sequence of positive integers k_n, $n = 1, 2, \dots$ is called a **Glasner set** if for every infinite set $A \subset [0, 1)$ and every $\varepsilon > 0$ there exists k_n such that the dilation $k_n A \bmod 1 = \{k_n x \bmod 1 : x \in A\}$ is ε–dense in $[0, 1]$, i.e., $k_n A \bmod 1$ intersects every subinterval of $[0, 1]$ of the length ϵ. The following sequences k_n, $n = 1, 2, \dots,$ are Glasner sets:

(i) $k_n = n$,

(ii) $k_n = P(n)$, where $P(x)$ is a non-constant polynomial with integer coefficients,

(iii) $k_n = P(p_n)$, where p_n is the increasing sequence of all primes and polynomial $P(x)$ is as in (ii).

A strictly increasing sequence of positive integers k_n, $n = 1, 2, \dots$, has **quantitative Glasner property** if for every given $\varepsilon > 0$ there exists an integer $s(\varepsilon)$ such that for any finite set $A \subset [0, 1)$ of cardinality at least $s(\varepsilon)$ there exists k_n such that the dilation $k_n A \bmod 1$ is ε-dense in $[0, 1)$. The following sequences k_n, $n = 1, 2, \dots$, have this property:

(iv) $k_n = n$ as in (i) with $s(\varepsilon) = [\varepsilon^{-2-\gamma}]$, where $\gamma > 0$ is arbitrary and $\varepsilon \le \varepsilon_0(\gamma)$,

(v) $k_n = P(n)$, where $P(x)$ is a non-constant polynomial with integer coefficients,

(vi) $k_n = P(p_n)$ as in (iii) with $s(\varepsilon) = [\varepsilon^{-2d-\delta}]$, where $d = \deg P(x)$, $\delta > 0$ arbitrary and $\varepsilon < \varepsilon_0\big(P(x), \delta\big)$,

(vii) k_n, $n = 1, 2, \dots$, is: $(*)$ uniformly distributed for each positive integer m (i.e., for each $i = 0, 1, \dots, m-1$ the relative density of $k_m \equiv i \pmod m$ is $1/m$), and $(**)$ for each irrational α, the sequence $k_n\alpha \bmod 1$ is uniformly distributed in $[0, 1]$. Here $s(\varepsilon) = \left[\varepsilon^{-2-3(\log\log(1/\varepsilon))^{-1}}\right] + 1$, for every $\varepsilon < \varepsilon_0$, where ε_0 depends on the sequence k_n, $n = 1, 2, \dots$

(viii) $k_n = \big[f(n)\big]$, where $f(x)$ denotes a non-polynomial entire function that is real on the real numbers and such that $|f(z)| = O\big(e^{(\log|z|)^\alpha}\big)$ with $\alpha < 4/3$ and $s(\varepsilon)$ is as in (vii).

(ix) $k_n = \big[f(p_n)\big]$, where f is as in (viii) and $s(\varepsilon)$ is as in (vii).

(x) $k_n = [n^\alpha]$ for any $\alpha \ge 1$ not an integer ≥ 2 and $s(\varepsilon)$ is as in (vii).

Open problem: For $\mathbf{x} = (x_1, \dots, x_n) \in [0, 1]^N$ and positive integers $k_1 < k_2 < \cdots < k_K$ define the N-dimensional sequence $k_1\mathbf{x}, k_2\mathbf{x}, \dots, k_K\mathbf{x}$ and let $D_K^{(2)}(k_n\mathbf{x})$ be its L^2 discrepancy. Generalizing O. Strauch (1989), H. Albrecher (2002) (cf. [SP, pp. 3–14]) proved, for the mean value of the L^2 discrepancy $D_N^{(2)}(k_n\boldsymbol{x})$, that

$$\int\limits_{[0,1]^s} D_N^{(2)}(k_n\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \sum_{m,n=1}^{K}\left(\frac{1}{3} + \frac{1}{12}\frac{(k_m, k_n)^2}{k_m k_n}\right)^N + \left(\frac{1}{2^N} - \left(\frac{5}{12}\right)^N\right) - \frac{1}{3^N},$$

where (k_m, k_n) is a g.c.d. of k_m and k_n. Find some connection between Glasner sets and mean values of such L^2 discrepancy.

Proposed by O. Strauch.

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1.8. Digitally shifted Hammersley sequences

Let $x = 0.x_1x_2\dots x_m$ and $y = 0.y_1y_2\dots y_m$ be two real numbers written in dyadic expansion. Define $x \oplus y = z = 0.z_1z_2\dots z_m$, where $z_i = x_i + y_i \pmod 2$, $i = 1, 2, \dots, m$. Let $\gamma_2(n)$ be the van der Corput radical inverse function defined by $\gamma_2(n) = 0.a_0a_1\dots a_{m-1}$, where $n = a_{m-1}a_{m-2}\dots a_0$ is a positive integer (again in dyadic expansion). Then for the L^2 discrepancy $D_N^{(2)}$ of the sequence

$$\left(\frac{n}{N}, \gamma_2(n) \oplus x\right), \quad n = 0, 1, \dots, N-1, \quad \text{with } N = 2^m$$

we have

$$\frac{m^2}{64} - \frac{19m}{192} - \frac{lm}{16} + \frac{l^2}{16} + \frac{5}{16} + \frac{m}{8.2^m} - \frac{l}{4.2^m} + \frac{5}{16.2^m} - \frac{1}{72.4^m} \le N^2 D_N^{(2)},$$

$$N^2 D_N^{(2)} \le \frac{m^2}{64} - \frac{19m}{192} - \frac{lm}{16} + \frac{l^2}{16} + \frac{l}{4} + \frac{7}{16} + \frac{m}{8.2^m} - \frac{l}{4.2^m} + \frac{3}{16.2^m} - \frac{1}{72.4^m},$$

where l denotes the number of zeros in the dyadic expansion of x. If m is even and $l = m/2$, then

$$D_N^{(2)} = \mathcal{O}\left(\frac{\log N}{N^2}\right)$$

which is the best possible. A similar situation holds in the case of odd m and $l = (m-1)/2$.

Problem. Find an exact formula for $N^2 D_N^{(2)}$.

NOTES. (I) See P. Kritzer and F. Pillichshammer [2005, Th. 2 and 3] for L^2 discrepancy bounds.
(II) For the L^2 discrepancy of the 2-dimensional Hammersley sequence (also called Roth sequence)

$$\left(\frac{n}{N}, \gamma_2(n)\right), \qquad n = 0, 1, \dots, N-1, \quad N = 2^m$$

the following exact formula

$$N^2 D_N^{(2)} = \frac{m^2}{64} + \frac{29m}{192} + \frac{3}{8} - \frac{m}{16.2^m} + \frac{1}{4.2^m} - \frac{1}{72.2^{2m}}.$$

was proved by I. V. Vilenkin (1967) and independently by J. H. Halton and S. K. Zaremba (1969).

Solution. According to P. Kritzer and F. Pillichshammer (2006)

$$N^2 D_N^{(2)} = \frac{m^2}{64} - \frac{19m}{192} - \frac{lm}{16} + \frac{l^2}{16} + \frac{l}{4} + \frac{3}{8} + \frac{m}{16.2^m} - \frac{l}{8.2^m} + \frac{1}{4.2^m} - \frac{1}{72.4^m}.$$

Proposed by O. Strauch.

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1.9. Block sequence

Let x_n, $n = 1, 2, \dots$ be an increasing sequence of positive integers, $\underline{d}(x_n)$ be the lower asymptotic density, $\overline{d}(x_n)$ be the upper asymptotic density of x_n, $n = 1, 2, \dots$, and $X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}\right)$. Let $G(X_n)$ be the set of all d.f.'s of the block sequence X_n, $n = 1, 2, \dots$, i.e., the set of all possible weakly limits $F(X_{n_k}, x) \to g(x)$ as $k \to \infty$, where

$$F(X_{n_k}, x) = \frac{\#\{i \le n_k; x_i/x_{n_k} < x\}}{n_k}.$$

$G(X_n)$ has the following properties:

(i) If $g(x) \in G(X_n)$ increases and is continuous at $x = \beta$ and $g(\beta) > 0$, then there exists $1 \le \alpha < \infty$ such that $\alpha g(x\beta) \in G(X_n)$. If every d.f. of $G(X_n)$ is continuous at 1, then $\alpha = 1/g(\beta)$.

(ii) Assume that all d.f.'s in $G(X_n)$ are continuous at 0 and $c_1(x) \notin G(X_n)$. Then for every $\tilde{g}(x) \in G(X_n)$ and every $1 \le \alpha < \infty$ there exists $g(x) \in G(X_n)$ and $0 < \beta \le 1$ such that $\tilde{g}(x) = \alpha g(x\beta)$ a.e.

(iii) Assume that all d.f.s in $G(X_n)$ are continuous at 1. Then all d.f.'s in $G(X_n)$ are continuous on $(0,1]$, i.e., only possible discontinuity is in 0.

(iv) If $\underline{d}(x_n) > 0$, then for every $g(x) \in G(X_n)$ we have $\big(\underline{d}(x_n)/\overline{d}(x_n)\big).\, x \le g(x) \le \big(\overline{d}(x_n)/\underline{d}(x_n)\big).\, x$ for every $x \in [0,1]$. Thus $\underline{d}(x_n) = \overline{d}(x_n) > 0$ implies u.d. of the block sequence X_n, $n = 1, 2, \dots$

(v) If $\underline{d}(x_n) > 0$, then every $g(x) \in G(X_n)$ is continuous on $[0,1]$.

(vi) If $\underline{d}(x_n) > 0$, then there exists $g(x) \in G(X_n)$ such that $g(x) \ge x$ for every $x \in [0,1]$.

(vii) If $\overline{d}(x_n) > 0$, then there exists $g(x) \in G(X_n)$ such that $g(x) \le x$ for every $x \in [0,1]$.

(viii) Assume that $G(X_n)$ is singleton, i.e., $G(X_n) = \big\{g(x)\big\}$. Then either $g(x) = c_0(x)$ for $x \in [0,1]$; or $g(x) = x^\lambda$ for some $0 < \lambda \le 1$ and $x \in [0,1]$. Moreover, if $\overline{d}(x_n) > 0$, then $g(x) = x$.

(ix) $\max_{g \in G(X_n)} \int_0^1 g(x)\, \mathrm{d}x \ge \frac{1}{2}$.

(x) Assume that every d.f. $g(x) \in G(X_n)$ has a constant value on the fixed interval $(u,v) \subset [0,1]$ (maybe different). If $\underline{d}(x_n) > 0$, then all d.f.'s in $G(X_n)$ has infinitely many intervals with constant values.

(xi) There exists an increasing sequence x_n, $n = 1, 2, \dots$, of positive integers such that $G(X_n) = \big\{h_\alpha(x); \alpha \in [0,1]\big\}$, where $h_\alpha(x) = \alpha$, $x \in (0,1)$ is the constant d.f.

(xii) There exists an increasing sequence x_n, $n = 1, 2, \dots$, of positive integers such that

$$c_1(x) \in G(X_n) \quad \text{but} \quad c_0(x) \notin G(X_n),$$

where $c_0(x)$ and $c_1(x)$ are one-jump d.f.'s with the jump of height 1 at $x = 0$ and $x = 1$, respectively.

(xiii) There exists an increasing sequence x_n, $n = 1, 2, \dots$, of positive integers such that $G(X_n)$ is non-connected.

(xiv) $G(X_n) = \{x^\lambda\}$ if and only if $\lim_{n\to\infty}(x_{k.n}/x_n) = k^{1/\lambda}$ for every $k = 1, 2, \dots$ Here as in (viii) we have $0 < \lambda \le 1$.

(xv) If $\underline{d}(x_n) > 0$, then all d.f.s $g(x) \in G(X_n)$ are continuous, nonsingular and bounded by $h_1(x) \le g(x) \le h_2(x)$, where

$$h_1(x) = \begin{cases} x\frac{\underline{d}}{\overline{d}} & \text{if } x \in \left[0, \frac{1-\overline{d}}{1-\underline{d}}\right], \\ \frac{\underline{d}}{\frac{1}{x} - (1-\underline{d})} & \text{otherwise,} \end{cases} \qquad h_2(x) = \min\left(x\frac{\overline{d}}{\underline{d}}, 1\right).$$

Furthermore $h_1(x)$ and $h_2(x)$ are optimal and $h_1(x) \notin G(X_n)$.

NOTES. The properties (i)–(x) can be found in O. Strauch and J. T. Tóth (2001, 2002); (xi), (xiii) in G. Grekos and O. Strauch (2007); (xii) was found by L. Mišík (2004, personal communication); (xiv) is in F. Filip and J. T. Tóth (2006); (xv) is in V. Baláž, L. Mišík, J. T. Tóth and O. Strauch (2009). For concrete examples, cf. [SP, p. 2–217, 2.22.6; p. 2–219, 2.22.7; p. 2–222, 2.22.8; p. 2–225, 2.22.9, 2.22.10; p. 2–226, 2.22.11].

Methods:

$\mathcal{Z}$-transform. For positive integers $x_0 < x_1 < x_2 < \cdots$ we can assign the complex function $f(z) = \sum_{n=0}^{\infty} \frac{x_n}{z^n}$. The following holds:

(i) $x_n \longrightarrow f(z) = \sum_{n=0}^{\infty} \frac{x_n}{z^n}$;

(ii) $x_0 + x_1 + \cdots + x_{n-1} \longrightarrow \frac{f(z)}{z-1}$;

(iii) $x_{n+1} - x_n \longrightarrow (z-1)f(z) - zx_0$;

(iv) $\frac{x_n}{n} \longrightarrow \int_z^{\infty} \frac{f(\xi)}{\xi} \mathrm{d}\xi$;

(v) $nx_n \longrightarrow -z\frac{\mathrm{d}}{\mathrm{d}z}f(z)$;

(vi) $(n-1)x_{n-1} \longrightarrow -\frac{\mathrm{d}}{\mathrm{d}z}f(z)$;

(vii) If $x_n \longrightarrow f(z)$ and $y_n \longrightarrow g(z)$, then for convolution $x_n * y_n = z_n$, where $z_n = x_0y_n + x_1y_{n-1} + \cdots + x_ny_0$ we have $x_n * y_n \longrightarrow f(z).g(z)$; If $f(z)$ is known, then there exists inverse transform

(viii) $x_n = \frac{1}{2\pi i} \oint_C f(z)z^{n-1}\,\mathrm{d}z = \sum_{i=1}^{k} \mathrm{res}_{z=z_i} f(z)z^{n-1}$;

(ix) $\frac{x_0+x_1+\cdots+x_{n-1}}{(n-1)x_{n-1}} = \frac{\sum_{i=1}^{k} \mathrm{res}_{z=z_i} \frac{f(z)}{z-1} z^{n-1}}{\sum_{i=1}^{k} \mathrm{res}_{z=z_i} \left(-\frac{\mathrm{d}}{\mathrm{d}z} f(z)\right) z^{n-1}} = \int_0^1 x\,\mathrm{d}F(X_n, x)$.

Problem. Using $\mathcal{Z}$-transform (vii) and (ix) for a study of $G(Z_n)$, where $z_n = x_n * y_n$.

Algorithm [V. Baláž, L. Mišík, O. Strauch and J. T. Tóth (2008)]: Let x_n, $n = 1, 2, \dots$ be an increasing sequence of positive integers. Put $x_0 = 0$ and $t_n = x_n - x_{n-1}$, $n = 1, 2, \dots$ For every $n = 1, 2, \dots$, from t_n we compute the finite sequence $t_1^{(n)}, t_2^{(n)}, \dots, t_n^{(n)}$ by the following procedure:

1^0. For $n = 1$, $t_1^{(1)} = t_1 = x_1$;

2^0. For $n = 2$, $t_1^{(2)} = t_1 + t_2 - 1 = x_2 - 1$ and $t_2^{(2)} = 1$;

3^0. Assume that for $n-1$ we have $t_i^{(n-1)}$, $i = 1, 2, \dots, n-1$, for n we put $t'_i = t_i^{(n-1)}$, $i = 1, 2, \dots, n-1$, and $t'_n = t_n$.

The following steps (a) and (b) produce new $t'_1, \dots, t'_n$.

(a) If there exists k, $1 \le k < n$, such that $t'_1 = t'_2 = \dots = t'_{k-1} > t'_k$ and $t'_n > 1$, then we put $t'_k := t'_k + 1$, $t'_n := t'_n - 1$ and $t'_i := t'_i$ in all other cases.

(b) If such k does not exist and $t'_n > 1$, then we put $t'_1 := t'_1 + 1$, $t'_n := t'_n - 1$ and $t'_i := t'_i$ in all other cases.

In the nth step we will repeat (a) and (b) and the algorithm ends, if $t'_n = 1$ and which gives the resulting $t_1^{(n)} := t'_1, \dots, t_n^{(n)} := t'_n$.

Assuming that $t_n \ne 1$ for infinitely many n, these $t_i^{(n)}$, $i = 1, 2, \dots, n$ can have two possible forms:

(A) $t_1^{(n)} = \dots = t_m^{(n)} = D_n > t_{m+1}^{(n)} \ge t_{m+2}^{(n)} = t_{m+3}^{(n)} = \dots t_n^{(n)} = 1$,

(B) $t_1^{(n)} = \dots = t_m^{(n)} = D_n > t_{m+1}^{(n)} = \dots = t_{m+s}^{(n)} = D_n - 1 \ge t_{m+s+1}^{(n)} = \dots$
$\dots = t_n^{(n)} = 1$,

where $m = m(n)$, $s = s(n)$, $D_1 \le D_2 \le \cdots$ and $D_n \ge 2$ starting from n with $t_n > 1$. Thus there are two possibilities:

(I) D_n is bounded;

(II) $D_n \to \infty$.

In the case (I) we have only the form (A) and $D_n = \text{const.} = c \ge 2$ for all sufficiently large n.

In the case (II) both cases (A) and (B) are possible. Further properties:

- $x_n = \sum_{i=1}^n t_i^{(n)}$ for $n = 1, 2, \dots$
- Denoting $x_j^{(n)} = \sum_{i=1}^j t_i^{(n)}$, then we have $x_j \le x_j^{(n)}$ for $j = 1, 2, \dots, n$.
- Putting $X_n^{(n)} = \left(\frac{x_1^{(n)}}{x_n^{(n)}}, \frac{x_2^{(n)}}{x_n^{(n)}}, \dots, \frac{x_n^{(n)}}{x_n^{(n)}} \right)$ then $F(X_n^{(n)}, x) \le F(X_n, x)$ for all $x \in [0, 1]$ and $n = 1, 2, \dots$
- Selecting a sequence of indices n_k such that
$$F(X_{n_k}, x) \to g(x) \quad \text{and} \quad F(X_{n_k}^{(n_k)}, x) \to \tilde{g}(x),$$
then we have $\tilde{g}(x) \le g(x)$ for all $x \in [0, 1]$.

Open problem is to execute Algorithm on a some number-theoretic sequence.

Examples. (I) O. Strauch and J. T. Tóth (2001): Put $x_n = p_n$, the nth prime and denote
$$X_n = \left(\frac{2}{p_n}, \frac{3}{p_n}, \dots, \frac{p_{n-1}}{p_n}, \frac{p_n}{p_n} \right).$$
The sequence of blocks X_n is u.d. and therefore the ratio sequence p_m/p_n, $m = 1, 2, \dots, n$, $n = 1, 2, \dots$ is u.d. in $[0, 1]$. This generalizes a result of A. Schinzel

(cf. W. Sierpiński [1964, p. 155]). Note that from u.d. of X_n applying the L^2 discrepancy of X_n we get the following interesting limit

$$\lim_{n\to\infty}\frac{1}{n^2p_n}\sum_{i,j=1}^{n}|p_i-p_j|=\frac{1}{3}.$$

(II) O. Strauch and J. T. Tóth (2001): Let γ, δ, and a be given real numbers satisfying $1\le\gamma<\delta\le a$. Let x_n be an increasing sequence of all integer points lying in the intervals

$$(\gamma,\delta),(\gamma a,\delta a),\dots,\left(\gamma a^k,\delta a^k\right),\dots$$

Then $G(X_n)=\left\{g_t(x);t\in[0,1]\right\}$, where $g_t(x)$ has constant values

$$g_t(x)=\frac{1}{a^i(1+t(a-1))}\quad\text{for}\quad x\in\frac{(\delta,a\gamma)}{a^{i+1}(t\delta+(1-t)\gamma)},\qquad i=0,1,2,\dots$$

and on the component intervals it has a constant derivative

$$4ex\ g'_t(x)=\frac{t\delta+(1-t)\gamma}{(\delta-\gamma)(\frac{1}{a-1}+t)}\quad\text{for}\quad x\in\frac{(\gamma,\delta)}{a^{i+1}(t\delta+(1-t)\gamma)},\qquad i=0,1,2,\dots$$

$$\text{and}\quad x\in\left(\frac{\gamma}{t\delta+(1-t)\gamma},1\right).$$

Here we write $(xz,yz)=(x,y)z$ and $(x/z,y/z)=(x,y)/z$. From it follows that the set $G(X_n)$ has the following properties:

(i) Every $g\in G(X_n)$ is continuous.

(ii) Every $g\in G(X_n)$ has infinitely many intervals with constant values, i.e., with $g'(x)=0$, and in the infinitely many complement intervals it has a constant derivative $g'(x)=c$, where $\frac{1}{\overline{d}}\le c\le\frac{1}{\underline{d}}$ and for lower $\underline{d}$ and upper $\overline{d}$ asymptotic density of x_n we have $\underline{d}=\frac{(\delta-\gamma)}{\gamma(a-1)}$, $\overline{d}=\frac{(\delta-\gamma)a}{\delta(a-1)}$.

(iii) The graph of every $g\in G(X_n)$ lies in the intervals $\left[\frac{1}{a},1\right]\times\left[\frac{1}{a},1\right]\cup\left[\frac{1}{a^2},\frac{1}{a}\right]\times\left[\frac{1}{a^2},\frac{1}{a}\right]\cup\dots$ Moreover, the graph g in $\left[\frac{1}{a^k},\frac{1}{a^{k-1}}\right]\times\left[\frac{1}{a^k},\frac{1}{a^{k-1}}\right]$ is similar to the graph of g in $\left[\frac{1}{a^{k+1}},\frac{1}{a^k}\right]\times\left[\frac{1}{a^{k+1}},\frac{1}{a^k}\right]$ with coefficient $\frac{1}{a}$. Using the parametric expression, it can be written for all $x\in\left(\frac{1}{a^{i+1}},\frac{1}{a^i}\right)$ that $g_t(x)=\frac{g_t(a^ix)}{a^i}$, $i=0,1,2,\dots$

(iv) $G(X_n)$ is connected and the upper distribution function $\overline{g}(x)=g_0(x)\in G(X_n)$ and the lower distribution function $\underline{g}(x)\notin G(X_n)$. The graph of $\underline{g}(x)$ on $\left[\frac{1}{a},1\right]\times\left[\frac{1}{a},1\right]$ coincides with the graph of $y(x)=\left(1+\frac{1}{\underline{d}}\left(\frac{1}{x}-1\right)\right)^{-1}$ on $\left[\frac{\gamma}{\delta},1\right]$, further, on $\left[\frac{1}{a},\frac{\gamma}{\delta}\right]$ we have $\underline{g}(x)=\frac{1}{a}$.

(v) $G(X_n)=\left\{\frac{g_0(x\beta)}{g_0(\beta)};\beta\in\left[\frac{1}{a},\frac{\delta}{a\gamma}\right]\right\}$.

(III) O. Strauch and G. Grekos (2007): Let x_n and y_n, $n = 1, 2, \dots$, be two strictly increasing sequences of positive integers such that for the related block sequences $X_n = \left(\frac{x_1}{x_n}, \dots, \frac{x_n}{x_n}\right)$ and $Y_n = \left(\frac{y_1}{y_n}, \dots, \frac{y_n}{y_n}\right)$, we have singleton $G(X_n) = \{g_1(x)\}$ and $G(Y_n) = \{g_2(x)\}$. Furthermore, let n_k, $k = 1, 2, \dots$, be an increasing sequence of positive integers such that $N_k = \sum_{i=1}^{k} n_i$ satisfies $\frac{n_k}{N_k} \to 1$. Denote by z_n the following increasing sequence of positive integers composed by blocks (here we use the notation $a(b, c, d, \dots) = (ab, ac, ad, \dots)$)

$$(x_1, \dots, x_{n_1}), x_{n_1}(y_1, \dots, y_{n_2}), x_{n_1}y_{n_2}(x_1, \dots, x_{n_3}), x_{n_1}y_{n_2}x_{n_3}(y_1, \dots, y_{n_4}), \dots$$

Then the sequence of blocks $Z_n = \left(\frac{z_1}{z_n}, \dots, \frac{z_n}{z_n}\right)$ has the set of d.f.s

$$\begin{aligned} G(Z_n) = &\left\{g_1(x), g_2(x), c_0(x)\right\} \\ &\cup \left\{g_1(xy_n);\ n = 1, 2, \dots\right\} \\ &\cup \left\{g_2(xx_n);\ n = 1, 2, \dots\right\} \\ &\cup \left\{\frac{1}{1+\alpha}c_0(x) + \frac{\alpha}{1+\alpha}g_1(x);\ \alpha \in [0, \infty)\right\} \\ &\cup \left\{\frac{1}{1+\alpha}c_0(x) + \frac{\alpha}{1+\alpha}g_2(x);\ \alpha \in [0, \infty)\right\}, \end{aligned}$$

where $g_1(xy_n) = 1$ if $xy_n \ge 1$, similarly for $g_2(xx_n)$.

Open problems:

1. Characterize a nonempty set H of d.f.s for which there exists an increasing sequence of positive integers x_n such that $G(X_n) = H$.

2. Probably $\frac{x_n}{x_{n+1}} \to 1$ implies that $G(X_n)$ is singleton.

Solution of 2. By F. Filip, L. Mišík and J. T. Tóth (2007) the solution is negative. They found counterexample:

Let a_k, n_k, $k = 1, 2, \dots$, and x_n, $n = 1, 2, \dots$ be three increasing integer sequences and $h_1 < h_2$ be two positive integers. Assume that

(i) $\frac{n_k}{n_{k+1}} \to 0$ for $k \to \infty$;

(ii) $\frac{a_k}{n_{k+1}} \to 0$ for $k \to \infty$;

(iii) for odd k we have

$$a_k^{h_2} \le x_{n_k} = (a_{k-1} + n_k - n_{k-1})^{h_1} \le (a_k + 1)^{h_2}$$

and

$$x_i = (a_k + i - n_k)^{h_2} \qquad \text{for} \quad n_k < i \le n_{k+1};$$

(iv) for even k we have

$$a_k^{h_1} \le x_{n_k} = (a_{k-1} + n_k - n_{k-1})^{h_2} \le (a_k + 1)^{h_1}$$

and

$$x_i = (a_k + i - n_k)^{h_1} \qquad \text{for} \quad n_k < i \le n_{k+1}.$$

Then $\frac{x_n}{x_{n+1}} \to 1$ and the set $G(X_n)$ of all distribution functions of the sequence of blocks X_n is $G(X_n) = G_1 \cup G_2 \cup G_3 \cup G_4$, where

$$G_1 = \left\{x^{\frac{1}{h_2}}.t; t \in [0,1]\right\},$$

$$G_2 = \left\{x^{\frac{1}{h_2}}(1-t) + t; t \in [0,1]\right\},$$

$$G_3 = \left\{\max(0, x^{\frac{1}{h_1}} - (1 - x^{\frac{1}{h_1}})u); u \in [0,\infty)\right\} \text{ and}$$

$$G_4 = \left\{\min(1, x^{\frac{1}{h_1}}.v); v \in [1,\infty)\right\}.$$

F. Filip, L. Mišík and J. T. Tóth (2007) also proved: If $G(X_n) = \{g(x)\}$ such that $g(x) < 1$ for $x \in [0,1)$, then $\frac{x_n}{x_{n+1}} \to 1$. This implies that for u.d. sequence X_n we have $\frac{x_n}{x_{n+1}} \to 1$.

3. Characterize increasing sequences x_n, $n = 1, 2, \dots$, of positive integers for which $G(X_n)$ is connected.

Notes. Some criterion of connectivity of $G(X_n)$ is given in G. Grekos and O. Strauch [2007, Th. 2] which is based on the relation $\tilde{g}(x) \prec g(x)$ defined on $G(X_n)$ if there exist α, β such that $\tilde{g}(x) = \alpha g(x\beta)$.

4. Prove or disprove: $G(X_n) \subset \{c_\alpha(x); \alpha \in [0,1]\} \Longrightarrow G(X_n) = \{c_0(x)\}$, for every increasing sequence x_n, $n = 1, 2, \dots$ of positive integers.

Notes. G. Grekos and O. Strauch (2007) proved that if $G(X_n) \subset \{c_\alpha(x); \alpha \in [0,1]\}$, then $c_0(x) \in G(X_n)$ and if $G(X_n)$ contains two different d.f.s, then also $c_1(x) \in G(X_n)$. Furthermore, $\underline{d}(x_n) = 0$ and $\overline{d}(x_n) > 0$ implies $c_1(x) \in G(X_n)$.

5. Prove or disprove: $\lim_{n\to\infty} \frac{x_n}{x_1+\cdots+x_n} = 0 \Longleftrightarrow c_0(x) \notin G(X_n)$. If it is true, then $c_0(x) \notin G(X_n)$ gives necessary and sufficient conditions that the sequence $Y_n, n = 1, 2, \dots$ of blocks

$$Y_n = \left(\frac{1}{x_n}, \frac{2}{x_n}, \dots, \frac{x_n}{x_n}\right)$$

is u.d. For a theory of blocks sequences Y_n, see Š. Porubský, T. Šalát and O. Strauch (1988).

6. There is open the theory of d.f. $G(X_n, Y_n)$ for two-dimensional blocks

$$(X_n, Y_n) = \left(\left(\frac{x_1}{x_n}, \frac{y_1}{y_n}\right), \left(\frac{x_2}{x_n}, \frac{y_2}{y_n}\right), \dots, \left(\frac{x_n}{x_n}, \frac{y_n}{y_n}\right)\right),$$

where x_n, $n = 1, 2, \dots$, and y_n, $n = 1, 2, \dots$ are increasing sequences of positive integers. It can be proved that the sequence $\left(\frac{p_i}{p_n}, \frac{i}{n}\right)$, $i = 1, 2, \dots, n$ is not u.d. in $[0,1]^2$. Here p_n, $n = 1, 2, \dots$, is the increasing sequence of all primes.

7. L. M i š í k: For every increasing sequence x_n, $n = 1, 2, \dots$, of positive integers there exists $g(x) \in G(X_n)$ such that $g(x) \geq x$ for all $x \in [0, 1]$. For $\underline{d}(x_n) > 0$ it holds by (vi).

Solution of 7. V. B a l á ž, L. M i š í k, J. T. T ó t h and O. S t r a u c h (2013): If $\underline{d}(x_n) = 0$ and we select n_k such that $\frac{n_k}{x_{n_k}} = \min_{i \leq n_k} \frac{i}{x_i}$ and that $F(X_{n_k}, x)$ $\to g(x)$, then $g(x) \geq x$ for $x \in [0, 1]$.

Proposed by O. Strauch.

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1.10. Logarithmic and trigonometric functions

[SP, p. 2–131 and 2–132]: Find the set $G(x_n)$ for the following sequences x_n, $n = 1, 2, \dots$:

(i) $x_n = (\log n) \cos(n\alpha) \bmod 1$,

(ii) $x_n = (\cos n)^n$,

(iii) $x_n = \cos(n + \log n) \bmod 1$.

Notes. (I) D. B e r e n d, M. D. B o s h e r n i t z a n and G. K o l e s n i k (1995) proved that (i) is everywhere dense in $[0, 1]$. They showed that there are uncountably many α's for which every of these sequences (i) is not u.d.

(II) The original problem of everywhere density in $[-1, 1]$ of (ii) was posed by M. B e n z e and F. P o p o v i c i (1996) and was solved by J. B u k o r (1997).

This problem was also solved (in some generality) in F. L u c a (1999). S. H a r t m a n (1949) proved that if $\frac{\alpha}{\pi}$ is irrational, then

$$\liminf_{n\to\infty}(\cos\alpha n)^n = \liminf_{n\to\infty}(\sin\alpha n)^n = -1.$$

(III) The sequence (iii) is not u.d., which was proved by L. K u i p e r s (1953).

Solution of (iii). S. S t e i n e r b e r g e r: The sequence $x_n = \cos(n+\log n) \bmod 1$ has the same a.d.f $g(x)$ as the sequence $\cos n \bmod 1$. It follows from that

(a) $\cos(n+\log n) = \cos 2\pi\left(\frac{n}{2\pi} + \frac{1}{2\pi}\log n\right) = \cos 2\pi z_n$, where

(b) $z_n = \frac{n}{2\pi} + \frac{1}{2\pi}\log n \bmod 1$ is u.d. sequence since $\frac{n}{2\pi}$ and $\frac{n}{2\pi} + \frac{1}{2\pi}\log n$ are u.d. simultaneously, see [SP, p. 2–27, 2.3.6.]

(c) Put $f(x) = \cos 2\pi x \bmod 1$ Then a.d.f. $g(x)$ of x_n is

$$g(x) = \left|f^{-1}\big([0,x)\big)\right| = \frac{1}{2} - \frac{1}{\pi}\arccos x + 1 - \frac{1}{\pi}\arccos(x-1).$$

(IV) D. B e r e n d and G. K o l e s n i k (2011): The sequence

$$P(n)\cos n\alpha \bmod 1, n = 1,2,\dots,$$

is completely u.d. for any non-constant polynomial $P(x)$ and α with $\cos\alpha$ transcendental. If $\cos\alpha$ is not transcendental D. B e r e n d and G. K o l e s n i k (2011) also proved: Let α be such that $e^{i\alpha}$ is an algebraic number of degree d which is not a root of unity. Then the sequence

(1) $\big(P(n)\cos n\alpha, P(n+1)\cos(n+1)\alpha, \dots, P(n+d-1)\cos(n+d-1)\alpha\big) \bmod 1$,

$n = 1,2,\dots$, is u.d. for any non-constant polynomial $P(x)$.

Open problem. D. B e r e n d and G. K o l e s n i k (2011): Let $P(x) = x$, $\alpha = \arccos 3/5$, i.e., $e^{i\alpha} = (3+4i)/5$ and denote

$$x_n = P(n)\cos n\alpha = n\frac{(3+4i)^n - (3-4i)^n}{2.5^n}.$$

Then by (1) the sequence $(x_n, x_{n+1}) \bmod 1$ is u.d., but

$$(x_n, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}) \bmod 1$$

is not u.d. The authors ask whether the sequences $(x_n, x_{n+1}, x_{n+2}) \bmod 1$ and $(x_n, x_{n+1}, x_{n+2}, x_{n+3}) \bmod 1)$ are u.d.

Solution of (ii). C h. A i s t l e i t n e r, M. H o f e r and M. M a d r i t s c h (2013): Let $x_n = \cos(\alpha n)^n \bmod 1$, $n = 1,2,\dots$. For $\frac{\alpha}{2\pi} \notin \mathbb{Q}$ we set $a = 3/4$, in the case $\frac{\alpha}{2\pi} = \frac{p}{q} \in \mathbb{Q}$ for p,q co-prime let

$$a = \begin{cases} \frac{q+1}{2q} + \frac{q-1}{4q} & \text{if } 4 \mid (q-1), \\ \frac{q-1}{2q} + \frac{q+1}{4q} & \text{if } 4 \nmid (q-1) \end{cases}$$

for q odd and let

$$a = \begin{cases} \frac{1}{2} + \frac{q-2}{4q} & \text{if } 4 \nmid q \text{ and } 8 \mid (q-2), \\ \frac{1}{2} + \frac{q+2}{4q} & \text{if } 4 \nmid q \text{ and } 8 \nmid (q-2), \\ \frac{q+2}{2q} + \frac{1}{4} & \text{if } 4 \mid q \text{ and } 8 \nmid q, \\ \frac{q+2}{2q} + \frac{q-4}{4q} & \text{if } 8 \mid q \end{cases}$$

for q even. Then a.d.f. of x_n is given by

$$g_a(x) = \begin{cases} 0 & \text{if } x = 0, \\ a & \text{if } 0 < x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

Partial solution of (i). 1^0. Ch. Aistleitner, M. Hofer and M. Madritsch (2013) proved: Let α be such that the discrepancy D_N of the sequence

$$\frac{\alpha}{2\pi} n \bmod 1, \qquad n = 1, 2, \dots, N$$

is of asymptotic order $D_N = o\left(\frac{1}{\log N}\right)$. Then the sequence $(\log n)\cos(n\alpha) \bmod 1$ is u.d. in $[0,1]$.

2^0. Let $x_n = (\log n)\cos(n\alpha) \bmod 1$, $n = 1, 2, \dots$, $\frac{\alpha}{2\pi} = \frac{p}{q}$, where p, q are coprime and let $N_1 < N_2 < \cdots$ be fixed integer sequence such that

$$\lim_{k\to\infty} \left\{\cos(\alpha i)\log N_k\right\} = \beta_i \qquad \text{for} \quad i = 1, \dots, q. \tag{1}$$

Then there exists d.f.

$$g(x) = \lim_{k\to\infty} F_{N_k}(x), \quad F_N(x) = \frac{\{n \le N; x_n \in [0,x)\}}{N},$$

such that

$$g(x) = \frac{1}{q}\sum_{i=1}^{q} h_{q,\beta_i,c_i}(x), \tag{2}$$

where

$$h_{q,\beta_i,c_i}(x) = \begin{cases} f_{\beta_i,c_i}(x+1-\nu_i) - f_{\beta_i,c_i}(1-\nu_i) & \text{if } 0 \le x \le \nu_i \text{ and } c_i > 0, \\ f_{\beta_i,c_i}(x-\nu_i) + 1 - f_{\beta_i,c_i}(1-\nu_i) & \text{if } \nu_i \le x \le 1 \text{ and } c_i > 0, \\ f_{\beta_i,c_i}(x+\nu_i) - f_{\beta_i,c_i}(\nu_i) & \text{if } 0 \le x \le 1-\nu_i \text{ and } c_i < 0, \\ f_{\beta_i,c_i}(x-(1-\nu_i)) + 1 - f_{\beta_i,c_i}(\nu_i) & \text{if } 1-\nu_i \le x \le 1 \text{ and } c_i < 0, \\ \mathbf{1}_{\{(0,1]\}}(x) & \text{if } c_i = 0, \end{cases}$$

where $\nu_i = \left\{|c_i|\log(q)\right\}$, $c_i = \cos(\alpha i)$ and

$$f_{\beta,c}(x) = \begin{cases} g_{\beta,c}(x) & \text{if } c > 0, \\ 1 - g_{\beta|c|}(1-x) & \text{if } c < 0, \\ \mathbf{1}_{\{(0,1]\}}(x) & \text{if } c = 0, \end{cases}$$

and

$$g_{\beta,c}(x) = \frac{e^{\frac{\min(x,\beta)}{c}} - 1}{e^{\frac{\beta}{c}}} + \frac{1}{e^{\frac{\beta}{c}}} \frac{e^{\frac{x}{c}} - 1}{e^{\frac{1}{c}} - 1}.$$

Moreover, the set $G(x_n)$ is the set of all d.f. of the form (2) for those $(\beta_1, \dots, \beta_q)$ for which a subsequence $(N_k)_{k\geq 1}$ satisfying (1) exists.

The authors note that for arbitrary q, it is a difficult problem to determine all possible vectors $(\beta_1, \dots, \beta_q)$ for which there exists $N_1 < N_2 < \cdots$ such that (1) holds, referred to K. Gristmair (1997).

Proposed by O. Strauch.

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1.11. Euler totient function

(cf. [SP, p. 2–191, 2.20.11]). If φ is the Euler totient function, then the sequence

$$\frac{\varphi(n)}{n}, \qquad n = 1, 2, 3, \dots,$$

has in $[0,1]$ singular a.d.f.

$$g_0(x).$$

NOTES. (I) I. J. Schoenberg (1928, 1936) proved that this sequence has continuous and strictly increasing a.d.f.

(II) P. Erdős (1939) showed that this a.d.f. is singular. Here a function is **singular**, if it is continuous, strictly monotone and has vanishing derivative almost everywhere on the interval of its definition.

(III) H. Davenport (1933) expressed

$$g_0(x) = \sum_{n=1}^{\infty} A_n(x),$$

where

$$A_n(x) = \frac{1}{a_n(x)} - \sum_{i<n} \frac{1}{[a_i(x), a_n(x)]} + \sum_{i<j<n} \frac{1}{[a_i(x), a_j(x), a_n(x)]} - \cdots$$

and $[a, b]$ is the least common multiple of a and b. Here $a_1(x) < a_2(x) < \cdots$ is the sequence of the all positive integers n for which $\frac{\varphi(n)}{n} \le x$ and for every divisor $d|n$, $d \ne n$ we have $\frac{\varphi(d)}{d} > x$. These n are called x-numbers. Directly from definition we have

(i) Every x-number is square-free.

(ii) Every square-free a is an x-number for some x.
Concretely, if $a = p_1 p_2 \dots p_m$, $p_1 < p_2 < \cdots < p_m$, p_i are primes, then a is x-number for every $x \in \left[\prod_{i=1}^{m}\left(1 - \frac{1}{p_i}\right), \prod_{i=1}^{m-1}\left(1 - \frac{1}{p_i}\right)\right)$

(iii) For every $i < j$ we have $a_i(x) \nmid a_j(x)$.

(iv) Let $p_1 < p_2 < \cdots$ be an increasing sequence of all primes and let $x \in \left[1 - \frac{1}{p_s}, 1\right)$. Then $a_1(x) = p_1 = 2$, $a_2(x) = p_2 = 3, \dots, a_s(x) = p_s$. If furthermore $x < 1 - \frac{1}{p_{s+1}}$, then for every $j > s$, the $a_j(x)$ cannot be a prime and $p_i \nmid a_j(x)$, $i = 1, 2, \dots, s$.

(v) If $x \in \left[\prod_{i=1}^{s}\left(1 - \frac{1}{p_i}\right), \prod_{i=1}^{s-1}\left(1 - \frac{1}{p_i}\right)\right)$, then $a_1(x) = \prod_{i=1}^{s} p_i$ (p_i as in (iv)).

(vi) For every $n = 1, 2, \dots$ and every $x \in (0, 1)$ we have

$$\frac{\varphi(n)}{n} \le x \iff \exists_{i=1,2,\dots} a_i(x)|n.$$

(vii) Assume that $x < x'$. Then for every x-number $a_i(x)$ there exists x'-number $a_j(x')$ such that $a_j(x')|a_i(x)$.

(III') Applying Davenport's method B. A. Venkov (1949) proved that

(i) $\left(1 - g_0(x)\right) \log \frac{1}{1-x} \to e^{-c}$ as $x \to 1$, where c is Euler's constant.

(ii) $x \log\log \frac{1}{g_0(x)} \to e^{-c}$ as $x \to 0$.

(iii) Let p be a prime. Then for every $1 - \frac{1}{p} \le x$ we have

$$\frac{1}{p} = g_0(x) - (p-1) g_0\left(x\left(1 - \frac{1}{p}\right)\right) + (p-1)^2 g_0\left(x\left(1 - \frac{1}{p}\right)^2\right) - \cdots$$

(iv) The function $g_0(x)$ at every $x = \frac{\varphi(n)}{n}$, $n = 1, 2, \dots$, has infinite left derivative.

(v) $\left(\int_0^1 x^s \mathrm{d} g_0(x)\right) \log s \to e^{-c}$ as $s \to \infty$ (s are positive integers).

(IV) A. S. Faĭnleĭb (1967) proved that

$$\frac{A([0,x);N;\varphi(n)/n)}{N} = g_0(x) + \mathcal{O}\left(\frac{1}{\log\log N}\right).$$

(V) W. Schwarz (1962) (c.f. A. G. Postnikov (1971, p. 267)) proved: Let $f(x)$ be a polynomial with integer coefficients having non-zero discriminant. Assume that g.c.d of coefficients of $f(x)$ is 1 and $f(n) > 0$ for $n = 1, 2, \dots$ Let $L(d)$ denote the number of solutions $f(n) \equiv 0 \pmod d$. Then

$$\frac{1}{N}\sum_{n=1}^{N}\frac{\varphi(f(n))}{f(n)} = \prod_{\substack{p=2\\ p-\text{prime}}}^{\infty}\left(1 - \frac{L(p)}{p^2}\right) + \mathcal{O}(\log^c N),$$

where $c > 0$ is a constant. This leads to

Open problem 1: Find a.d.f (if exists) of the sequence

$$\frac{\varphi(f(n))}{f(n)}, \qquad n = 1, 2, \dots$$

(VI) O. Strauch (1996) proved that

$$\int_0^1 g_0^2(x)\,\mathrm{d}x = 1 - \frac{6}{\pi^2} - \frac{1}{2}\lim_{N\to\infty}\frac{1}{N^2}\sum_{m,n=1}^{N}\left|\frac{\varphi(m)}{m} - \frac{\varphi(n)}{n}\right|$$

and he gave an estimate

$$\frac{2}{\pi^4} \le \lim_{N\to\infty}\frac{1}{N^2}\sum_{m,n=1}^{N}\left|\frac{\varphi(m)}{m} - \frac{\varphi(n)}{n}\right| \le 2\frac{6}{\pi^2}\left(1 - \frac{6}{\pi^2}\right). \tag{1}$$

Open problem 2: Find an estimation of the limit

$$\lim_{N\to\infty}\frac{1}{N^2}\sum_{m,n=1}^{N}\left|\frac{\varphi(m)}{m} - \frac{\varphi(n)}{n}\right| = L$$

better as (1), where $L \in [0.021, 0.392]$.

• J.-Ch. Schlage-Puchta (2009) send a method which gives $L \in [0.27425, 0.274465]$.

(VI') The aim of this problem is to find $\int_0^1 g_0^2(x)\mathrm{d}x$. It is motivated by the paper of O. Strauch (1994) about three dimensional body Ω of points of the form

$$\left(\int_0^1 g(x)\,\mathrm{d}x, \int_0^1 xg(x)\,\mathrm{d}x, \int_0^1 g^2(x)\,\mathrm{d}x\right),$$

where $g(x)$ runs the set of all d.f.'s. The points achieved for singulars $g(x)$ are interior points of Ω. Using an expression of the boundary of Ω in Problem 1.23.2 we can find

$$0.250 < \int_0^1 g_0^2(x)\,\mathrm{d}x < 0.307.$$

(VII) F. Luca ([a]2003) proved that, if $M_n = 2^n - 1$ is the nth Mersenne number then the subsequence $\varphi(M_n)/M_n$ is dense in $[0,1]$ and has an a.d.f. ([b]2005).

(VII') F. Luca and I. E. Shparlinski (2007) proved the existence of the moment

$$\frac{1}{N}\sum_{n=0}^{N-1}\left(\frac{\varphi(F_n)}{F_n}\right)^k = \Gamma_k + O_k\left(\frac{(\log N)^k}{N}\right)$$

for all $k = 1, 2, \dots$ with some positive constant Γ_k. Thus the sequence

$$\frac{\varphi(F_n)}{F_n}, \qquad n = 0, 1, 2, \dots$$

has an a.d.f. F. Luca in ([a]2003) also proved that $\varphi(F_n)/F_n$ is dense in $[0,1]$.

(VIII) (See [SP, p. 1–13]). I. J. Schoenberg (1959) introduced the following summation method: the sequence x_n is called **φ-convergent** to α if the sequence $y_n = \frac{1}{n}\sum_{d|n}\varphi(d)x_d$ converges to α. Schoenberg's Theorem 2 (1959) shows that the φ-convergence of x_n implies the classical convergence of x_{n_k} (to the same limit) for every sequence n_k for which $\liminf_{k\to\infty}\frac{\varphi(n_k)}{n_k} > 0$. Since a 0–1 φ-convergent sequence has the φ-limit 0 or 1, no φ-u.d. sequence exists (E. Kováč (2005)).

Open problem 3: Find a sequence x_n for which $y_n = \frac{1}{n}\sum_{d|n}\varphi(d)x_d \bmod 1$ is u.d. in $[0,1]$.

In connection of this we mentioned (cf. A. G. Postnikov [1971, p. 219, Th. 6b], [SP, p. 2–189, 2.20.8]): Let $f(n)$ be an arithmetical function which satisfies

(i) $f(n) = \sum_{d|n}\Phi(d)$,

(ii) $\sum_{n=1}^{\infty}\frac{|\Phi(d)|}{d} < \infty$

for some arithmetical function Φ. Then the sequence

$$f(n), \qquad n = 1, 2, \dots,$$

has the a.d.f.

$$g(x)$$

defined on $(-\infty,\infty)$.

• For $x_n = \frac{\varphi(n)}{n}$ and an interval $(k, k+N]$ define the step d.f.

$$F_{(k,k+N]}(x) = \frac{\#\{n \in (k, k+N]; x_n \in [0, x]\}}{N}.$$

Open problem 4: Find all possible limits of step d.f.s $F_{(k,k+N]}(x)$, for sequences of intervals $(k, k+N]$.

(IX) P. Erdős (1946) proved:

(i) If $\frac{\log\log\log k}{N} \to 0$ as $N \to \infty$, ten $F_{(k,k+N]}(x) \to g_0(x)$ for $x \in [0,1]$ and by Chinese remainder theorem he found k and N such that $\frac{\log\log\log k}{N} \to \frac{1}{2}$ and $\frac{1}{N}\sum_{k<n\le k+N} \frac{\varphi(n)}{n} < \frac{1}{2} < \frac{1}{N}\sum_{n=1}^{N} \frac{\varphi(n)}{n} = \frac{6}{\pi^2} + O\big(\frac{\log N}{N}\big)$, thus $F_{(k,k+N]}(x) \not\to g_0(x)$.

(ii) For a proof of (i) he used

$$\left(\frac{1}{N}\sum_{k<n\le k+N}\left(\frac{\varphi(n(t))}{n(t)}\right)^s - \frac{1}{N}\sum_{n=1}^{N}\left(\frac{\varphi(n)}{n}\right)^s\right) \to 0,$$

where $n(t) = \prod_{p|n, p\le t} p$, p are primes and $t = N$.

(X) V. Baláž, P. Liardet and O. Strauch (2007) proved:

(i) Necessary and sufficient condition: For any two sequences N and k of positive sequences, $N \to \infty$, we have $F_{(k,k+N]}(x) \to g_0(x)$, for every $x \in [0,1]$, if and only if, for every $s = 1, 2, \dots$, $\frac{1}{N}\sum_{k<n\le k+N}\sum_{N<d|n} \Phi(d) \to 0$, were (cf. A. G. Postnikov (1971, p. 360)) $\Phi(d) = \prod_{p|d}\big(\big(1-\frac{1}{p}\big)^s - 1\big)$ for the squarefree d and $\Phi(d) = 0$ in others, where p denotes a prime. In quantitative form:

$$\frac{1}{N}\sum_{k<n\le k+N}\sum_{N<d|n}\Phi(d) = \frac{1}{N}\sum_{k<n\le k+N}\left(\frac{\varphi(n)}{n}\right)^s - \frac{1}{N}\sum_{n=1}^{N}\left(\frac{\varphi(n)}{n}\right)^s + O\left(\frac{3^s(1+\log N)^s}{N}\right).$$

Using this they found that for $k = \prod_{p\le e^{e^{e^N}}} p$ we have $F_{(k,k+N]}(x) \to g_0(x)$ as $N \to \infty$ and contrary to (IX)(i) we have $\frac{\log\log\log k}{N} \to \infty$.

(ii) A quantitative form of Erdős' (IX)(ii): For every integer k, N and $t = N$ we have

$$\frac{1}{N}\sum_{k<n\le k+N}\left(\frac{\varphi(n(t))}{n(t)}\right)^s = \frac{1}{N}\sum_{n=1}^{N}\left(\frac{\varphi(n)}{n}\right)^s + O\left(\frac{3^s(1+\log N)^s}{N}\right)$$

for $s = 1, 2, \dots$

(iii) This implies that every d.f. $g(x)$, $F_{(k,k+N]} \to g(x)$ on $(0,1)$ must satisfies
$$\int_0^1 x^s \,\mathrm{d}g(x) \le \int_0^1 x^s \,\mathrm{d}g_0(x),$$
for every $s = 1, 2, \dots$

(iv) By Chinese theorem it can be found a sequence of intervals $(k, k+N]$ such that $F_{(k,k+N]}(x) \to c_0(x)$, where d.f. $c_0(x)$ has a step 1 in $x = 0$.

(v) Assume that $F_{(k,k+N]}(x) \to g(x)$ for all $x \in (0,1)$. Then
$$g_0(x) \le g(x) \le g_0(x) + \prod_{i=1}^{s} \left(1 - \frac{1}{p_i}\right) \tag{1}$$
for $x \in (0,1)$, where $p_1, p_2, \dots$ is the increasing sequence of all primes and $1 - \frac{1}{p_s} \le x$. By an anonymous referee in all cases the right hand side of (1) is ≥ 1.

(XI) A. Schinzel and Y. Wang (1958) proved that for any given $(\alpha_1, \alpha_2, \dots \dots, \alpha_{N-1}) \in [0, \infty)^{N-1}$ we can select a sequence of k such that
$$\left(\frac{\varphi(k+2)}{\varphi(k+1)}, \frac{\varphi(k+3)}{\varphi(k+2)}, \dots, \frac{\varphi(k+N)}{\varphi(k+N-1)}\right) \to (\alpha_1, \alpha_2, \dots, \alpha_{N-1}).$$
Select a subsequence of k such that $\frac{\varphi(k+1)}{k+1} \to \alpha$. Then
$$\left(\frac{\varphi(k+1)}{k+1}, \frac{\varphi(k+2)}{k+2}, \dots, \frac{\varphi(k+N)}{k+N}\right) \to (\alpha, \alpha\alpha_1, \alpha\alpha_1\alpha_2, \dots, \alpha\alpha_1\alpha_2\dots\alpha_{N-1}).$$
Now, for arbitrary d.f. $\tilde{g}(x)$ there exists a sequence α_n, $n = 1, 2, \dots$ in $(0, \infty)$ such that for every $n = 1, 2, \dots$ we have $\alpha_1\alpha_2\dots\alpha_n \in (0,1)$ and that the sequence $\alpha_1\alpha_2\dots\alpha_n$, $n = 1, 2, \dots$, has asymptotic d.f. $\tilde{g}(x)$. Then there exists $\alpha \in (0,1]$ and a sequence of intervals $(k, k+N]$ such that $F_{(k,k+N]} \to g(x)$ and for $x \in (0,1)$ we have
$$g(x) = \begin{cases} \tilde{g}\left(\frac{x}{\alpha}\right) & \text{if } x \in [0, \alpha), \\ 1 & \text{if } x \in [\alpha, 1]. \end{cases}$$

Open problem 5: Find a distribution of the sequence
$$\left(\frac{\varphi(n)}{n}, \frac{\varphi(n+1)}{n+1}\right), \qquad n = 1, 2, \dots$$

Problems 1–5 proposed by O. Strauch.

Open problem 6 proposed by F. Luca:

(i) Is the sequence of general term $\big(\varphi(1) + \dots + \varphi(n)\big)/n$ uniformly distributed modulo 1?

(ii) Is the sequence of general term $\big(\varphi(1)\varphi(2)\cdots\varphi(n)\big)^{1/n}$ uniformly distributed modulo 1?

Regarding (i) above, R. Balasubramanian and F. Luca (2007) have shown that the set of n such that $\big(\varphi(1)+\cdots+\varphi(n)\big)/n$ is an integer is of asymptotic density zero.

Solution of 6. J.-M. Deshouillers and H. H. Iwaniec (2008) gave positive answer to (i) and conditional positive answer to (ii), they proved:

(XII) Let $\nu(n)$ be an arithmetic function which is completely multiplicative and satisfies the conditions

(i) $|\nu(p)|\le\nu$ for some positive number ν and every prime p,

(ii) $\sum_{d\le x}\mu(d)\nu(d)\ll x\,(\log x)^{-A}$ for every positive A,

where the implied constant depends only on ν and A. Define the arithmetic function ϕ by $\phi(m)=m\prod_{p|m}\big(1-\frac{\nu(p)}{p}\big)$. Then, if the number $\alpha=\frac{1}{2}\prod_p\big(1-\frac{\nu(p)}{p^2}\big)$ is irrational, the sequence $\frac{1}{n}\sum_{m\le n}\phi(m)$, $n=1,2,\dots$, is u.d. modulo one.

Notes. For classical Euler $\varphi(n)$ function corresponding $\alpha=\frac{3}{\pi^2}$.

(XIII) Let $\nu(n)$ be a completely multiplicative function such that

(i) $-\nu\le\nu(p)<\min\{p,\nu\}$ for some positive ν and every prime p,

(ii) that there exist real numbers β and λ such that $\prod_{p\le n}\big(1-\frac{\nu(p)}{p}\big)=\beta(\log n)^{-\lambda}\big(1+O\big(\frac{1}{\log n}\big)\big)$, where the implied constant depends only on ν.

Again as in (I), we define the strongly multiplicative function ϕ by $\phi(m)=m\prod_{p|m}\big(1-\frac{\nu(p)}{p}\big)$, and we let $\alpha=\frac{1}{e}\prod_p\big(1-\frac{\nu(p)}{p}\big)^{\frac{1}{p}}$. If α is irrational, then the sequence $\big(\prod_{m\le n}\phi(m)\big)^{\frac{1}{n}}$, $n=1,2,\dots$, is u.d. modulo one.

(XIV) Let the arithmetical function ν satisfy (i) and (ii) in (XIII). If α is rational and ν takes only algebraic values, then the sequence $\big(\prod_{m\le n}\phi(m)\big)^{\frac{1}{n}}$, $n=1,2,\dots$, is not uniformly distributed modulo one. By the authors comments, for the classical Euler $\varphi(n)$ the arithmetic property of corresponding

$$\alpha=\frac{1}{e}\prod_p\left(1-\frac{1}{p}\right)^{\frac{1}{p}}$$

is an **open problem**. This constant is very likely to be irrational: R. Bumby showed that if α is rational, then its denominator has at least 20 decimal digits. A special case of (XIV) shows that if the constant α is rational, then the sequence $\big(\prod_{m\le n}\varphi(m)\big)^{\frac{1}{n}}$, $n=1,2,\dots$, is not u.d. modulo 1.

(XV) F. Luca, V. J. Mejía Huguet and F. Nicolae (2009) show that

$$\left(\frac{\varphi(F_{n+1})}{\varphi(F_n)}, \frac{\varphi(F_{n+2})}{\varphi(F_n)}, \cdots \frac{\varphi(F_{n+k})}{\varphi(F_n)}\right), \qquad n = 1, 2, \dots$$

is dense in $[0, \infty)^k$, $k = 1, 2, \dots$. The authors have the following comments:

– for any positive integer k and every permutation $(i_1, \dots, i_k)$ there exist infinitely many integers n such that $\varphi(F_{n+i_1}) < \varphi(F_{n+i_2}) < \cdots < \varphi(F_{n+i_k})$.

– P. Erdős, K. Győry and Z. Papp (1980) call two arithmetic functions $f(n)$ and $g(n)$ *independent* if for every permutations $(i_1, \dots, i_k)$ and $(j_1, \dots, j_k)$ of $(1, \dots, k)$, there exist infinitely many integers n such that both

$$f(n + i_1) < f(n + i_2) < \cdots < f(n + i_k),$$
$$g(n + j_1) < g(n + j_2) < \cdots < g(n + j_k).$$

– $\varphi(n)$ and Carmichael $\lambda(n)$ are independent (N. Doyon and F. Luca (2006)).

– $\sigma\big(\varphi(n)\big)$ and $\varphi\big(\sigma(n)\big)$ are independent (M. O. Hername and F. Luca (2009)).

Open problems in F. Luca, V.J. Mejía Huguet and F. Nicolae (2009):

– Are the functions $\varphi(F_n)$ and $F_{\varphi(n)}$ independent?

– Are the functions $\varphi(F_n)$ and $\varphi(M_n)$ independent?

Submitted by O. Strauch.

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1.12. van der Corput sequence in the base q

[SP, p. 2–102, 2.11]. Let $q \geq 2$ be an integer and

$$n = a_0(n) + a_1(n)q + \cdots + a_{k(n)}(n)q^{k(n)},\ a_j(n) \in \{0, 1, \ldots, q-1\},\ a_{k(n)} > 0,$$

be the q-adic digit expansion of integer n in the base q. Then the van der Corput sequence $\gamma_q(n)$, $n = 0, 1, 2, \ldots$, in the base q defined by

$$\gamma_q(n) = \frac{a_0(n)}{q} + \frac{a_1(n)}{q^2} + \cdots + \frac{a_{k(n)}(n)}{q^{k(n)+1}}$$

is u.d.

Open problem: Find the distribution function of the sequence

$$\big(\gamma_q(n), \ldots, \gamma_q(n+s-1)\big), \qquad n = 0, 1, 2, \ldots,\ \text{in } [0,1]^s.$$

NOTES. (I) The $\gamma_q(n)$ is called the **radical inverse function** of the natural q-adic digit expansion of n.

(II) The Halton sequence in the bases $q_1, \ldots, q_s$ is defined by

$$\mathbf{x}_n = \big(\gamma_{q_1}(n), \ldots, \gamma_{q_s}(n)\big), \qquad n = 0, 1, 2, \ldots$$

For the pairwise coprime bases $q_1, \ldots, q_s$ the Halton sequence is u.d., cf. [SP, p. 3–72].

(i) The Halton sequence $\mathbf{x}_n$ is u.d. in $[0,1)^s$ if and only if the bases $q_1, \ldots, q_s$ are coprime, see P. Hellekalek and H. Niederreiter (2011).

(III) Van der Corput sequence $\gamma_q(n)$, $n = 0, 1, \ldots, N-1$ has discrepancy

$$D_N^*\big(\gamma_q(n)\big) < \frac{1}{N}\left(\frac{q\log(qN)}{\log q}\right),$$

i.e., it is low discrepancy sequence, but for $s = 2$ we have, see O. Blažeková (2007),

$$D_N\big(\gamma_q(n), \gamma_q(n+1)\big) = \frac{1}{4} + O\Big(D_N\big(\gamma_q(n)\big)\Big),$$

$$D_N^*\Big(\big(\gamma_q(n), \gamma_q(n+1)\big)\Big) = \max\left(\frac{1}{q}\left(1 - \frac{1}{q}\right), \frac{1}{4}\left(1 - \frac{1}{q}\right)^2\right) + O\Big(D_N\big(\gamma_q(n)\big)\Big).$$

Thus, van der Corput sequence is not pseudo-random.

(IV) Discrepancy bounds of the van der Corput and Halton sequences can be found in the added bibliography.

(V) **Solution for** $s = 2$ is given in J. Fialová and O. Strauch (2010): Every point $\big(\gamma_q(n), \gamma_q(n+1)\big)$, $n = 0, 1, 2, \ldots$, lie on the line segment

$$Y = X - 1 + \frac{1}{q^k} + \frac{1}{q^{k+1}}, \quad X \in \left[1 - \frac{1}{q^k}, 1 - \frac{1}{q^{k+1}}\right]$$

for $k = 0, 1, \ldots$ and let T be their union. Because $\gamma_q(n)$ is u.d., then the sequence $\big(\gamma_q(n), \gamma_q(n+1)\big)$ has a.d.f. $g(x,y)$ of the form

$$g(x,y) = \Big|\mathrm{Project}_x\Big(([0,x) \times [0,y)) \cap T\Big)\Big|,$$

where $\mathrm{Project}_x$ is a projection of a two dimensional set to the x-axis. It is a copula and $g(x,y)$ can be computed explicitly as

$$g(x,y) = \begin{cases} 0 & \text{if } (x,y) \in A, \\ 1-(1-y)-(1-x) = x+y-1 & \text{if } (x,y) \in B, \\ y - \frac{1}{q^i} & \text{if } (x,y) \in C_i, \\ x - 1 + \frac{1}{q^{i-1}} & \text{if } (x,y) \in D_i, \end{cases}$$

$i = 1, 2, \ldots$, where

(VI) **Formal solution.** Ch. Aisleitner and M. Hofer (2013): Let T denote von Neuman-Kakutani transformation described in Fig. 1. Define an s-dimensional curve $\big\{\gamma(t); t \in [0,1)\big\}$, where $\gamma(t) = \big(t, T(t), T^2(t), \ldots, T^{s-1}t\big)$. Then the searched a.d.f. is

$$g(x_1, x_2, \ldots, x_s) = \Big|\big\{t \in [0,1]; \gamma(t) \in [0,x_1] \times [0,x_2] \times \cdots \times [0,x_s]\big\}\Big|,$$

where $|X|$ is the Lebesgue measure of set X. An explicit formula of a such a.d.f. for $s = 4$ is **open.**

(VI') For an arbitrary continuous $F(x_1, x_2, \ldots, x_s)$ we have

$$\int_{[0,1]^s} F(x_1, x_2, \ldots, x_s)\, \mathrm{d}g(x_1, x_2, \ldots, x_s) = \int_0^1 F\big(x, T(x), T^2(x), \ldots, T^{s-1}(x)\big)\, \mathrm{d}x.$$

Proof. Put $\gamma(n) = x_n$. Then

$$\big(\gamma_q(n), \ldots, \gamma_q(n+s-1)\big) = \big(x_n, T(x_n), T^2(x_n), \ldots, T^{s-1}(x_n)\big)$$

and by Weyl's limit relation

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} F\big(\gamma_q(n), \ldots, \gamma_q(n+s-1)\big)$$
$$= \int_{[0,1]^s} F(x_1, x_2, \ldots, x_s)\, \mathrm{d}g\,(x_1, x_2, \ldots, x_s)$$

and

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} F\big(x_n, T(x_n), \ldots, T^{s-1}(x_n)\big)$$

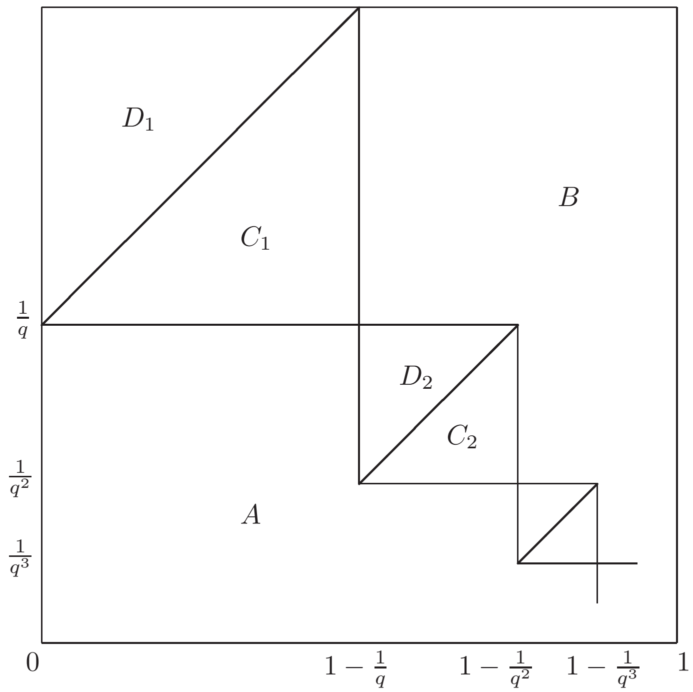

Figure 1. Line segments containing $\big(\gamma_q(n), \gamma_q(n+1)\big), n = 1, 2, \dots$ The graph of von Neumann-Kakutani transformation.

$$= \int_0^1 F\big(x, T(x), \dots, T^{s-1}(x)\big)\, \mathrm{d}x.$$

(VII) **A.d.f. of** $\big(\gamma_q(n), \gamma_q(n+2)\big)$, $n = 1, 2, \dots$ All terms of the sequence

$$\big(\gamma_q(n), \gamma_q(n+2)\big), \qquad n = 1, 2, \dots,$$

lie in the line segments

$$Y = X + \frac{2}{q}, \qquad X \in \left[0, 1 - \frac{2}{q}\right), \qquad \text{or}$$

$$Y = X + \frac{1}{q} + \frac{1}{q^{i+1}} + \frac{1}{q^{i+2}} - 1, \qquad X \in \left[1 - \frac{1}{q} - \frac{1}{q^{i+1}}, 1 - \frac{1}{q} - \frac{1}{q^{i+2}}\right), \qquad \text{or}$$

$$Y = X + \frac{1}{q} + \frac{1}{q^{i+1}} + \frac{1}{q^{i+2}} - 1, \qquad X \in \left[1 - \frac{1}{q^{i+1}}, 1 - \frac{1}{q^{i+2}}\right)$$

for $i = 0, 1, \dots$ Divide $[0,1]^2$ by the Fig. 2 then we have the following explicit form of a.d.f. $g(x,y)$ of the sequence $\big(\gamma_q(n), \gamma_q(n+2)\big)$.

$$g(x,y)=\begin{cases} x & \text{if} \quad (x,y)\in D_0,\\ y-\frac{2}{q} & \text{if} \quad (x,y)\in C_0,\\ 0 & \text{if} \quad (x,y)\in A_0,\\ y+x-1 & \text{if} \quad (x,y)\in B_0,\\ x-1+\frac{2}{q} & \text{if} \quad (x,y)\in E_0,\\ y & \text{if} \quad (x,y)\in F_0,\\ 0 & \text{if} \quad (x,y)\in A',\\ x+y-1+\frac{1}{q} & \text{if} \quad (x,y)\in B',\\ x-1+\frac{1}{q}+\frac{1}{q^i} & \text{if} \quad (x,y)\in D'_i,\\ y-\frac{1}{q^{i+1}} & \text{if} \quad (x,y)\in C'_i,\\ \frac{1}{q} & \text{if} \quad (x,y)\in A'',\\ x+y-1 & \text{if} \quad (x,y)\in B'',\\ x-1+\frac{1}{q}+\frac{1}{q^i} & \text{if} \quad (x,y)\in D''_i,\\ y-\frac{1}{q^{i+1}} & \text{if} \quad (x,y)\in C''_i. \end{cases}$$

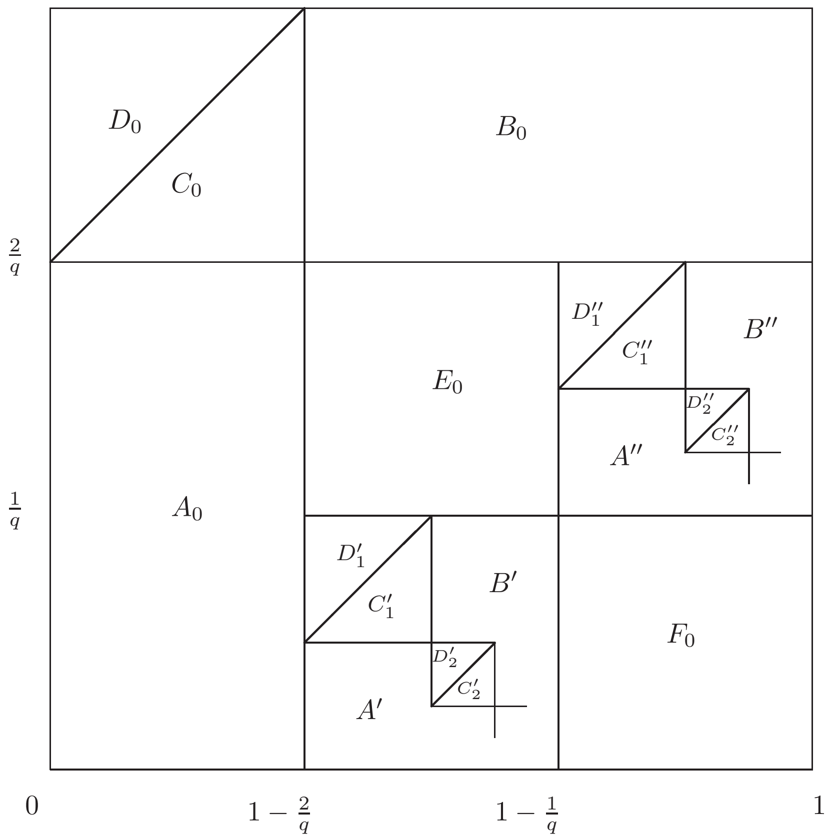

Figure 2.

(VIII) **A.d.f. of** $\big(\gamma_q(n),\gamma_q(n+1),\gamma_q(n+2)\big)$**,** $n=1,2,\dots$ An explicit form of $g(x,y,z)$ is given in J. Fialová, L. Mišík and O. Strauch (2013) and it have 27 possibilities. For example, if $q\ge 3$, then

$$g(x,x,x)=\begin{cases}0 & \text{if } x\in\left[0,\frac{2}{q}\right],\\ x-\frac{2}{q} & \text{if } x\in\left[\frac{2}{q},1-\frac{1}{q}\right],\\ 3x-2 & \text{if } x\in\left[1-\frac{1}{q},1\right].\end{cases}$$

(IX) As an applications, by the Weyl limit relation, we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}F\big(\gamma_q(n),\gamma_q(n+1),\gamma_q(n+2)\big)$$
$$=\int_0^1\int_0^1\int_0^1 F(x,y,z)\,\mathrm{d}_x\,\mathrm{d}_y\,\mathrm{d}_x\,g(x,y,z),$$

where $F(x,y,z)$ is an arbitrary continuous function in $[0,1]^3$.
(X) For the right-hand side of (IX) we can using

$$\int_0^1\int_0^1\int_0^1 F(x,y,z)\,\mathrm{d}_x\,\mathrm{d}_y\,\mathrm{d}_z\,g(x,y,z)=F(1,1,1,)$$
$$-\int_0^1 g(1,1,z)\,\mathrm{d}_z\,F(1,1,z)-\int_0^1 g(1,y,1)\,\mathrm{d}_y\,F(1,y,1)-\int_0^1 g(x,1,1)\,\mathrm{d}_x\,F(x,1,1)$$
$$+\int_0^1\int_0^1 g(1,y,z)\,\mathrm{d}_y\,\mathrm{d}_z\,F(1,y,z)+\int_0^1\int_0^1 g(x,1,z)\,\mathrm{d}_x\,\mathrm{d}_z\,F(x,1,z)$$
$$+\int_0^1\int_0^1 g(x,y,1)\,\mathrm{d}_x\,\mathrm{d}_y\,F(x,y,1)$$
$$-\int_0^1\int_0^1\int_0^1 g(x,y,z)\,\mathrm{d}_x\,\mathrm{d}_y\,\mathrm{d}_z\,F(x,y,z).$$

(XI) **Example.** Put $F(x,y,z)=\max(x,y,z)$. Then by (X)

$$\int_0^1\int_0^1\int_0^1 F(x,y,z)\,\mathrm{d}_x\,\mathrm{d}_y\,\mathrm{d}_z\,g(x,y,z)$$

$$= 1 - \int_0^1\int_0^1\int_0^1 g(x,y,z)\,\mathrm{d}_x\,\mathrm{d}_y\,\mathrm{d}_z\,F(x,y,z)$$

$$= 1 - \int_0^1 g(x,x,x)\,\mathrm{d}\,x$$

and for $q \geq 3$ we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \max\bigl(\gamma_q(n),\gamma_q(n+1),\gamma_q(n+2)\bigr) = \frac{1}{2} + \frac{2}{q} - \frac{3}{q^2}.$$

(XII) **Example.** Put $F(x,y,z) = \min(x,y,z)$. Then by (X) we have

$$= \int_0^1\int_0^1\int_0^1 F(x,y,z)\,\mathrm{d}_x\,\mathrm{d}_y\,\mathrm{d}_x\,g(x,y,z)$$

$$1 - 3\cdot\frac{1}{2} + 2\cdot\int_0^1 g(x,x,1)\,\mathrm{d}x + \int_0^1 g(x,1,x)\,\mathrm{d}x - \int_0^1 g(x,x,x)\,\mathrm{d}x$$

which implies

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \min\bigl(\gamma_2(n),\gamma_2(n+1),\gamma_2(n+2)\bigr) = \begin{cases} \frac{1}{2} - \frac{2}{q} + \frac{3}{q^2} & \text{if } q \geq 4,\\ \frac{1}{6} & \text{if } q = 3,\\ \frac{3}{16} & \text{if } q = 2. \end{cases}$$

(XIII) **Example.** Put $F(x,y,z) = xyz$. By (X) we have

$$\int_0^1\int_0^1\int_0^1 F(x,y,z)\,\mathrm{d}_x\,\mathrm{d}_y\,\mathrm{d}_x\,g(x,y,z) = 1 - 3\cdot\frac{1}{2} + 2\cdot\int_0^1\int_0^1 g(x,y,1)\,\mathrm{d}x\,\mathrm{d}y$$
$$+ \int_0^1\int_0^1 g(x,1,z)\,\mathrm{d}x\,\mathrm{d}z - \int_0^1\int_0^1\int_0^1 g(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

and we find

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \gamma_q(n)\cdot\gamma_q(n+1)\cdot\gamma_q(n+2) = \frac{q^4 - 3q^3 + 3q^2 + 2q + 2}{4q^4 + 4q^3 + 4q^2}$$

for $q \geq 3$.

Submitted by O. Strauch.

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1.13. Sequences of differences

[SP, p. 2–7, 2.1.7]. The sequence

$$x_n \in [0,1), \qquad n = 1, 2, \ldots,$$

is u.d. if and only if the sequence

$$|x_m - x_n|, \qquad m, n = 1, 2, \ldots,$$

has the a.d.f.

$$g(x) = 2x - x^2.$$

Here the double sequence $|x_m - x_n|$, for $m, n = 1, 2, \ldots$, is ordered to an ordinary sequence y_n in such a way that the first N^2 terms of y_n are $|x_m - x_n|$ for $m, n = 1, 2, \ldots, N$.

Open problem: Assuming u.d. of x_n, $n = 1, 2, \ldots$ find a.d.f of the sequences

- $||x_m - x_n| - |x_k - x_l||$, $m, n, k, l = 1, 2, \ldots,$
- $|||x_m - x_n| - |x_k - x_l|| - ||x_i - x_j| - |x_r - x_s|||$, $m, n, k, l, i, j, r, s = 1, 2, \ldots,$ etc.

NOTES. We can use the following method: Let us denote by $g_j(x)$ an asymptotic distribution function of the sequence of jth differences (thus $g_1(x) = 2x - x^2$). For kth moment we have

$$\int_0^1 x^k \mathrm{d}g_{j+1}(x) = \int_0^1\int_0^1 |x-y|^k \mathrm{d}g_j(x)\,\mathrm{d}g_j(y).$$

For $j = 1$ we have

$$\int_0^1\int_0^1 |x-y|^k \mathrm{d}g_1(x)\,\mathrm{d}g_1(y) = \frac{8}{(k+1)(k+2)(k+4)}.$$

which implies

$$g_2(x) = \frac{8}{3}x - 2x^2 + \frac{1}{3}x^4.$$

Conjecture proposed by S. Steinerberger (2010): The density function $\frac{\mathrm{d}g_j(x)}{\mathrm{d}x}$ of the a.d.f. $g_j(x)$ of jth iterated differences is of the form

$$\frac{\mathrm{d}g_j(x)}{\mathrm{d}x} = \frac{2^{2^j+j-1}}{2^j!}(x-1)^j p(x),$$

where $p(x)$ is a polynomial with integer coefficients. What can be said about $p(x)$?

For $j = 3$ he found

$$\frac{\mathrm{d}g_3(x)}{\mathrm{d}x} = \frac{8}{315}(x-1)^3(-132 - 116x - 36x^2 + 3x^3 + x^4).$$

Proposed by O. Strauch.

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1.14. Bernoulli numbers

Open problem: For Bernoulli numbers B_{2n} find the distribution

$$B_{2n} \bmod 1 \qquad n = 1, 2, \dots$$

NOTES. (I) By von Staudt-Clausen formula $B_{2n} = A_{2n} - \sum_{(p-1)|2n} \frac{1}{p}$, where p are primes and A_{2n} are integers.
(II) F. Luca's comment: This problem was studied by P. Erdős and S. S. Wagstaff, Jr. (1980). They proved that $\sum_{(p-1)|2n} \frac{1}{p}$ is everywhere dense in $[5/6, \infty)$.

Submitted by O. Strauch.

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1.15. Ratio sequences

[SP, p. 2–215, 2.22.2]. For an increasing sequence of positive integers x_n let $\underline{d}(x_n)$, and $\overline{d}(x_n)$ denote the lower and upper asymptotic density of x_n, resp., and $d(x_n)\big(= \underline{d}(x_n) = \overline{d}(x_n)\big)$ its asymptotic density if it exists. The double sequence, called the **ratio sequence of** x_n,

$$\frac{x_m}{x_n}, \qquad m, n = 1, 2, \dots,$$

is everywhere dense in $[0, \infty)$ assuming that one of the following conditions holds:

(i) $d(x_n) > 0$,

(ii) $\overline{d}(x_n) = 1$,

(iii) $\underline{d}(x_n) + \overline{d}(x_n) \geq 1$,

(iv) $\underline{d}(x_n) \geq 1/2$,

(v) $A\big([0, x); x_n\big) \sim \frac{cx}{\log^\alpha x}$, where $c > 0, \alpha > 0$ are constant, $A\big([0, x); x_n\big) = \#\big\{n \in \mathbb{N}\,;\, x_n \in [0, x)\big\}$, and $\sim$ denotes the asymptotically equivalence (i.e., the ratio of the left and the right-hand side tends to 1 as $x \to \infty$).

Notes. (I) (i), (ii) and (v) were proved by T. Šalát (1969), for (iii) see O. Strauch and J. T. Tóth (1998) and (iv) follows from (iii).
(II) O. Strauch and J. T. Tóth (1998, Th. 2) proved that if the interval $(\alpha, \beta) \subset [0, 1]$ has an empty intersection with $\frac{x_m}{x_n}$ for $m, n = 1, 2, \dots$, then

$$\underline{d}(x_n) \leq \frac{\alpha}{\beta} \min\big(1 - \overline{d}(x_n), \overline{d}(x_n)\big), \qquad \overline{d}(x_n) \leq 1 - (\beta - \alpha). \tag{1}$$

(III) S. Konyagin (1999, personal communication) improved the second inequality to

$$\overline{d}(x_n) \leq \frac{1 - \beta}{1 - \alpha\beta}. \tag{2}$$

Problem. Find a best possible estimation of $\overline{d}(x_n)$.
Solution. G. Grekos (2006, personal communication) notes that in (2) the equation is valid for the sequence x_n, $n = 1, 2, \dots$ defined in O. Strauch and J. T. Tóth (1998, Ex. 1) such that x_n is the sequence of all integer points lying in the intervals

$$(\gamma, \delta), (\gamma a, \delta a), (\gamma a^2, \delta a^2), \dots, (\gamma a^n, \delta a^n), \dots,$$

where γ, δ and a are positive real numbers satisfying $\gamma < \delta$ and $a > 1$. In this case the lower $\underline{d}$ and upper $\overline{d}$ asymptotic density of x_n can be given explicitly by

$$\underline{d}(x_n) = \frac{(\delta - \gamma)}{\gamma(a-1)}, \qquad \overline{d}(x_n) = \frac{(\delta - \gamma)a}{\delta(a-1)},$$

and an interval (α, β) which does not contain points $\frac{x_m}{x_n}$, $m, n = 1, 2, \dots$ is

$$(\alpha, \beta) = \left(\frac{\delta}{\gamma a}, \frac{\gamma}{\delta}\right), \tag{3}$$

assuming $\delta/\gamma < \sqrt{a}$. All others subintervals of $[0,1]$ with empty intersection have the form $(\alpha/a^i, \beta/a^i)$, $i = 1, 2, \dots$. For interval (3) we have $\overline{d}(x_n) = \frac{1-\beta}{1-\alpha\beta}$ then (2) is the best possible.

(V) O. Strauch and Tóth (1998, Th. 6) also proved that (1) and (2) are also valid for interval (α, β) containing no accumulation points of $\frac{x_m}{x_n}$, $m, n = 1, 2, \dots$. If $X \subset [0,1]$ is an union of such intervals, then

$$\overline{d}(x_n) \le 1 - |X|,$$

where $|X|$ denotes the Lebesgue measure of X. Its improvement is **open**.

Proposed by O. Strauch.

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1.16. Continued fractions

[SP, p. 2–264, 2.26.8]. Let $\theta = [0; a_1, a_2, \dots]$ be an irrational number in $[0,1]$ given by its continued fraction expansion and let $p_n(\theta)/q_n(\theta)$, $n = 0, 1, 2, \dots$, be the corresponding sequence of its convergents. An **open problem** is to find, for the sequence

$$x_n = q_n(\theta) \pmod 2$$

the frequency of each possible block $(\dots, 0, \dots, 1, \dots, 0, \dots)$ of length s which occurs in x_n as $(x_{n+1}, \dots, x_{n+s})$ for a special class of θ (e.g., with bounded a_i).

NOTES. R. Moeckel (1982) proved that, for almost all θ, the three possible blocks $(0,1)$, $(1,0)$ and $(1,1)$ of length $s = 2$ occur in x_n with equal frequencies. The blocks of lengths $s = 3$ and $s = 4$ are investigated in V. N. Nolte (1990).

Proposed by O. Strauch.

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1.17. Strong uniform distribution

Say a sequence of integers $(a_n)_{n=1}^{\infty}$ is in $\mathbf{A}^*$ for a class of measurable functions $\mathbf{A}$ on $[0,1)$ if

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f\big(\{a_n x\}\big) = \int_0^1 f(t)\,\mathrm{d}t \quad \text{a.e.}$$

Here of course, for a real number y we have used $\{y\}$ to denote its fractional part. In a paper solving a well known classical problem of A. Khinchin's [Kh], J. M. Marstrand [M] also showed that if $q_1,\dots,q_k$ is a finite list of coprime natural numbers all greater than one then the semigroup it generates, $\mathbf{m} = (m_l)_{l=1}^{\infty} = \big\{q_1^{i_1}\cdots q_k^{i_k} : (i_1,\dots,i_k) \in (\mathbb{Z}_0^+)^k\big\}$ when ordered by size is in $(L^\infty)^*$. To do this he invoked D. Birkhoff's pointwise ergodic theorem [W] and the following lemma:

For strictly increasing sequences of natural numbers $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ if $G(u) = \big\{(r,s) : a_r b_s \le u\big\}$, $(u = 1,2,\dots)$ and $f \in L^\infty$ then

$$\lim_{u\to\infty}\frac{1}{|G(u)|}\sum_{(r,s)\in G(u)} f\big(\{a_r b_s x\}\big) = \int_0^1 f(t)\,\mathrm{d}t \quad \text{a.e.}$$

Here for a finite set A we have used $|A|$ to denote its cardinality. R. C. Baker [B] asked if $\mathbf{m}$ is in $(L^1)^*$. This was proved by the author [Na1] using A. A. Tempelman's generalization of Birkhoff's pointwise ergodic theorem [T]. With a view to applications to sequences other than $\mathbf{m}$, it would be interesting to know if an L^1 version of Marstrand's lemma is true. Some partial results in this direction are known. In [Na2] we show that for strictly increasing sequences of natural numbers $a = (a_r)_{r=1}^{\infty}$ and $b = (b_s)_{s=1}^{\infty}$, both of which are $(L^p)^*$ sequences for all $p > 1$, if there exists $C > 0$ such that

$$|\{r : a_r \le u\}||\{s : b_s \le u\}| \;\le\; C\big|\big\{(r,s) : a_r b_s \le u\big\}\big|, \tag{1}$$

for $(u = 1,2,\dots)$ then $a \circ b = \big\{a_r b_s : (r,s) \in \mathbf{N}^2\big\}$ (the sequence of products of pairs of elements in a and b) once ordered by size is also an $(L^p)^*$ sequence. An open question is whether this result from [Na2] is true for $p = 1$. We have the following partial result. Let

$$a_1 = (a_{1,i})_{i=1}^{\infty}, \dots, a_k = (a_{k,i})_{i=1}^{\infty},$$

denote finitely many $(L^1)^*$ sequences, and for a sequence a, let

$$G_a(u) = |\{i : a_i \leq u\}|.$$

Also let $a_1 \circ \cdots \circ a_k$ denote the set

$$\{b_1 \cdots b_k : b_1 \in a_1, \ldots, b_k \in a_k\},$$

counted with multiplicity and ordered by absolute value. Suppose there exists $K > 0$ such that for all $u \geq 1$

$$|G_{a_1}(u)| \cdots |G_{a_k}(u)| \leq K |G_{a_1 \circ \cdots \circ a_k}(u)|.$$

Then if $\log_+ |x| = \log \max(1, |x|)$ we show that $a_1 \circ \cdots \circ a_k$ is an $\left(L(\log_+ L)^{k-1}\right)^*$ sequence [Na3]. To be more specific, we ask if the space $\left(L(\log_+ L)^{k-1}\right)^*$ can be replaced by $(L^1)^*$. A second open question is whether a condition like (1) necessary for any of these results. As Marstrand observed the answer is no when $p = \infty$. It might be the case that if for fixed $p \in [1, \infty]$ if all three sequences $a = (a_r)_{r=1}^{\infty}$, $b = (b_s)_{=1}^{\infty}$ and $a \circ b$ are $(L^p)^*$ then (1) automatically holds. This too is unknown.

Proposed by R. Nair.

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1.18. Algebraic dilatations

Let $\theta_1, \ldots, \theta_k$ denote a finite set of real algebraic numbers all greater than 1 and let $(m_l)_{l=1}^{\infty}$ denote the semigroup generated multiplicatively by these numbers given an order consistent with the magnitude of the elements of this semigroup.

Suppose $f \in L^p\big([0,1)\big)$ for some $p \in [1,\infty)$. Under what conditions on p and the sequence $(m_l)_{l=1}^{\infty}$ is it the case that

$$\lim_{L\to\infty} \frac{1}{L}\sum_{l=1}^{L} f\big(\{m_l x\}\big) = \int_0^1 f(t)\mathrm{d}t \quad \text{a.e.}$$

This question is probably best understood by comparison with the cases where the $\theta_1,\dots,\theta_k$ are coprime natural numbers, in which case the result is true for $p = 1$ [Na1] and the case where $k = 1$ in which case the result is known for $p = 2$ [Bo][vPS]. In unpublished work the author has also shown that the result is true for $p > 1$ if only one of the set $\theta_1,\dots,\theta_k$ is not a rational integer. It seems likely some other arithmetic condition on the $\theta_1,\dots,\theta_k$ is necessary. For instance is the result true for instance when $p > 1$ assuming the natural logarithms of the numbers $\theta_1,\dots,\theta_k$ are linearly independent over the rationals? Are there circumstances where the assumption that k is finite can be dropped? The answer is yes for $\theta_1,\theta_2,\dots$ chosen to be some rapidly growing rational primes as shown in [L].

Proposed by R. Nair.

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1.19. Subsequence ergodic theorems

Suppose (X,β,μ) is a measure space. Say T a map from X to itself is measurable if $T^{-1}A := \{x : Tx \in A\} \in \beta$ for all A in the σ-algebra β. We say a measurable map T from X to itself is measure preserving if $\mu(T^{-1}A) = \mu(A)$ for all $A \in \beta$. Building on earlier work of J. Bourgain [Bo1], [Bo2], [Bo3], the author showed [Na4] [Na5] that if ϕ is a non-constant polynomial mapping the natural numbers to themselves, $(p_n)_{n=1}^{\infty}$ is the sequence of rational primes and if for $p > 1$ we have $f \in L^p(X,\beta,\mu)$ then $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f\big(T^{\phi(p_n)}x\big)$ exists μ almost everywhere. See also [Wi]. Our first question is what happens when $p = 1$. The analogous question for the case where p_n is the n^{th} natural number is also open. For the second question suppose $T_1,\dots,T_k$ is a finite set of commuting measure preserving maps on (X,β,μ) and that $\phi_1,\dots,\phi_k$ are nonconstant polynomials mapping the natural numbers to themselves and for $p > 1$ that $f \in L^p(X,\beta,\mu)$.

Then is it the case that $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f\big(T_1^{\phi_1(p_n)} \cdots T_k^{\phi_k(p_n)} x\big)$ μ exists μ almost everywhere? In the case where p_n is the n^{th} natural number this is a theorem, at least when $p = 2$ [Bo2].

Proposed by R. Nair.

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1.20. Square functions for subsequence ergodic averages

For a probability space (X, β, μ), measure preserving $T\colon X \to X$, a non-constant polynomial ϕ mapping the natural numbers to themselves and a μ measurable function f defined on X and a strictly increasing sequence of integers $(N_k)_{k=1}^{\infty}$, set $A_N f(x) = \frac{1}{N}\sum_{n=1}^{N} f\big(T^{\phi(n)}x\big)$ $(N = 1, 2, \dots)$ and set

$$S(f)(x) = \left(\sum_{k=1}^{\infty} |A_{N_{k+1}} f(x) - A_{N_k} f(x)|^2\right)^{\frac{1}{2}}.$$

In the situation where $\phi(n) = n$ and $(N_k)_{k=1}^{\infty}$ is any strictly increasing sequence of natural numbers it is shown in [JOR] that there exists $C > 0$ such that $\mu\big(\{x : S(f)(x) > \lambda\}\big) \le C\frac{||f||_1}{\lambda}$. One implication of this is that for any $p > 1$ there exists $C_p > 0$ such that $||S(f)||_p \le C_p||f||_p$. Results of this sort provide an alternative means, to almost everywhere convergence, of measuring the stability of the averages $(A_N f)_{N=1}^{\infty}$. Ideally we would like to prove an analogue of the [JOR] inequality for general ϕ or if it were not true to find out the extent to which it was and how and when it fails. One approach to questions of these sorts is via spectral theory and this reduces to a study of the behaviour of exponential sums of the form $a_n(\alpha) = \frac{1}{n}\sum_{l=1}^{n} e^{2\pi i\phi(n)\alpha}$ $(n = 1, 2, \dots)$. In the case where $\phi(n) = n$ these are for each α, averages of geometric progressions and consequently have well understood distributions. For more general ϕ these are

Weyl sums, are more complicated and our understanding is less complete. What is known follows from an application of the Hardy-Littlewood circle method. Using this, for $N_k = k$ $(k = 1, 2, \dots)$, the author has shown that for each $p > 1$ there exists $C_p > 0$ such that we have $||S(f)||_p \le C_p ||f||_p$. The same inequality is also true in the case $p = 2$ where $1 < a \le \frac{N_{k+1}}{N_k} < b$ for some a and b. These results are not as yet published [Na6] though see [NW] where variants of the method involved appear.

Proposed by R. Nair.

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1.21. Sets of integers of positive density

In this section for simplicity we confine attention to sets of integers of positive density in $\mathbb{N}$, though much of what we deal with can be meaningfully discussed in higher dimensions. For a set of natural numbers $S \subset \mathbb{N}$ and a countable collection $\mathbf{I} = (I_n)_{n\ge1}$ of intervals with $|I_n|$ tending to infinity as n does, let

$$B(S, I) = \limsup_{n\to\infty} \frac{|S \cap I_n|}{|I_n|}.$$

We call

$$B(S) = \sup_{\mathbf{I}} B(S, \mathbf{I}),$$

where the supremum is taken over all collections $\mathbf{I}$, the Banach density of S. The second definition we need is that of upper density defined as

$$d^*(S) = \limsup_{n\to\infty} \frac{|S \cap [0, n)|}{n}.$$

If the limit exists, we call $d^*(S)$ the density of S and denote it by $d(S)$. Plainly $B(S) > 0$ if $d^*(S) > 0$. A sequence of integers $\mathbf{k} = \big(k(n)\big)_{n\ge1}$ is called *intersective* if given any $S \subset \mathbb{N}$ with $B(S) > 0$ there exist a and b in S such that $a - b$ is in $\mathbf{k}$. A sequence of integers $\mathbf{k} = \big(k(n)\big)_{n\ge1}$ is called a set of *Poincaré recurrence* if given any (X, β, μ) any measure preserving transformation $T\colon X \to X$, and any A in β with $\mu(A) > 0$ there exists k in $\mathbf{k}$ such that

$$\mu(A \cap T^k A) > 0.$$

The terminology is motivated by the fact that as proved by H. Poincaré, $\mathbf{k}$ is a set of Poincaré recurrence in the case $k(n) = n$. What makes ergodic theory relevant to the study of intersectivity is that as proved by A. Bertrand-Mathis [Be] and H. Furstenberg [F] a sequence on integers is a set of intersectivity if and only if it is a set of Poincaré recurrence. Furstenberg used this viewpoint to show that when $k(n) = n^2$ $(n = 1, 2, \dots)$ then $\mathbf{k} = (k(n))_{n=1}^{\infty}$ is intersective. We say a sequence $\mathbf{k} = (k_n)_{n=1}^{\infty}$ is S. Hartman uniformly distributed [Ha] if for every non-integer θ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i\theta k_n} = 0.$$

It can be shown that any $\mathbf{k}$ that is Hartman uniformly distributed is also intersective [Bo]. E. Szemeredi [S] showed that if $B(S) > 0$ then S contains arithmetic progressions of arbitrary length. This was shown by H. Furstenberg to be a consequence of the fact that given any (X, β, μ) any measure preserving transformation $T\colon X \to X$, and any A in β with $\mu(A) > 0$ there exists a natural number n such that [F]

$$\mu\big(A \cap T^n A \cap \dots \cap T^{n(k-1)} A\big) > 0. \tag{1}$$

Motivated by (1) it would be interesting to decide whether if $\mathbf{k}$ is Hartman uniformly distributed then given any $S \subset \mathbb{N}$ with $B(S) > 0$ there exist $R \subset \mathbb{N}$ with $d(R) > 0$ existing and $d(R) \geq B(S)$ such that for any finite subset $\big\{h(1), \dots, h(l)\big\}$ of R,

$$B\Big(S \cap S + k\big(h(1)\big) \cap \dots \cap S + k\big(h(l)\big)\Big) > 0.$$

The question arose in a conversation with H. Furstenberg at the Newton Institute in 2000. So far it has been possible to prove this hypothesis subject to conditions similar to but stronger than S. Hartman uniform distribution [Na7] [NW] [NZ]. A theorem of V. Bergelson and A. Leibman [BL] is that given any polynomials $P_1(x), \dots, P_k(x)$ with integer coefficients such that $0 = P_1(0) = P_2(0) = \dots = P_k(0)$, any (X, β, μ) any measure preserving transformation $T\colon X \to X$ and any A in β with $\mu(A) > 0$ there exists a natural number n such that

$$\mu\big(A \cap T^{P_1(n)} A \cap \dots \cap T^{P_k(n)} A\big) > 0.$$

This implies both Szemeredi's theorem and the intersectivity of $\mathbf{k}$ where

$$k(n) = n^2.$$

It can also be shown [Na8] that given any polynomial $Q(x)$ such that for each non-zero integer m there exists another integer $l(m)$ with $\big(m, Q(l(m))\big) = 1$ any probability space (X, β, μ) any measure preserving transformation $T\colon X \to X$, any A in β with $\mu(A) > 0$ there exists a prime p such that

$$\mu\big(A \cap T^{Q(p)}A\big) > 0.$$

This leads one to conjecture that given polynomials $Q_1(x), \dots, Q_k(x)$, all with the property assumed for $Q(x)$, then given any probability space (X, β, μ), any measure preserving transformation $T\colon X \to X$, and any A in β with $\mu(A) > 0$, there exists a prime number p such that

$$\mu\big(A \cap T^{Q_1(p)}A \cap \dots \cap T^{Q_k(p)}A\big) > 0.$$

If proved this conjecture would imply analogues of both the Szemeredi theorem and the Furstenberg intersectivity phenomenon with the integer n chosen to be a prime. Examples of polynomials Q are not quite as easy to construct as examples of polynomials satisfying the Bergelson-Leibman condition but they include

$$Q(x) = x^n + 1 \quad \text{or} \quad x^n - 1$$

for any natural number n. Recently B. Green and T. Tao have proved that the set of primes contain arithmetic progressions of arbitrary length. It becomes natural to ask if, by analogy with the Bergelson-Leibman theorem, the primes contain arithmetic progressions of arbitrary length whose common difference is the value $P(a)$ say, for a given integer valued polynomial $P(x)$ with $P(0) = 0$ and some natural number a. We might also ask whether the primes contain arithmetic progressions of arbitrary length whose common difference is $Q(p)$ for a given polynomial Q such that for each non-zero integer m there exists another integer $l(m)$ with $\big(m, Q(l(m))\big) = 1$ and some prime p.

Proposed by R. Nair.

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1.22. Uniform distribution of the weighted sum-of-digits function

Let $\gamma = (\gamma_0, \gamma_1, \dots)$ be a sequence in $\mathbb{R}$ and let $q \in \mathbb{N}$, $q \geq 2$. For $n \in \mathbb{N}_0$ with base q representation $n = n_0 + n_1 q + n_2 q^2 + \cdots$ define the *weighted q-ary sum-of-digits function* by

$$s_{q,\gamma}(n) := \gamma_0 n_0 + \gamma_1 n_1 + \gamma_2 n_2 + \cdots$$

For $d \in \mathbb{N}$, weight-sequences $\gamma^{(j)} = \big(\gamma_0^{(j)}, \gamma_1^{(j)}, \dots\big)$ in $\mathbb{R}$ and $q_j \in \mathbb{N}$, $q_j \geq 2$, $j \in \{1, \dots, d\}$, define

$$s_{q_1,\dots,q_d,\gamma}(n) := \big(s_{q_1,\gamma^{(1)}}(n), \dots, s_{q_d,\gamma^{(d)}}(n)\big),$$

where $\boldsymbol{\gamma} = (\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_1, \dots)$ and $\boldsymbol{\gamma}_k = \big(\gamma_k^{(1)}, \dots, \gamma_k^{(d)}\big)$ for $k \in \mathbb{N}_0$.

Open question: Let $q_1, \dots, q_d \geq 2$ be pairwisely coprime integers. Under which conditions on the weight-sequences $\gamma^{(j)} = \big(\gamma_0^{(j)}, \gamma_1^{(j)}, \dots\big)$ in $\mathbb{R}$, $j \in \{1, \dots, d\}$, is the sequence

$$s_{q_1,\dots,q_d,\boldsymbol{\gamma}}(n) \bmod 1, \qquad n = 0, 1, 2, \dots \tag{1}$$

u.d. mod 1?

Proposed by F. Pillichshammer.

Notes. (I) For example if $\gamma_k^{(j)} = q_j^{-k-1}$ for all $j \in \{1, \dots, d\}$ and all $k \in \mathbb{N}_0$, then we obtain the d-dimensional van der Corput-Halton sequence which is well known to be uniformly distributed modulo one.

(II) If $\gamma_k^{(j)} = q_j^k \alpha_j$ for all $j \in \{1, \dots, d\}$ and all $k \in \mathbb{N}_0$, then the sequence (1) is the sequence $\big(\{n(\alpha_1, \dots, \alpha_d)\}\big)_{n \geq 0}$ which is well known to be uniformly distributed modulo one if and only if $1, \alpha_1, \dots, \alpha_d$ are linearly independent over $\mathbb{Q}$.

(III) If $\gamma_k^{(j)} = \alpha_j \in \mathbb{R}$ for all $j \in \{1, \dots, d\}$ and all $k \in \mathbb{N}_0$, then it was shown by M. Drmota and G. Larcher (2001) that the sequence (1) is u.d. mod 1 if and only if $\alpha_1, \dots, \alpha_d \in \mathbb{R} \setminus \mathbb{Q}$.

(IV) For $q_1 = \cdots = q_d = q$ it was shown by F. Pillichshammer (2007) that the sequence (1) is u.d. mod 1 if and only if for every $\boldsymbol{h} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$ one of the following properties holds: Either

$$\sum_{\substack{k=0 \\ \langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle q \notin \mathbb{Z}}}^{\infty} \|\langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle\|^2 = \infty$$

or there exists a $k \in \mathbb{N}_0$ such that $\langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle \notin \mathbb{Z}$ and $\langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle q \in \mathbb{Z}$. Here $\| \cdot \|$ denotes the distance to the nearest integer, i.e., for $x \in \mathbb{R}$, $\|x\| = \min_{k \in \mathbb{Z}} |x - k|$ and $\langle \cdot, \cdot \rangle$ is the usual inner product.

(V) The generalization can be found in R. Hofer, G. Larcher and F. Pillichshammer (2007), where a similar result was proved with the weighted sum-of-digits function replaced by a generalized weighted digit-block-counting function.

(VI) R. Hofer (2007) proved: Let $q_1, \dots, q_d \geq 2$ be pairwise coprime integers and $\gamma^{(1)}, \dots, \gamma^{(d)}$ be given weight sequences in $\mathbb{R}$. If for each dimension $j \in \{1, \dots, d\}$ the following sum

$$\sum_{i=0}^{\infty} \left\| h \left(\gamma_{2i+1}^{(j)} - q_j \gamma_{2i}^{(j)} \right) \right\|^2$$

is divergent for every nonzero integer h, then the sequence (1) is u.d. in $[0,1)^d$.

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1.23. Moment problem

1.23.1. Truncated Hausdorff moment problem

Recovered a d.f. $g(x)$, given its moments

$$s_n = \int_0^1 x^n \mathrm{d}g(x), \qquad n = 1, 2, \dots, N. \tag{1}$$

The set of all points $(s_1, s_2, \dots, s_N)$ in $[0,1]^N$ for which there exists d.f. $g(x)$ satisfying (1) is called the Nth moment space $\boldsymbol{\Omega}_N$. It can be shown:

(i) the point $(s_1, s_2, \dots, s_N)$ belongs to the moment space $\boldsymbol{\Omega}_N$ if and only if $\sum_{i=0}^{k} (-1)^i \binom{k}{i} s_{i+j} \geq 0$ for all $j, k = 0, 1, 2, \dots, N$.

(ii) $\boldsymbol{\Omega}_N$ is a simply connected, convex, and closed subset of $[0,1]^N$.

(iii) If the point $(s_1, s_2, \dots, s_N)$ belongs to the interior of the moment space $\boldsymbol{\Omega}_N$ the truncated moment problem (1) has infinitely many solutions.

(iv) If $(s_1, s_2, \dots, s_N)$ belongs to the boundary of the $\mathbf{\Omega}_N$, the (1) has a unique solution $g(x)$.

(v) If the sequence $x_n \in [0,1)$, $n = 1, 2, \dots$, satisfies $\lim \frac{1}{N}\sum_{n=1}^N x_n = s_1$, $\lim \frac{1}{N}\sum_{n=1}^N x_n^2 = s_2, \dots, \lim \frac{1}{N}\sum_{n=1}^N x_n^N = s_N$, where $(s_1, s_2, \dots, s_N)$ belongs to the boundary of the $\mathbf{\Omega}_N$, then x_n has an a.d.f. $g(x)$.

Exact characterization of the moment space $\mathbf{\Omega}_N$ can be found in S. Karlin and L. S. Shapley (1953), also see G. A. Athanassoulis and P. N. Gavriliadis (2002).

1.23.2. L^2 moment problem

Given a triple of numbers $(X_1, X_2, X_3) \in [0,1]^3$ O. Strauch (1994) gave a complete solution to the moment problem

$$(X_1, X_2, X_3) = \left(\int_0^1 g(x)\,\mathrm{d}x, \int_0^1 xg(x)\,\mathrm{d}x, \int_0^1 g^2(x)\,\mathrm{d}x\right)$$

in d.f. $g(x)\colon [0,1] \to [0,1]$. He expresses the boundary of the body

$$\Omega = \left\{\left(\int_0^1 g(x)\,\mathrm{d}x, \int_0^1 xg(x)\,\mathrm{d}x, \int_0^1 g^2(x)\,\mathrm{d}x\right);\ g \text{ is d.f.}\right\}$$

as $\Pi_1, \dots, \Pi_6$ surfaces and the curve Π_7 such that for $(X_1, X_2, X_3) \in \cup_{i=1}^6 \Pi_i$ the moment problem has unique solution, for $(X_1, X_2, X_3) \in \Pi_7$ exactly two solutions, and in the interior of Ω has infinitely many solutions (see [SP, 2–20, 2.2.21] for exact results). Now, if a sequence x_n, $n = 1, 2, \dots$, in $[0,1]$ has limits

$X_1 = 1 - \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N x_n$,
$X_2 = \frac{1}{2} - \frac{1}{2}\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N x_n^2$ and
$X_3 = 1 - \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N x_n - \frac{1}{2}\lim_{N\to\infty} \frac{1}{N^2}\sum_{m,n=1}^N |x_m - x_n|$

and if $(X_1, X_2, X_3) \in \bigcup_{1\le i\le 7} \Pi_i$, then the sequence x_n possess an asymptotic distribution function $g(x)$.

Open problem is to solve the moment problem

$$(X_1, X_2, X_3, X_4) = \left(\int_0^1 g(x)\,\mathrm{d}x, \int_0^1 xg(x)\,\mathrm{d}x, \int_0^1 x^2 g(x)\,\mathrm{d}x, \int_0^1 g^2(x)\,\mathrm{d}x\right).$$

E.g., for $g(x) = 2x - x^2$ it has the unique solution.

Proposed by O. Strauch.

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1.24. Scalar product

Let $\mathbf{x}_n = (x_{n,1}, \dots, x_{n,s})$ and $\mathbf{y}_n = (y_{n,1}, \dots, y_{n,s})$ be infinite sequences in $[0,1)^s$. Assume that the sequence $(\mathbf{x}_n, \mathbf{y}_n)$, $n = 1, 2, \dots$, is u.d. in $[0,1]^{2s}$. Then the sequence of the inner (i.e., scalar) products

$$x_n = \mathbf{x}_n \cdot \mathbf{y}_n = \sum_{i=1}^{s} x_{n,i} y_{n,i}, \qquad n = 1, 2, \dots$$

has the a.d.f. $g_s(x) = \big|\{(\mathbf{x}, \mathbf{y}) \in [0,1]^{2s};\ \mathbf{x} \cdot \mathbf{y} < x\}\big|$ on the interval $[0, s]$, and

$$g_s(x) = (-1)^s \int\limits_{\substack{x_1 + \cdots + x_s < x \\ x_1 \in [0,1], \dots, x_s \in [0,1]}} 1 \,.\, \log x_1 \dots \log x_s \, \mathrm{d}x_1 \dots \mathrm{d}x_s.$$

For $x \in [0,1]$ we have

$g_1(x) = x - \log x$,

$g_2(x) = \frac{x^2}{2}\big((\log x)^2 - 3\log x + \frac{7}{2} - \frac{1}{6}\pi^2\big)$,

$g_3(x) = \frac{x^3}{27}\big(-\frac{9}{2}(\log x)^3 + \frac{99}{4}(\log x)^2 + \big(-\frac{255}{4} + \frac{9}{4}\pi^2\big)\log x + \frac{575}{8} - \frac{33}{8}\pi^2 - 9\zeta(3)\big)$,

$g_s(x) = (-1)^s x^s \sum_{j=0}^{s} \binom{s}{j} (\log x)^{s-j} \frac{1}{(s-j)!} \cdot \int_{[0,1]^j} \prod_{i=1}^{j} \big(\log x_1 + \cdots + \log x_{j-1}$
$\quad + \log(1 - x_j)\big) x_1^{s-1} \dots x_j^{s-j} \mathrm{d}x_1 \dots \mathrm{d}x_j$.

Open is the explicit formula of $g_s(x)$ for $x \in [1, s]$.

NOTES. (I) O. Strauch (2003). The formula for $g_s(x)$ with $x \in [0,1]$ was proved by L. Habsieger (Bordeaux) (personal communication). A motivation is an application of $g_s(x)$ to one-time pad cipher, see O. Strauch (2004).

(II) E. Hlawka (1982) investigated the question of the distribution of the scalar product of two vectors on an s-dimensional sphere and also the problem of the associated discrepancies.

Proposed by O. Strauch.

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1.25. Determinant

Let $\mathbf{x}_n^{(i)} = \left(x_{n,1}^{(i)}, \dots, x_{n,s}^{(i)}\right)$, $i = 1, \dots, s$, be infinite sequences in the s-dimensional ball $B(r)$ with the center at $(0, \dots, 0)$ and radius r. Assume that these sequences are u.d. and statistically independent in $B(r)$, i.e., $\left(\mathbf{x}_n^{(1)}, \dots, \mathbf{x}_n^{(s)}\right)$ is u.d. in $B(r)^s$. Then the sequence

$$x_n = \left|\det\left(\mathbf{x}_n^{(1)}, \dots \mathbf{x}_n^{(s)}\right)\right|$$

has the a.d.f. $g_s(r, x)$ defined on the interval $[0, r^s]$ by

$$g_s(r, x) = \frac{\left|\{(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(s)}) \in B(r)^s;\ |\det(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(s)})| < x\}\right|}{|B(r)|^s},$$

and for $\lambda = \frac{x}{r^s}$ there exists $\widetilde{g}_s(\lambda)$ such that $g_s(r, x) = \widetilde{g}_s(\lambda)$ if $\lambda \in [0, 1]$. Here we have

$\widetilde{g}_1(\lambda) = \lambda,$

$\widetilde{g}_2(\lambda) = \frac{2}{\pi}(1 + 2\lambda^2) \arcsin \lambda + \frac{6}{\pi}\lambda\sqrt{1 - \lambda^2} - 2\lambda^2,$

$\widetilde{g}_3(\lambda) = 1 + \frac{9}{4}\lambda \int_\lambda^1 \frac{\arccos x}{x} \mathrm{d}x - \frac{3}{4}\lambda^3 \arccos \lambda - \sqrt{1 - \lambda^2} + \frac{7}{4}\lambda^2\sqrt{1 - \lambda^2}.$

Open is the explicit form of $\widetilde{g}_s(\lambda)$ for $s > 3$. A further open question is the explicit form of the a.d.f. of the above sequence with $[0, 1]^s$ instead of $B(r)$.

NOTES. (I) O. Strauch (2003).
(II) Note that the integral in $\widetilde{g}_3(\lambda)$ cannot be expressed as a finite combination of elementary functions, cf. I. M. Ryshik and I. S. Gradstein [1951, p. 122].
(III) The d.f.'s $\widetilde{g}_s(\lambda)$ and $g_s(x)$ from 1.24 form the basis of a new one-time pad cryptosystem introduced in O. Strauch (2002).

Proposed by O. Strauch.

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1.26. Ramanujan function

Let $\tau(n)$ be the Ramanujan function given by

$$q\prod_{n\geq 1}(1-q^n)^{24}=\sum_{n\geq 1}\tau(n)q^n.$$

Is $\{\tau(n+1)/\tau(n)\}_{n\in\mathbb{N}}$ dense in $\mathbb{R}$?

Proposed by F. Luca.

1.27. Apéry sequence

Let $(A_n)_{n\geq 0}$ be the Apéry sequence given by

$$A_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2.$$

Let $\mathcal{P}=\{p\ prime\ :\ p\mid A_n\ for\ some\ n\}$.

(i) Is it true that $\mathcal{P}$ misses infinitely many primes?

(ii) For a positive real number x let $\mathcal{P}(x)=\mathcal{P}\cap[1,x]$. Find lower bounds for $\#\mathcal{P}(x)$.

Regarding (ii), it follows from the results of F. Luca (2007) that $\#\mathcal{P}(x)\gg\log\log x$.

Proposed by F. Luca.

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1.28. Urban's conjecture

Let $k\in\mathbb{N}$ be fixed, and let λ_i,μ_i, for $1\leq i\leq k$ be real algebraic numbers with absolute values greater than 1. Assume that, for $1=1,2,\dots,k$, the pairs λ_i,μ_i are multiplicatively independent (i.e., they are not integers m,n such that $\lambda_i^m=\mu_i^n$), and $(\lambda_i,\mu_i)\neq(\lambda_j,\mu_j)$ for $i\neq j$. Then for any real numbers $\theta_1,\dots,\theta_k$ with at least one $\theta_i\notin\mathbb{Q}\big(\cup_{i=1}^k\{\lambda_i,\mu_i\}\big)$ the double sequence

$$\sum_{i=1}^{k}\lambda_i^m\mu_i^n\theta_i \bmod 1,\qquad m,n=1,2,\dots$$

is dense in $[0,1]$.

NOTES. (I) R. Urban (2007). In a first step he proved (Theorem 1.6):
Let λ_1,μ_1 and λ_2,μ_2 be two distinct pairs of multiplicatively independent real algebraic integers of degree 2, with absolute values greater than 1, such that the absolute values of their conjugates $\tilde{\lambda}_1,\tilde{\mu}_1,\ \tilde{\lambda}_2,\tilde{\mu}_2$ are also greater than 1. Let

$\mu_1 = g_1(\lambda_1)$ for some $g_1 \in \mathbb{Z}[x]$ and $\mu_2 = g_2(\lambda_2)$ for some $g_2 \in \mathbb{Z}[x]$. Assume that at least one element in each pair λ_i, μ_i has all positive powers irrational. Assume further that there exist $k, lk', l' \in \mathbb{N}$ such that

(a) $\min\big(|\lambda_2|^k|\mu_2|^l, |\tilde\lambda_2|^k|\tilde\mu_2|^l\big) > \max\big(|\lambda_1|^k|\mu_1|^l, |\tilde\lambda_1|^k|\tilde\mu_1|^l\big)$ and

(b) $\min\big(|\lambda_1|^{k'}|\mu_1|^{l'}, |\tilde\lambda_1|^{k'}|\tilde\mu_1|^{l'}\big) > \max\big(|\lambda_2|' k|\mu_2|^{l'}, |\tilde\lambda_2|^{k'}|\tilde\mu_2|^{l'}\big)$.

Then for any real numbers θ_1, θ_2 with at least one $\theta_i \neq 0$ the sequence

$$\lambda_1^m \mu_1^n \theta_1 + \lambda_2^m \mu_2^n \theta_2 \bmod 1, \qquad m, n = 1, 2, \dots$$

is dense in $[0, 1]$. For illustration

$$\left(\sqrt{23}+1\right)^m \left(\sqrt{23}+2\right)^n \theta_1 + \left(\sqrt{61}+1\right)^m \left(\sqrt{61}-6\right)^n \theta_2 \bmod 1, \qquad m, n = 1, 2, \dots$$

is dense in $[0, 1]$, assuming $(\theta_1, \theta_2) \neq (0, 0)$.

R. Urban note that (a) and (b) hold, when

$$|\lambda_2| > |\tilde\lambda_2| > |\lambda_1| > |\tilde\lambda_1| > 1 \quad \text{and} \quad |\mu_1| > |\tilde\mu_1| > |\mu_2| > |\tilde\mu_2| > 1.$$

He also note that Theorem 1.6 can be extended in the case when not all of λ_i, μ_i are of degree 2, but if λ_i, μ_i are rationals, then θ_i must be irrational. As example, for every $\theta_2 \neq 0$, the sequence

$$\left(3+\sqrt{3}\right)^m 2^n + 5^m 7^n \theta_2 \sqrt{2} \bmod 1, \qquad m, n = 1, 2, \dots$$

is dense in $[0, 1]$.

(II) The Conjecture is motivated by H. Furstenberg's (1967) result: If $p, q > 1$ are multiplicatively independent integers, i.e., they are not both integer powers of some integer, then for every irrational θ the double sequence

$$p^n q^m \theta \bmod 1, \qquad m, n = 1, 2, \dots$$

is everywhere dense in $[0, 1]$.

(III) Further generalization was given by B. Kra (1999): For positive integers $1 < p_i < q_i$, $i = 1, 2, \dots, k$, assume that all pairs p_i, q_i are multiplicatively independent and pairs $(p_i, q_i) \neq (p_j, q_j)$ for $i \neq j$. Then for distinct $\theta_1, \dots, \theta_k$ with at least one irrational θ_i the sequence

$$\sum_{i=1}^{K} p_i^n q_i^m \theta_i \bmod 1, \qquad m, n = 1, 2, \dots \tag{1}$$

is dense in $[0, 1]$.

(IV) D. Berend in MR1487320 (99j:11079) reformulated Kra's result:

Let p_i, q_i integers and θ_i real, $i = 1, 2, \dots, k$. If p_1 and q_1 are multiplicatively independent, θ_1 is irrational, and pairs $(p_i, q_i) \neq (p_1, q_1)$ for $i \geq 2$, then the sequence (1) is dense in $[0, 1]$.

(V) Precisely, H. Furstenberg (1967) proved: Let S be a non-lacunary semigroup of rational integers. Then $S\alpha \bmod 1$ is dense in $[0,1]$ for any irrational α.

D. Berend (1987) extends it:

Let K be a real algebraic number field and S a subsemigroup of the multiplicative group of K such that

(i) $S \subset (-\infty,-1) \cup (1,\infty)$,

(ii) there exit multiplicatively independent $\lambda, \mu \in S$ (i.e., there exist no integers m and n, not both of which are 0, with $\lambda^m = \mu^n$),

(iii) $\mathbb{Q}(S) = K$. Then for every $\alpha \notin K$ the set $S\alpha \bmod 1$ is dense in $[0,1]$. If, moreover

(iv) $S \not\subset PS(K)$, then $S\alpha \bmod 1$ is dense in $[0,1]$ for every $\alpha \neq 0$.

Here, if $[K:\mathbb{Q}] = m$ denotes by $PS(K)$ the semigroup of all Pisot or Salem number of degree m over $\mathbb{Q}$.

Furthermore, if $S\alpha \bmod 1$ is dense in $[0,1]$ for every $\alpha \notin K$ or for all $\alpha \neq 0$, then S has a subsemigroup having the same property generated by two elements.

(VI) D. Berend (1987a): Let p, q, and c be non-zero integers with p and q multiplicatively independent, ξ an irrational and β arbitrary. Then the set

$$\left\{p^m q^n \xi + c^{m+n}\beta : m,n \in \mathbb{N}\right\}$$

is dense modulo 1.

(VII) R. Urban (2009): Let $a_1 > a_2 > 1$ and $b_1 > b_2 > 1$ be two pairs of multiplicatively independent integers, and let c be a positive real number. Suppose that $a_1 < b_1$ and $a_2 > b_2$. Then, for any real numbers ξ_1, ξ_2 with at least one ξ_i irrational, there exists $q \in \mathbb{N}$ such that for any real number β, the set

$$\left\{a_1^m a_2^n q\xi_1 + b_1^m b_2^n q\xi_2 + c^{m+n}\beta : m,n \in \mathbb{N}\right\}$$

is dense modulo 1.

Submitted by O. Strauch.

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1.29. Extreme values of $\int_0^1\int_0^1 F(x,y)\,\mathrm{d}_x\,\mathrm{d}_y\,g(x,y)$ for copulas $g(x,y)$

Let $F(x,y)$ be a Riemann integrable function defined on $[0,1]^2$ and x_n, y_n, $n=1,2,\dots$, be two u.d. sequences in $[0,1)$. A problem is to find limit points of the sequence

$$\frac{1}{N}\sum_{n=1}^{N}F(x_n,y_n),\qquad N=1,2,\dots \tag{1}$$

Applying Helly theorems we obtain limit points of (1) form the set

$$\left\{\int_0^1\int_0^1 F(x,y)\,\mathrm{d}_x\,\mathrm{d}_y\,g(x,y);g(x,y)\in G((x_n,y_n))\right\}, \tag{2}$$

where $G(x_n,y_n)$ is the set of all d.f.'s of the two-dimensional sequence (x_n,y_n), $n=1,2,\dots$ In this case, two-dimensional sequence (x_n,y_n) need not be u.d. but every d.f. $g(x,y)\in G((x_n,y_n))$ satisfies

(i) $g(x,1)=x$ for $x\in[0,1]$ and

(ii) $g(1,y)=y$ for $y\in[0,1]$.

The d.f. $g(x,y)$ which satisfies (i) and (ii) is called *copula* and a basic theory of copulas can be found in R. B. Nelsen (1999), see [OP, 2.3, Deterministic analysis of sequences].

Open problem: Find extreme values of $\int_0^1\int_0^1 F(x,y)\,\mathrm{d}_x\,\mathrm{d}_y\,g(x,y)$, where $g(x,y)$ is a copula.

NOTES. (I) Firstly, for $F(x,y)=|x-y|$, this problem was formulated by F. Pillichshammer and S. Steinerberger (2009). They proved: *Let x_n and y_n be two uniformly distributed sequences in* $[0,1)$. *Then*

$$\limsup_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}|x_n-y_n|\le\frac{1}{2}$$

and in particular,

$$\limsup_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}|x_{n+1}-x_n|\le\frac{1}{2}$$

and this result is best possible. They also found

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}|x_{n+1}-x_n|=\frac{2(b-1)}{b^2}$$

for van der Corput sequence x_n in the base b and

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}|x_{n+1}-x_n| = 2\{\alpha\}\bigl(1-\{\alpha\}\bigr)$$

for $x_n = n\alpha \bmod 1$, where α is irrational.

(II) Secondly, S. Steinerberger (2009) study (1) for $F(x,y) = f_1(x)f_2(x)$ and for u.d. sequences $x_n = \Phi(z_n)$ and $y_n = \Psi(z_n)$, where $\Phi(x)$ and $\Psi(x)$ are uniformly distributed preserving (u.d.p.) functions and z_n is a u.d. sequence. For u.d.p. see this [OP, 2.1 Uniform distribution theories]. He proved: Let $f\colon [0,1]\to\mathbb{R}$ be a Lebesgue measurabble function, we see it as random variable, and $g(x) = |f^{-1}([0,x))|$ be its d.f. and put $f^*(x) = g^{-1}(x)$. If d.f. $g(x)$ does not have inverse function we put $f^*(x) = \inf\bigl\{t\in\mathbb{R}; g(t)\geq x\bigr\}$.

Let f_1, f_2 be Riemann integrable functions on $[0,1]$. Let Φ,Ψ be arbitrary u.d.p. transformations. Then

$$\int_0^1 f_1^*(x)f_2^*(1-x)\,\mathrm{d}x \leq \int_0^1 f_1\bigl(\Phi(x)\bigr)f_2\bigl(\Psi(x)\bigr)\mathrm{d}x \leq \int_0^1 f_1^*(x)f_2^*(x)\,\mathrm{d}x$$

and these bounds are best possible. Also, every number within the bounds is attained by some u.d.p. Φ,Ψ. In his proof Steinerberger used the Hardy-Littlewood inequality [Hardy, Littlewood and Pólya (1934), Th. 378]

$$\int_0^1 f_1(x)f_2(x)\,\mathrm{d}x \leq \int_0^1 f_1^*(x)f_2^*(x)\,\mathrm{d}x.$$

(III) J. Fialová and O. Strauch (2010) proved: *Let $F(x,y)$ be a Riemann integrable function defined on $[0,1]^2$. For differential of $F(x,y)$ let us assume that $\mathrm{d}_x\,\mathrm{d}_y\,F(x,y) > 0$ for every $(x,y)\in[0,1]^2$. Then*

$$\max_{g(x,y)\text{-}copula}\int_0^1\int_0^1 F(x,y)\,\mathrm{d}_x\,\mathrm{d}_y\,g(x,y) = \int_0^1 F(x,x)\,\mathrm{d}x,$$

$$\min_{g(x,y)\text{-}copula}\int_0^1\int_0^1 F(x,y)\,\mathrm{d}_x\,\mathrm{d}_y\,g(x,y) = \int_0^1 F(x,1-x)\,\mathrm{d}x,$$

where, precisely, max *is attained in* $g(x,y) = \min(x,y)$ *and* min *in* $g(x,y) = \max(x+y-1,0)$, uniquely. In proof they used expression

$$\int_0^1\!\!\int_0^1 F(x,y)\,\mathrm{d}_x\,\mathrm{d}_y\,g(x,y) = F(1,1) - \int_0^1 g(1,y)\,\mathrm{d}_y\,F(1,y) - \int_0^1 g(x,1)\,\mathrm{d}_x\,F(x,1) + \int_0^1\!\!\int_0^1 g(x,y)\,\mathrm{d}_x\,\mathrm{d}_y\,F(x,y)$$

which holds for every Riemann integrable function $F(x,y)$ and d.f. $g(x,y)$ which have no any common discontinuity points. And then they used Fréchet-Hoeffding bounds [R. B. Nelsen (1999), p. 9] $\max(x+y-1,0) \le g(x,y) \le \min(x,y)$ which holds for every $(x,y) \in [0,1]^2$ and for every copula $g(x,y)$.

(IIIa) Using Sklar theorem that every d.f. $g(x,y)$ can be express as $g(x,y) = c\big(g(x,1), g(1,y)\big)$ for related copula $c(x,y)$ J. Fialová and O. Strauch extend: *Let us assume that $F(x,y)$ is a continuous function such that*

$$\mathrm{d}_x\mathrm{d}_y F(x,y) > 0 \qquad \textit{for every} \quad (x,y) \in (0,1)^2.$$

Then for the extremes of integral $\int_0^1\int_0^1 F(x,y)\,\mathrm{d}_x\,\mathrm{d}_y\,g(x,y)$ for $g(x,y)$ for which $g(x,1) = g_1(x)$ and $g(1,y) = g_2(y)$ we have

$$\max_{g(x,y)} \int_0^1\!\!\int_0^1 F(x,y)\,\mathrm{d}_x\,\mathrm{d}_y\,g(x,y) = \int_0^1 F\big(g_1^{-1}(x), g_2^{-1}(x)\big)\mathrm{d}x,$$

$$\min_{g(x,y)} \int_0^1\!\!\int_0^1 F(x,y)\,\mathrm{d}_x\,\mathrm{d}_y\,g(x,y) = \int_0^1 F\big(g_1^{-1}(x), g_2^{-1}(1-x)\big)\mathrm{d}x,$$

where the maximum is attained in $g(x,y) = \min\big(g_1(x), g_2(y)\big)$ and the minimum in $g(x,y) = \max\big(g_1(x)+g_2(y)-1, 0\big)$, uniquely.

(IIIb) J. Fialová and O. Strauch (2011) criterion: *Assume that a copula $g(x,y)$ maximize $\int_0^1\int_0^1 F(x,y)\,\mathrm{d}_x\,\mathrm{d}_y\,g(x,y)$ and let $[X_1,X_2]\times[Y_1,Y_2]$ be an interval in $[0,1]^2$ such that the differential*

$$g(X_2,Y_2) + g(X_1,Y_1) - g(X_2,Y_1) - g(X_1,Y_2) > 0.$$

If for every interior point (x,y) of $[X_1,X_2]\times[Y_1,Y_2]$ the differential $\mathrm{d}_x\,\mathrm{d}_y\,F(x,y)$ has a constant sign, then

(i) *if $\mathrm{d}_x\mathrm{d}_y F(x,y) > 0$ then*

$$g(x,y) = \min\big(g(x,Y_2) + g(X_1,y) - g(X_1,Y_2), g(x,Y_1) + g(X_2,y) - g(X_2,Y_1)\big)$$

(ii) *if $\mathrm{d}_x\mathrm{d}_y F(x,y) < 0$ then*

$$g(x,y) = \max\big(g(x,Y_2) + g(X_2,y) - g(X_2,Y_2), g(x,Y_1) + g(X_1,y) - g(X_1,Y_1)\big)$$

for every $(x,y) \in [X_1,X_2]\times[Y_1,Y_2]$.

(IV) J. Fialová and O. Strauch (2010) also consider $F(x,y)$ in the form $F(x,y) = f(x)y$ and study the limit points of

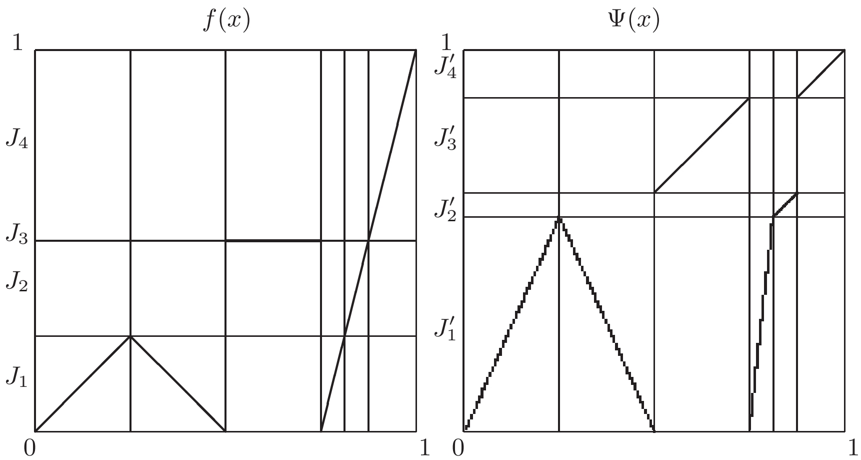
$$\frac{1}{N}\sum_{n=1}^{N} f(x_n)y_n,$$

where x_n is u.d. sequence and u.d. sequence y_n is given by $y_n = \Phi(x_n)$, where $\Phi(x)$ is a u.d.p. This problem is equivalent to find

$$\max_{\Phi(x)-\text{u.d.p.}} \int_0^1 f(x)\Phi(x)\,\mathrm{d}x, \qquad \min_{\Phi(x)-\text{u.d.p.}} \int_0^1 f(x)\Phi(x)\,\mathrm{d}x.$$

J. Fialová (2010) solve this problem for piecewise linear $f(x)$. Here the function $f\colon [0,1] \to [0,1]$ is *piecewise linear* (p.l.) if there exists a system of ordinate intervals J_j, $j = 1,2,\dots,u$ which are disjoint and fulfilled the whole interval $[0,1]$, and a corresponding system of abscissa intervals $I_{j,i}$, $i = 1,2,\dots,l_j$, such that $f(x)/I_{j,i}$ is the increasing or decreasing diagonal of $I_{j,i} \times J_j$. J. Fialová (2010) proved: *Let $f\colon [0,1] \to [0,1]$ be a p.l. function with ordinate decomposition J_j, $j = 1,2,\dots n$, and abscissa decomposition $I_{j,i}$, $i = 1,2,\dots l_j$. Define a p.l. function $\Psi(x)$ in the same abscissa decomposition $I_{j,i}$, but in a new ordinate decomposition J'_j, with the lengths $|J'_j| = \sum_{i=1}^{l_j} |I_{j,i}|$, $j = 1,2,\dots n$, ordered similarly as J_j, $j = 1,2,\dots,u$. Put the graph of $\Psi(x)/I_{j,i}$ on $I_{j,i} \times J'_j$ as the diagonal $\nearrow$ or $\searrow$ if and only if $f(x)/I_{j,i}$ is the $\nearrow$ or $\searrow$ diagonal. Note that if $f(x)$ is a constant on the interval $I_{j,i}$, then J_i is a point, and the graph $\Psi(x)/I_{j,i}$ can be defined arbitrary, either increasing or decreasing in $I_{j,i} \times J'_j$. Then $\Psi(x)$ is the best u.d.p. approximation of $f(x)$.*

For example



(IVa) The result in (IV) correspond (II) since we have $\Psi(x) = g_f\big(f(x)\big)$. But J. F i a l o v á used

$$\int_0^1 \big(f(x) - \Psi(x)\big)^2 \mathrm{d}x = \int_0^1 f^2(x)\,\mathrm{d}x - 2\int_0^1 f(x)\Psi(x)\,\mathrm{d}x + \int_0^1 \Psi^2(x)\,\mathrm{d}x,$$

$\int_0^1 \Psi^2(x)\mathrm{d}x = \frac{1}{3}$ which gives

$$\max \int_0^1 f(x)\Psi(x)\,\mathrm{d}x = \min \int_0^1 \big(f(x) - \Psi(x)\big)^2\,\mathrm{d}x.$$

(V) S. S t e i n e r b e r g e r (2009) generalized **open problem** to give bounds for

$$\int_0^1 f_1\big(\Phi_1(x)\big) f_2\big(\Phi_2(x)\big) \dots f_s\big(\Phi_s(x)\big)\mathrm{d}x$$

of Riemann integrable $f_1, \dots, f_s$ and u.d.p. maps $\Phi_1, \dots, \Phi_s$. He proved the following partial results:

a) $\max_{\Phi_1,\dots,\Phi_s} \int_0^1 f_1\big(\Phi_1(x)\big) f_2\big(\Phi_2(x)\big) \dots f_s\big(\Phi_s(x)\big)\mathrm{d}x \le \big(\prod_{i=1}^n \int_0^1 |f_i(x)|^s \mathrm{d}x\big)^{\frac{1}{s}}$.

b) $\min_{\Phi_1,\dots,\Phi_n} \int_0^1 \Phi_1(x)\Phi_2(x)\dots\Phi_s(x)\,\mathrm{d}x \ge \frac{1}{e^s}$.

c) $\min_{\Phi_1,\dots,\Phi_s} \int_0^1 \Phi_1(x)\Phi_2(x)\dots\Phi_s(x)\,\mathrm{d}x \le e^{\frac{1}{6s}} \frac{s}{s-2}\frac{4}{\pi}\frac{1}{e^s}$.

Comments: Let x_n, $n = 1, 2, \dots$, be a u.d. sequence in $[0, 1)$ and $g(t_1, \dots, t_s)$ be an a.d.f. of the s-dimensional sequence $\big(\Phi_1(x_n), \dots, \Phi_s(x_n)\big)$, $n = 1, 2, \dots$ We have

$$g(t_1, \dots, t_s) = \big|\Phi_1^{-1}\big([0, t_1)\big) \cap \dots \cap \Phi_s^{-1}\big([0, t_s)\big)\big|,$$

$g(1 \dots, t_i, 1 \dots, 1) = t_i$ for $i = 1, \dots, s$, i.e., it is a copula and

$$\int_0^1 f_1\big(\Phi_1(x)\big) \dots f_s\big(\Phi_s(x)\big)\mathrm{d}x = \int_{[0,1]^s} f_1(t_1) \dots f_s(t_s)\,\mathrm{d}g(t_1, \dots, t_s).$$

(VI) Thus we arrive at the **open problem:**
Find extreme values of $\int_{[0,1]^s} F(\mathbf{x})\,\mathrm{d}g(\mathbf{x})$, where $g(\mathbf{x})$ is an s-dimensional copula.

(VII) S. S t e i n e r b e r g e r (2010) generalized (I) to give bounds for the asymptotic behavior of

$$\frac{1}{N}\sum_{n=1}^N ||\mathbf{x}_n - \mathbf{y}_n||,$$

where $\mathbf{x}_n, \mathbf{y}_n$ are u.d. sequences in a bounded Jordan measurable domain Ω. E.g., for s-dimensional ball he found the sharp inequality

$$\frac{1}{N}\sum_{n=1}^{N}||\mathbf{x}_n - \mathbf{y}_n|| \le \frac{2s}{s+1}.$$

(VIII) **Open problem:** Transform the theory of d.f.'s to the multidimensional unit sphere $\mathbb{S}$ and find extremes of the energy integral

$$\int_{\mathbb{S}} ||\mathbf{x} - \mathbf{y}||^s \mathrm{d}g(\mathbf{x})\,\mathrm{d}g(\mathbf{y}).$$

The expository paper on Riesz energy can be found in J. B r a u c h a r t (2011).
(IX) See also problem 1.37.

Submitted by O. Strauch.

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1.30. Niederreiter-Halton (NH) sequence

Directly from R. H o f e r and G. L a r c h e r (2010): Niederreiter-Halton (NH) sequence is a combination of different digital $(\mathbf{T}_i, w_i)$-sequences in different prime bases $q_1, \dots, q_r$ with $w_1 + \cdots + w_r = s$ into a single sequence in $[0, 1)^s$.

Finite row NH sequence is a NH sequence if all generating matrices of the component digital $(\mathbf{T}_i, w_i)$-sequences have each row containing only finitely many entries different from zero.

Infinite row NH sequence is a (NH) sequence which is not finite row.

Digital $(\mathbf{T}, s)$-sequence over $\mathbb{F}_q$.

- Let s be a dimension;
- q be a prime;
- Represent $n = n_0 + n_1 q + n_2 q^2 + \cdots$ in base q;
- Let $C_1, \dots, C_s$ be $\mathbb{N} \times \mathbb{N}$-matrices in the finite field $\mathbb{F}_q$;
- $C_i \cdot (n_0, n_1, \dots)^T = (y_0^{(i)}, y_1^{(i)}, \dots)^T \in \mathbb{F}_q^{\mathbb{N}}$;
- $x_n^{(i)} := \frac{y_0^{(i)}}{q} + \frac{y_1^{(i)}}{q^2} + \cdots$;
- The sequence $\mathbf{x}_n = \left(x_n^{(1)}, \dots, x_n^{(s)}\right)$ is said to be $(\mathbf{T}, s)$-sequence if for every $m \in \mathbb{N}$ there exists $\mathbf{T}(m)$ such that $0 \le \mathbf{T}(m) \le m$ and for all $d_1 + \cdots + d_s = m - \mathbf{T}(m)$ and the $\big(m - \mathbf{T}(m)\big) \times m$-matrix consisting of
  the upper left $d_1 \times m$-submatrix of C_1
  the upper left $d_2 \times m$-submatrix of C_2
  ...
  the upper left $d_s \times m$-submatrix of C_s
  has rank $m - \mathbf{T}(m)$.

If $\mathbf{T}$ is minimal we speak **strict digital $(\mathbf{T}, s)$-sequence**.

Open Problem: Determine whether the following two-dimensional NH sequences in base 3 and, respectively, 2 are low-discrepancy sequences (i.e., $D_N^* = O((\log N)^s)/N))$ or not:

1. $C^{(1)}$ is the unit matrix in $\mathbb{F}_3$ and

$$C^{(2)} = \begin{pmatrix} 1 & \underbrace{00\dots0}_{l_1} & 1 & & 0 & 0 & \dots & & \\ 0 & 1 & & \underbrace{00\dots\dots0}_{l_2} & 1 & 0 & 0 & \dots & \\ 0 & 0 & 1 & & \underbrace{00\dots\dots\dots0}_{l_3} & 1 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

in $\mathbb{F}_2$ with $l_1, l_2, l_3, \dots$ arbitrary but $\lim_{n\to\infty} l_n = \infty$.

2. $C^{(1)}$ is the unit matrix in $\mathbb{F}_3$ and

$$C^{(2)} = \begin{pmatrix} 1 & \overbrace{00\dots0}^{l_1} & 1 & \overbrace{00\dots0}^{l_2} & 1 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

in $\mathbb{F}_2$ and the first row contains infinitely many $1'$s but with density 0.

3.

$$C^{(1)} = C^{(2)} = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

but $C^{(1)}$ in $\mathbb{F}_3$ and $C^{(2)}$ in $\mathbb{F}_2$.

NOTES. (I) A basic example is the Halton sequence which is a combination of s digital $(0,1)$-sequences in different prime bases $q_1, \dots, q_s$ generated by the unit matrices in $\mathbb{F}_{q_i}$ for each i.
(II) General NH sequences were first investigated by R. Hofer, P. Kritzer, G. Larcher and F. Pillichshammer (2009) and R. Hofer (2009) and she proved: NH sequence is u.d. if and only if each $(\mathbf{T}_i, w_i)$ is u.d.
(III) A strictly digital $(\mathbf{T}, s)$-sequence is u.d. if and only if

$$\lim_{m \to \infty} \big(m - \mathbf{T}(m)\big) = \infty.$$

If $\mathbf{T}(m) \le t$ for all m, then $(\mathbf{T}, s)$-sequence is (t, s)-sequence.
(IV) R. Hofer and G. Larcher (2010) give concrete examples of digital $(0, s)$-sequences generated by matrices with finite rows.

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1.31. Gauss-Kuzmin theorem and $g(x) = g_f(x)$

Denote

$$f(x) = 1/x \bmod 1,$$

$$g_f(x) = \int_{f^{-1}([0,x))} 1.\mathrm{d}g\,(x),$$

$$(g_{f^n})_f(x) = g_{f^{n+1}}(x),$$

$$g_0(x) = \frac{\log(1+x)}{\log 2}.$$

The problem is to find all solutions $g(x)$ of the functional equation $g(x) = g_f(x)$ for $x \in [0,1]$. It is equivalent to

$$g(x) = \sum_{n=1}^{\infty} g\left(\frac{1}{n}\right) - g\left(\frac{1}{n+x}\right) \qquad \text{for d.f. } g(x), x \in [0,1]. \tag{1}$$

The following is known:

(I) $g_0(x)$ satisfies (1).

(II) Gauss-Kuzmin theorem: If $g(x) = x$, then $g_{f^n}(x) \to g_0(x)$ and the rate of convergence is $O\big(q^{\sqrt{n}}\big)$, $0 < q < 1$.

(III) Theorem in (II) was proved by R. Kuzmin (1928) assuming for a starting function $g(x)$
   (i) $0 < g'(x) < M$ and
   (ii) $|g''(x)| < \mu$.
   Thus, if $g(x)$ satisfies (i), (ii), and (1), then $g(x) = g_0(x)$.

(IV) Theorem (II) was inspired by Gauss. He conjectured $m_n(x) \to g_0(x)$, where $m_n(x) = \big|\{\alpha \in [0,1]; 1/r_n(\alpha) < x\}\big|$ and for continued fraction expansion $\alpha = \big[a_0(\alpha); a_1(\alpha), a_2(\alpha), \dots\big]$, $r_n(\alpha) = \big[a_{n+1}(\alpha); a_{n+2}(\alpha), \dots,\big]$. In this case $m_n(x) = g_{f^n}(x)$ for $g(x) = x$, since $f\big(1/r_n(\alpha)\big) = 1/r_{n+1}(\alpha)$.

(V) For starting point $x_0 \in [0,1]$ we define the iterate sequence x_n as

$$x_1 = f(x_0),\ x_2 = f\big(f(x_0)\big),\ x_3 = f\Big(f\big(f(x_0)\big)\Big), \dots$$

   Then a.d.f. $g(x)$ solves (1). For example, the sequence $x_1 = 1/r_1$, $x_2 = 1/r_2, \dots$ for $\frac{\sqrt{5}-1}{2} = [0;1,1,1,\dots]$ produces solution $g(x) = c_{\frac{\sqrt{5}-1}{2}}(x)$.

(VI) **Chain of solutions.** If d.f. $g_1(x)$ solve the equation $g_f(x) = g(x)$ and $(g_2)_f(x) = g_1(x)$, then $g_2(x)$ solve $g_f(x) = g(x)$, again. The $g_2(x)$ can be found as solution

$$g_1(x) = \sum_{n=1}^{\infty} g_2\left(\frac{1}{n}\right) - g_2\left(\frac{1}{n+x}\right).$$

   From it

$$g_2\left(\frac{1}{x+1}\right) = 1 - g_1(x) + \sum_{n=2}^{\infty} g_2\left(\frac{1}{n}\right) - g_2\left(\frac{1}{n+x}\right)$$

   and thus it is suffice to define a non-increasing $g_2(x)$ on $[0,1/2)$, such that
   (i) $g_2(0) = 0$,
   (ii) $\sum_{n=2}^{\infty} g_2\big(\frac{1}{n}\big) - g_2\big(\frac{1}{n+x}\big) \le \sum_{n=1}^{\infty} g_1\big(\frac{1}{n}\big) - g_1\big(\frac{1}{n+x}\big)$,
   (iii) $\sum_{n=2}^{\infty} g_2'\big(\frac{1}{n}\big)\frac{1}{(x+n)^2} \le \sum_{n=1}^{\infty} g_2'\big(\frac{1}{n}\big)\frac{1}{(x+n)^2}$.

(VII) If d.f. $g_1(x)$ and $g_2(x)$ satisfy (1) and $g_1(x) = g_2(x)$ for $x \in [0, 1/2)$, then $g_1(x)$ and $g_2(x)$ coincide on the whole interval $[0, 1]$. Other sets of uniqueness are $\left[0, \frac{1}{n+1}\right) \cup \left(\frac{1}{n}, 1\right]$ for arbitrary positive integer n.

(VIII) If $x_n \in [0, 1)$ has a.d.f. $g_1(x)$, then $\left\{\frac{1}{x_n}\right\}$ has a.d.f. $g_2(x) = \sum_{n=1}^{\infty} g_1\left(\frac{1}{n}\right) - g_1\left(\frac{1}{n+x}\right)$.

Submitted by O. Strauch.

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1.32. Benford law

Let $b \geq 2$ be an integer considered as a base for the development of positive real number $x > 0$ and $M_b(x)$ be a mantissa of x defined by $x = M_b(x) \times b^{n(x)}$ such that $1 \leq M_b(x) < b$ holds, where $n(x)$ is a uniquely determined integer. Let $K = k_1 k_2 \dots k_r$ be a positive integer expressed in the base b, that is

$$K = k_1 \times b^{r-1} + k_2 \times b^{r-2} + \cdots + k_{r-1} \times b + k_r,$$

where $k_1 \neq 0$ and at the same time $K = k_1 k_2 \dots k_r$ is considered as an r-consecutive block of integers in the base b. We have

$$\begin{aligned} & K \leq M_b(x) \times b^{r-1} < K + 1 \\ & \iff \frac{K}{b^{r-1}} \leq M_b(x) < \frac{K+1}{b^{r-1}} \\ & \iff \log_b\left(\frac{K}{b^{r-1}}\right) \leq \log_b\bigl(M_b(x)\bigr) < \log_b\left(\frac{K+1}{b^{r-1}}\right) \\ & \iff \log_b\left(\frac{K}{b^{r-1}}\right) \leq \log_b x \bmod 1 < \log_b\left(\frac{K+1}{b^{r-1}}\right). \end{aligned} \tag{1}$$

Definition. A sequence x_n, $n = 1, 2, \dots$, of positive real numbers satisfies Benford law (abbreviated to B.L.) of order r if for every r-digits number $K = k_1 k_2 \dots k_r$ we have

$$\lim_{N\to\infty} \frac{\#\{n \leq N;\ \text{first } r \text{ digits of } M_b(x_n) \text{ are equal to } K\}}{N} = \log_b(K+1) - \log_b K.$$

Here "the first r digits of $M_b(x_n) = K$ is the same as" the first r digits (starting a non-zero digit) of $x_n = K$.

Definition. If a sequence x_n, $n = 1, 2, \dots$, satisfies B.L. of order r, for every $r = 1, 2, \dots$, then it is called that x_n satisfies strong B.L. or extended or generalized B.L. In the following we will described it as B.L.

From (1) directly follows:

(I) **Theorem.** *A sequence x_n, $x_n > 0$, $n = 1, 2, \dots$, satisfies B.L. if and only if the sequence* $\log_b x_n \bmod 1$ *is u.d. in* $[0, 1)$.

(II) **Theorem.** *For every K and r there rexists infinitely many n such that the first r digits (starting a non-zero digit) of $x_n = K$ if and only if* $\log_b x_n \bmod 1$ *is dense in* $[0, 1)$.

Characterization u.d. of $\log_b x_n \bmod 1$ using d.f.'s in $G(x_n \bmod 1)$
In V. B a l á ž, K. N a g a s a k a and O. S t r a u c h (2010) is proved:

(III) **Theorem.** *Let x_n, $n = 1, 2, \dots$, be a sequence in $(0, 1)$ and $G(x_n)$ be the set of all d.f.s of x_n. Assume that every d.f. $g(x) \in G(x_n)$ is continuous at $x = 0$. Then the sequence x_n satisfies B.L. in the base b if and only if for every $g(x) \in G(x_n)$ we have*

$$x = \sum_{i=0}^{\infty} \left(g\left(\frac{1}{b^i}\right) - g\left(\frac{1}{b^{i+x}}\right) \right) \qquad \text{for} \quad x \in [0, 1]. \tag{2}$$

Find all solutions of (2). Some examples are:

$$g(x) = \begin{cases} x \text{ if } x \in \left[0, \frac{1}{b}\right], \\ 1 + \log_b x + (1 - x)\frac{1}{b-1} \text{ if } x \in \left[\frac{1}{b}, 1\right]. \end{cases}$$

$$\tilde{g}(x) = \begin{cases} 0 \text{ if } x \in \left[0, \frac{1}{b}\right], \\ 1 + \frac{\log x}{\log b} \text{ if } x \in \left[\frac{1}{b}, 1\right], \end{cases}$$

$$g^*(x) = \begin{cases} 0 \text{ if } x \in \left[0, \frac{1}{b^2}\right], \\ 2 + \frac{\log x}{\log b} \text{ if } x \in \left[\frac{1}{b^2}, \frac{1}{b}\right], \\ 1 \text{ if } x \in \left[\frac{1}{b}, 1\right] \end{cases}$$

$$g^{**}(x) = \begin{cases} 0 \text{ if } x \in \left[0, \frac{1}{b^3}\right], \\ 3 + \frac{\log x}{\log b} \text{ if } x \in \left[\frac{1}{b^3}, \frac{1}{b^2}\right], \\ 1 \text{ if } x \in \left[\frac{1}{b^2}, 1\right]. \end{cases}$$

(IV) Simple results:

(i) Fibonacci numbers F_n, $n!$, n^n, n^{n^2}, satisfy B.L.

(ii) The positive sequences x_n and $1/x_n$, $n = 1, 2, \dots$ satisfy B.L. in the base b simultaneously.

(iii) The positive sequences x_n and nx_n, $n = 1, 2, \dots$ satisfy B.L. in the base b simultaneously.

(iv) For a sequence $x_n > 0$, $n = 1, 2, \dots$, assume that
 – $\lim_{n\to\infty} x_n = \infty$ monotonically,
 – $\lim_{n\to\infty} \log \frac{x_{n+1}}{x_n} = 0$ monotonically.

Then the sequence x_n satisfies B.L. in every base b if and only if

$$\lim_{n\to\infty} n \log \frac{x_{n+1}}{x_n} = \infty.$$

(v) Assume $x_n > 0$, $n = 1, 2, \dots$. If for every $k = 1, 2, \dots$ the ratio sequence x_{n+k}/x_n, $n = 1, 2, \dots$, satisfies B.L. in the base b, then the original sequence x_n, $n = 1, 2, \dots$ also satisfies B.L. in the base b, see A. I. Pavlov (1981).

(IX) J. L. Brown, Jr. and R. L. Duncan (1970): Let x_n be a sequence generated by the recursion relation

$$x_{n+k} = a_{k-1}x_{n+k-1} + \cdots + a_1 x_{n+1} + a_0 x_n, \qquad n = 1, 2, \dots,$$

where $a_0, a_1, \dots, a_{k-1}$ are non-negative rationals with $a_0 \neq 0$, k is a fixed integer, and $x_1, x_2, \dots, x_k$ are starting points. Assume that the characteristic polynomial

$$x^k - a_{k-1}x^{k-1} - \cdots - a_1 x - a_0$$

has k distinct roots $\beta_1, \beta_2, \dots, \beta_k$ satisfying

$$0 < |\beta_1| < \cdots < |\beta_k|$$

and such that none of the roots has magnitude equal to 1.

Then $\log x_n \bmod 1$ is u.d.

Furthermore, the general solution of the recurrence is $x_n = \sum_{j=1}^{k} \alpha_j \beta_j^n$ and if j_0 is the largest value of j for which $\alpha_j \neq 0$ and if $\log_b \beta_{j_0}$ is irrational, then also

$$\log_b x_n \bmod 1 \quad \text{is u.d.},$$

i.e., x_n satisfies B.L. in the base b. This implies that Fibonacci and Lucas numbers obey B.L. what rediscovered L. C. Washington (1981).

Submitted by O. Strauch.

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1.33. The integral $\int_{[0,1]^s} F(\mathbf{x},\mathbf{y})\,\mathrm{d}g(\mathbf{x})\,\mathrm{d}g(\mathbf{y})$ for d.f.'s $g(\mathbf{x})$

• Let $F(x,y)$ be a real continuous symmetric function defined on $[0,1]^2$ and let $G(F)$ be a set of all d.f.s $g(x)$ satisfying

$$\int_0^1\int_0^1 F(x,y)\,\mathrm{d}g(x)\,\mathrm{d}g(y)=0. \tag{1}$$

The study of $G(F)$ is motivated by the fact that for every sequence $x_n\in[0,1)$ we have

$$G(x_n)\subset G(F)\iff \lim_{N\to\infty}\frac{1}{N^2}\sum_{m,n=1}^N F(x_m,x_n)=0,$$

where $G(x_n)$ is the set of all d.f.s of the sequence x_n, $n=1,2,\dots$ [1]

This immediately follows from Riemann-Stieltjes integral

$$\int_0^1\int_0^1 F(x,y)\,\mathrm{d}F_N(x)\,\mathrm{d}F_N(y)=\frac{1}{N^2}\sum_{m,n=1}^N F(x_m,x_n),$$

where $F_N(x)=\frac{1}{N}\#\{n\le N; x_n<x\}$. Assuming $\lim_{k\to\infty}F_{N_k}(x)=g(x)$ for all continuity points x of g, then Helly-Bray lemma implies

$$\lim_{k\to\infty}\int_0^1\int_0^1 F(x,y)\,\mathrm{d}F_{N_k}(x)\,\mathrm{d}F_{N_k}(y)=\int_0^1\int_0^1 F(x,y)\,\mathrm{d}g(x)\,\mathrm{d}g(y).$$

Open problem is to solve $\int_0^1\int_0^1 F(x,y)\,\mathrm{d}g(x)\,\mathrm{d}g(y)=0$ in d.f.'s $g(x)$. The multi-dimensional case is mentioned in Problem 2.2 (II).

Partial results:

(I) Let us denote

$$F_{\tilde g}(x,y)=\int_0^1\tilde g^2(t)\,\mathrm{d}t-\int_x^1\tilde g(t)\,\mathrm{d}t-\int_y^1\tilde g(t)\,\mathrm{d}t+1-\max(x,y).$$

---

[1] Define $G(F=A)$ is the set of all d.f.s $g(x)$ for which $\int_0^1\int_0^1 F(x,y)\,\mathrm{d}g(x)\,\mathrm{d}g(y)=A$, and $G(A\le F\le B)$ is the set of all d.f.s $g(x)$ for which $A\le\int_0^1\int_0^1 F(x,y)\,\mathrm{d}g(x)\,\mathrm{d}g(y)\le B$. Then again $G(x_n)\subset G(F=A)$ is equivalent to $\lim_{N\to\infty}\frac{1}{N^2}\sum_{m,n=1}^N F(x_m,x_n)=A$, and $G(x_n)\subset G(A\le F\le B)$ is equivalent to $A\le\liminf_{N\to\infty}\frac{1}{N^2}\sum_{m,n=1}^N F(x_m,x_n)$ and $\limsup_{N\to\infty}\frac{1}{N^2}\sum_{m,n=1}^N F(x_m,x_n)\le B$.

From the relation

$$\int_0^1 \big(g(x)-\tilde{g}(x)\big)^2\,\mathrm{d}x = \int_0^1\!\!\int_0^1 F_{\tilde{g}}(x,y)\,\mathrm{d}g(x)\,\mathrm{d}g(y),$$

we see that the moment problem (1) with $F(x,y) = F_{\tilde{g}}(x,y)$ has the unique solution $g(x) = \tilde{g}(x)$.

(II) Let $F\colon [0,1]^2 \to \mathbb{R}$ be a continuous and symmetric function. For every distribution functions $g(x)$, $\tilde{g}(x)$ we have

$$\begin{aligned}\int_0^1\!\!\int_0^1 F(x,y)\,\mathrm{d}g(x)\,\mathrm{d}g(y) = 0 \Longleftrightarrow &\int_0^1\!\!\int_0^1 F(x,y)\,\mathrm{d}\tilde{g}(x)\,\mathrm{d}\tilde{g}(y)\\ &= \int_0^1 \big(g(x)-\tilde{g}(x)\big)\left(2\,\mathrm{d}_x F(x,1) - \int_0^1 \big(g(y)+\tilde{g}(y)\big)\,\mathrm{d}_y\,\mathrm{d}_x F(x,y)\right).\end{aligned}$$

Especially, putting $\tilde{g}(x) = c_0(x)$, we have

$$\begin{aligned}&\int_0^1\!\!\int_0^1 F(x,y)\,\mathrm{d}g(x)\,\mathrm{d}g(y) = 0\\ &\qquad\Longleftrightarrow F(0,0) = \int_0^1 \big(g(x)-1\big)\left(2\,\mathrm{d}_x F(x,1) - \int_0^1 \big(g(y)+1\big)\,\mathrm{d}_y\,\mathrm{d}_x F(x,y)\right).\end{aligned}$$

• A symmetric continuous $F(x,y)$ defined on $[0,1]^2$ is called *copositive* if

$$\int_0^1\!\!\int_0^1 F(x,y)\,\mathrm{d}g(x)\,\mathrm{d}g(y) \geq 0$$

for all distribution functions $g\colon [0,1] \to [0,1]$.

(III) Let $F(x,y)$ be a copositive function having continuous $F'_x(x,1)$ a.e., and let $g_1(x)$ be a strictly increasing solution of the moment problem (1). Then for every strictly increasing d.f. $g(x)$ we have

$$\int_0^1\!\!\int_0^1 F(x,y)\,\mathrm{d}g(x)\,\mathrm{d}g(y) = 0 \Leftrightarrow F'_x(x,1) = \int_0^1 g(y)\,\mathrm{d}_y F'_x(x,y) \quad \text{a.e. on} \quad [0,1],$$

Proposed by O. Strauch.

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1.34. Comparison of random sequences using the game theory

A finite two-person zero-sum matrix game with the payoff matrix $\mathbf{A}$.

$$\mathbf{A}=\begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,m}\\ a_{1,1} & a_{1,2} & \dots & a_{1,m}\\ \dots & \dots & \dots & \dots\\ a_{m,1} & a_{m,2} & \dots & a_{m,m}\end{pmatrix}$$

In this form, Player I chooses a row, Player II chooses a column, and II pays I the entry in the chosen row and column. Note that the entries of the matrix are the winnings of the row chooser and losses of the column chooser. This pure strategy for Player I of choosing row i may be represented as the $\mathbf{e}_i=(0,\dots,0,1,0,\dots,0)$, the unit vector with a 1 in the ith position and 0-f's elsewhere. Similarly, the pure strategy for II of choosing the jth column may be represented by $\mathbf{e}_j=(0,\dots,0,1,0,\dots,0)$ and the payoff to I is

$$\mathbf{e}_i\mathbf{A}\mathbf{e}_j^T=a_{i,j}.$$

Now, let Player I use a sequence $\mathbf{e}_n^{(I)}$, $n=1,2,\dots$, of pure strategy and Player II a sequence $\mathbf{e}_n^{(II)}$, $n=1,2,\dots$, of pure strategy. Then the mean-value of the payoff of I after N games is

$$\frac{1}{N}\sum_{n=1}^{N}\mathbf{e}_n^{(I)}\mathbf{A}\big(\mathbf{e}_n^{(II)}\big)^T.$$

Assume that there exist densities

$$\lim_{N\to\infty}\frac{1}{N}\#\Big\{n\le N;\mathbf{e}_n^{(I)}=\mathbf{e}_i\Big\}=p_i,\qquad i=1,2,\dots,m,$$
$$\lim_{N\to\infty}\frac{1}{N}\#\Big\{n\le N;\mathbf{e}_n^{(II)}=\mathbf{e}_i\Big\}=q_i,\qquad i=1,2,\dots,m.$$

The vector

$$\mathbf{p} = (p_1, p_2, \dots, p_m)$$

is called a mixed strategy for Player I. Similarly,

$$\mathbf{q} = (q_1, q_2, \dots, q_n)$$

is a mixed strategy for Player II. If the sequences $\mathbf{e}_n^{(I)}$ and $\mathbf{e}_n^{(II)}$ are statistically independent, then we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbf{e}_n^{(I)} \mathbf{A} \left(\mathbf{e}_n^{(II)}\right)^T = \mathbf{pAq}^T = \sum_{i,j=1}^{m} p_i a_{i,j} q_j. \tag{1}$$

The mixed strategies $\mathbf{p}$ and $\mathbf{q}$ can be computed optimally such that $\mathbf{pAq}^T = 0$, but independence of $\mathbf{e}_n^{(I)}$ and $\mathbf{e}_n^{(II)}$ is a **problem**. Player with better sequence can be found payoff positive.

Now, we transform the above matrix game to continuous case: Put

$$\begin{aligned} I_{i,j} = &[p_1 + p_2 + \dots + p_{i-1}, p_1 + p_2 + \dots + p_i) \\ &\times [q_1 + q_2 + \dots + q_{j-1}, q_1 + q_2 + \dots + q_j). \end{aligned}$$

Define $F(x, y)$ on $[0, 1]^2$ such that

$$F(x, y) = a_{i,j} \qquad \text{if} \quad (x, y) \in I_{i,j}, \quad i, j, = 1, 2, \dots, m. \tag{1'}$$

Let Player I use u.d. sequence x_n and Player II u.d. sequence y_n, $n = 1, 2, \dots$ If $(x_n, y_n) \in I_{i,j}$ the Player I choices pure strategy $\mathbf{e}_i$ and Player II pure strategy $\mathbf{e}_j$. Then mean value of the payoff of the Player I is

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} F(x_n, y_n) = \int_0^1\!\!\int_0^1 F(x, y)\, \mathrm{d}g(x, y), \tag{2}$$

where $g(x, y)$ is a d.f. of the sequence (x_n, y_n), $n = 1, 2, \dots$.

Example: Odd or Even. Players I and II simultaneously call out one of the numbers one or two. Player I wins if the sum of the numbers is odd. Player II wins if the sum of the numbers is even. The payoff matrix $\mathbf{A}$ is

$$\mathbf{A} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

and the function $F(x, y)$ corresponding to (1) is

$F(x, y)$

1	-1
-1	1

Let x_n and y_n, $n = 1, 2, \dots$ be u.d. sequences such that $(x_n, y_n$ has a.d.f. $g(x, y)$.

(i) If $g(x, y) = xy$ then $\int_0^1 \int_0^1 F(x, y)\,\mathrm{d}x\,\mathrm{d}y = 0$.

(ii) If $g(x, y) = \min(x, y)$ then $\int_0^1 \int_0^1 F(x, y)\,\mathrm{d}\min(x, y) = \int_0^1 F(x, x)\,\mathrm{d}x = -1$.

(iii) If $g(x, y) = \max(x + y - 1, 0)$ then $\int_0^1 \int_0^1 F(x, y)\,\mathrm{d}\max(x + y - 1, 0) = \int_0^1 F(x, 1 - x)\,\mathrm{d}x = 1$.

(iv) If $(x_n, y_n) = \big(\gamma_q(n), \gamma_q(n+1)\big)$, where $\gamma_q(n)$ is the van der Corput sequence in the base q, then

$\int_0^1 \int_0^1 F(x, y)\,\mathrm{d}g(x, y) = \frac{4}{q} - 1$. Thus for $q = 2$ Player I wins prize 1 and in the case $q > 5$ he loses.

Proposed by O. Strauch.

1.35. Oscillating sums

Directly from J. Arias de Reyna and J. van de Lune (2008):

- $S_\alpha(n) = \sum_{j=1}^{n} (-1)^{[j\alpha]}$ where α is any real number.
- Denoted by $t_0 = 0, t_1, t_2, \dots$ the sequence of those n for which $S_\alpha(n)$ assumes a value for the first time, i.e., is larger/smaller than ever before.
- Let $\alpha = [a_0; a_1, a_2, \dots]$ and $\beta = \alpha/2 = [b_0; b_1, b_2, \dots]$ be simple continued fraction expansins.

Open Problem: Determine whether the t_k is recurrent and the sequence $\mathrm{sign}\big(S(t_k)\big)$ is to be purely periodic.

Notes. (I) H. D. Ruderman (1977) proposed and D. Borwein (1978) solved (among other) that the series $\sum_{n=1}^{\infty} (-1)^{[n\sqrt{2}]}/n$ converges.
(II) P. Bundschuh (1977) proved that the series $\sum_{n=1}^{\infty} (-1)^{[n\alpha]}/n$ converges for numbers α with bounded b_i of $\beta = \alpha/2 = [b_0; b_1, b_2, \dots]$.
(III) J. Schoissengeier (2007) proved that the series

$$\sum_{n=1}^{\infty} (-1)^{[n\alpha]}/n \qquad \text{and} \qquad \sum_{k=0, 2\nmid q_k}^{\infty} (-1)^k (\log b_{k+1})/q_k$$

converge simultaneously. Here $\frac{p_k}{q_k}$ are convergents of $\beta = \alpha/2 = [b_0; b_1, b_2, \dots]$.
(IV) A. E. Brouwer and J. van de Lune (1976) have shown that $S_\alpha(n) \geq 0$ for all n if and only if the partial quotients a_{2i} of $\alpha = [a_0; a_1, a_2, \dots]$ are even for all $i \geq 0$.
(V) J. Arias de Reyna and J. van de Lune (2008) proved that $S_\alpha(n)$ is not bounded, so that the corresponding sequence t_k actually is an infinite sequence. They also prove that for every $j \geq 1$ there is an index k such that $t_j - t_{j-1} = Q_k$, where P_k/Q_k is a certain convergent of $\alpha = [a_0; a_1, a_2, \dots]$. They also give a fast algorithm for the computation of $S_\alpha(n)$ for any irrational α and for very large n in terms of $\beta = \alpha/2 = [b_0; b_1, b_2, \dots]$, e.g., $S_{\sqrt{2}}(10^{1000}) = -10$, $S_{\sqrt{2}}(10^{10000}) = 166$, $S_\pi(10^{10000}) = 11726$.

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1.36. Discrepancy system in the unit cube

Let S^{n-1} be the unit sphere of the n-dimensional euclidean space $\mathbb{R}^n$ and a cap is a portion of the sphere cut of by hyperplane. P. Gruber (2009) discus the problem whether the family of all caps of given size is a discrepancy system. A. Volčič (2011) in the planar case proved that the family of all arcs of S^1 of a constant length l is a discrepancy system if $\frac{l}{2\pi}$ is irrational. For rational $\frac{l}{2\pi}$ there exists a non-uniformly distributed sequence x_m, $m = 1, 2, \dots$, in S^1 such that $\frac{\{m \leq N; x_n \in C\}}{N} \to \frac{l}{2\pi}$ for every arc $C \subset S^1$ of the length l. In the case $n - 1 \geq 2$, Volčič (2011) proved that if s is a zero of a $d + 2$ dimensional Legendre polynomial of even degree, then x_m need not be uniformly distributed even if $\frac{\#\{n \leq N; x_m \in C_s\}}{N} \to P(C_s)$ for any spherical cap $C_s(u) = \{v \in S^{n-1}; u.v \geq s\}$. Here P is the the normalized Hausdorff measure on the sphere and $u.v$ is the usual scalar product in $\mathbb{R}^n$. In his proof he used P. Ungar result (1954) that $\int_{C_s} f dP = 0$ for all spherical caps C_s need not imply $f = 0$.

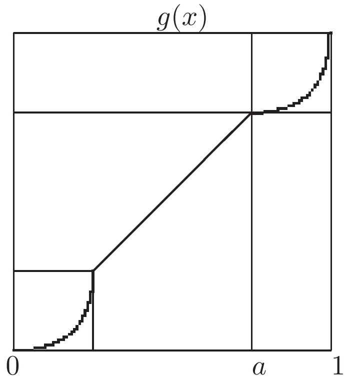
(I) **Open Problem:** Discuss a similar problem in $[0,1]^s$. For example:

(II) Let x_n, $n = 1, 2, \dots$, be a sequence in $[0,1)$ such that $\frac{\#\{n\le N; x_n\in I\}}{N} \to |I|$ for all intervals $I \subset [0,1]$ of the fixed length $|I| = a$. Then x_n need not be u.d.

P r o o f. All $g(x) \in G(x_n)$ must satisfy

$$g(x+a) = g(x) + a \qquad \text{for} \quad x \in [0, 1-a]. \tag{1}$$

The following d.f. $g(x)$ satisfies (1) but $g(x) \neq x$.



Proposed by O. Strauch.

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VOLČIČ, A.: *On Grubner's problem concerning uniform distribution on the sphere,* Arch. Math. **97** (2011), 385–390.

1.37. Extremes of $\int_0^1 \int_0^1 F(x,y)\,\mathrm{d}_x\,\mathrm{d}_y\,g(x,y)$ attained at shuffles of M

(I) R. B. N e l s e n [1999, p. 59, 3.2.3.]: Let I_i, $i = 1, 2, \dots, n$ be a decomposition of the unit interval $[0,1]$, let π be a permutation of $(1, 2, \dots, n)$, and let $T\colon [0,1] \to [0,1]$ be an one-to-one map whose graph T is formed by diagonals or anti-diagonals of squares $I_i \times I_{\pi(i)}$, $i = 1, 2, \dots, n$. Then the copula $C(x,y)$ defined by

$$C(x,y) = \Big|\mathrm{Project}_x\Big(\big([0,x)\times[0,y)\big)\cap T\Big)\Big|$$

is called the shuffle of M.

(II) M. H o f e r and M. R. I a c ò (2013) proved: Let $(a_{i,j})$, $i, j = 1, 2, \dots, n$ be a real-valued $n \times n$ matrix. Let

$$I_{i,j} = \left[\frac{i-1}{n}, \frac{i}{n}\right] \times \left[\frac{j-1}{n}, \frac{j}{n}\right], \qquad i, j = 1, 2, \dots, n$$

and let the piecewise constant function $F(x,y)$ be defined as

$$F(x,y) = a_{i,j} \qquad \text{if} \quad (x,y) \in I_{i,j}, \quad i,j = 1,2,\dots,n.$$

Then

$$\max_{g(x,y)\text{-copula}} \int_0^1\!\!\int_0^1 F(x,y)\,\mathrm{d}_x\,\mathrm{d}_y g(x,y) = \frac{1}{n}\sum_{i=1}^{n} a_{i,\pi^*(i)}.$$

Here $\pi^*(i)$ maximizes $\sum_{i=1}^{n} a_{i,\pi(i)}$, where π is a permutation of $(1,2,\dots,n)$. The maximum is attained at $g(x,y) = C(x,y)$, where $C(x,y)$ is the shuffle of M whose graph T is formed by diagonals or anti-diagonals in $I_{i,\pi^*(i)}$, $i = 1,2,\dots,n$.

(III) Applying (II) M. Hofer and M. R. Iacò approximate extremes of

$$\int_0^1\!\!\int_0^1 F(x,y)\,\mathrm{d}g(x,y)$$

which respect to copulas $g(x,y)$ by the following: For continuous $F(x,y)$ on $[0,1]^2$ define piecewise constant functions $F_1(x,y)$, $F_2(x,y)$ as

$$F_1(x,y) = \min_{(u,v)\in I_{i,j}} F(u,v) \quad \text{if } (x,y) \in I_{i,j}, \quad i,j = 1,2,\dots,n,$$

$$F_2(x,y) = \max_{(u,v)\in I_{i,j}} F(u,v) \quad \text{if } (x,y) \in I_{i,j}, \quad i,j = 1,2,\dots,n,$$

where

$$I_{i,j} = \left[\frac{i-1}{n}, \frac{i}{n}\right] \times \left[\frac{j-1}{n}, \frac{j}{n}\right].$$

Let $C_0(x,y), C_1(x,y), C_2(x,y)$ be copulas such that

$$C_1(x,y) \text{ maximizes } \int_0^1\!\!\int_0^1 F_1(x,y)\,\mathrm{d}g(x,y),$$

$$C_2(x,y) \text{ maximizes } \int_0^1\!\!\int_0^1 F_2(x,y)\,\mathrm{d}g(x,y)$$

and

$$C_0(x,y) \text{ maximizes } \int_0^1\!\!\int_0^1 F(x,y)\,\mathrm{d}g(x,y).$$

over all copulas $g(x,y)$. Then

$$\int_0^1\!\!\int_0^1 F(x,y)\mathrm{d}C_0(x,y) = \lim_{n\to\infty}\int_0^1\!\!\int_0^1 F_1(x,y)\,\mathrm{d}C_1(x,y) = \lim_{n\to\infty}\int_0^1\!\!\int_0^1 F_2(x,y)\,\mathrm{d}C_2(x,y).$$

(IV) **Open problem:** Using (III) and a numerical experiment the authors conjecture that the extreme of $\int_0^1\int_0^1 \sin\big(\pi(x+y)\big)\,\mathrm{d}_x\,\mathrm{d}_y g(x,y)$ is

$$\max_{g(x,y)\text{-copula}} \int_0^1\int_0^1 \sin\big(\pi(x+y)\big)\,\mathrm{d}_x\,\mathrm{d}_y g(x,y) = \frac{3}{4\sqrt{2}} - \frac{1}{2\pi}$$

and it is attained at shuffle of M formed by anti-diagonal of $\left[0,\frac{3}{4}\right]\times\left[0,\frac{3}{4}\right]$ and by diagonal $\left[\frac{3}{4},1\right]\times\left[\frac{3}{4},1\right]$. Compare with Problem 1.29.

(V) The copula in (IV) satisfies (IIIb) in 1.29 which is a necessary condition for a copula maximizing a related integral.

(VI) Note that if x_n, $n=1,2,\dots$, is a u.d. sequence, then two-dimensional sequence $\big(x_n,T(x_n)\big)$ has a.d.f $C(x,y)$ and thus for every continuous $F(x,y)$ we have

$$\int_0^1\int_0^1 F(x,y)\,\mathrm{d}C(x,y) = \int_0^1 F\big(x,T(x)\big)\,\mathrm{d}x.$$

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1.38. Two-dimensional Benford's law

Let $x_n>0$, $y_n>0$, $n=1,2,\dots$ and $b>1$ be an integer base, K_1,K_2 be positive integers, and

$$K_1 = k_1^{(1)}k_2^{(1)}\dots k_{r_1}^{(1)} \text{ in base b},\qquad K_2 = k_1^{(2)}k_2^{(2)}\dots k_{r_2}^{(2)} \text{ in base b},$$

$$u_1=\log_b\left(\frac{K_1}{b^{r_1-1}}\right),\qquad u_2=\log_b\left(\frac{K_1+1}{b^{r_1-1}}\right),$$

$$v_1=\log_b\left(\frac{K_2}{b^{r_2-1}}\right),\qquad v_2=\log_b\left(\frac{K_2+1}{b^{r_2-1}}\right).$$

(I) As in Problem 1.32 we have

first r_1 digits (starting a non-zero digit) of $x_n = K_1 \iff \{\log_b x_n\}\in[u_1,u_2)$,

first r_2 digits (starting a non-zero digit) of $y_n = K_2 \iff \{\log_b y_n\}\in[v_1,v_2)$.

Denote [2]

$$F_N(x,y)=\frac{\#\{n\le N;\{\log_b x_n\}<x \text{ and } \{\log_b y_n\}<y\}}{N}.$$

(II) From definition of d.f.'s the following holds:

Let $g(x,y)\in G\big(\{\log_b x_n\},\{\log_b y_n\}\big)$ and $\lim_{k\to\infty}F_{N_k}(x,y)=g(x,y)$ for $(x,y)\in[0,1]^2$. Then

$$\lim_{k\to\infty}\frac{\#\{n\le N_k;\ \text{first } r_1 \text{ digits of } x_n=K_1 \text{ and } \text{ first } r_2 \text{ digits of } y_n=K_2}{N_k}$$
$$=g(u_2,v_2)+g(u_1,v_1)-g(u_2,v_1)-g(u_1,v_2).$$

(III) As example we give:

$$G\Big(\big\{\log_b n\big\},\{\log_b(n+1)\}\Big)$$
$$=\bigg\{g_u(x,y)=\frac{b^{\min(x,y)}-1}{b-1}\frac{1}{b^u}+\frac{b^{\min(x,y,u)}-1}{b^u};u\in[0,1]\bigg\}.$$

By the Sklar theorem

$$g_u(x,y)=\min\big(g_u(x),g_u(y)\big),\ \text{where}$$

$$g_u(x)=\frac{b^x-1}{b-1}\cdot\frac{1}{b^u}+\frac{b^{\min(x,u)}-1}{b^u}.$$

Put $x_n=\log_b n \bmod 1$ and $y_n=\log_b(n+1)\bmod 1$. Then by (II)

$$\lim_{k\to\infty}\frac{\#\{n\le N_k;\ \text{first } r_1 \text{ digits of } x_n=K_1 \text{ and first } r_2 \text{ digits of } y_n=K_2}{N_k}$$
$$=g_u(u_2,v_2)+g_u(u_1,v_1)-g_u(u_2,v_1)-g_u(u_1,v_2).$$

If $K_1=K_2$ then $=g_u(u_2)-g_u(u_1)$. It can be found directly.

In the following examples we use statistical independent sequences: Let $x_n\in[0,1)$, $n=1,2,\dots$, be an u.d. sequence. Then

(IV) x_n and $\log_b n \bmod 1$ are statistically independent (G. Rauzy (1973) see [SP, p. 2–27, 2.3.6.].

(V) x_n and $\log_b(n\log n)\bmod 1$ are statistically independent Y. Ohkubo (2011).

(VI) x_n and $\log_b p_n \bmod 1$ are statistically independent (Y. Ohkubo (2011)).

(VII) The sequences $\log_b n \bmod 1$, $\log_b p_n \bmod 1$ and $\log_b \log n \bmod 1$ have the same set of d.f.'s. (Y. Ohkubo (2011).

(VIII) From (IV) it follows: Let $x_n\in[0,1)$, $n=1,2,\dots$, be u.d. sequence.

[2] In the following the sentence "*starting a non-zero digit*" we will not mention.

Then

$$G\big(x_n, \{\log_b n\}\big) = \big\{g_u(x,y) = x.g_u(y); u \in [0,1]\big\},$$

where

$$g_u(x) = \frac{b^x - 1}{b-1} \cdot \frac{1}{b^u} + \frac{b^{\min(x,u)} - 1}{b^u}$$

and $F_{N_k}(x,y) \to g_u(x,y)$ if $\{\log_b N_k\} \to u$.

(IX) Let $x_n \in [0,1)$, $n = 1,2,\dots$, be u.d. sequence. Then

$$G\big(x_n, \{\log_b p_n\}\big) = \big\{g_u(x,y) = x.g_u(y); u \in [0,1]\big\},$$

where

$$g_u(x) = \frac{b^x - 1}{b-1} \cdot \frac{1}{b^u} + \frac{b^{\min(x,u)} - 1}{b^u}$$

and $F_{N_k}(x,y) \to g_u(x,y)$ if $\{\log_b N_k\} \to u$.

(X) We have

$$G\big(\{\log_b F_n\}, \{\log_b p_n\}\big) = \big\{x.g_u(y); u \in [0,1]\big\}$$

and let $\{\log_b N_k\} \to u$. Then

$$\lim_{k\to\infty} \frac{\#\{n \le N_k;\ \text{first } r_1 \text{ digits of } F_n = K_1 \text{ and first } r_2 \text{ digits of } p_n = K_2\}}{N_k}$$

$$= u_2 g_u(v_2) + u_1 g_u(v_1) - u_2 g_u(v_1) - u_1 g_u(v_2),$$

where F_n is the sequence of Fibonacci numbers and p_n is the increasing sequence of all primes and

$$u_1 = \log_b\left(\frac{K_1}{b^{r_1-1}}\right), \qquad u_2 = \log_b\left(\frac{K_1+1}{b^{r_1-1}}\right),$$

$$v_1 = \log_b\left(\frac{K_2}{b^{r_2-1}}\right), \qquad v_2 = \log_b\left(\frac{K_2+1}{b^{r_2-1}}\right),$$

$$g_u(x) = \frac{b^x - 1}{b-1} \cdot \frac{1}{b^u} + \frac{b^{\min(x,u)} - 1}{b^u}.$$

(XI) Problem 1.38 is inspired by the result of F. Luca and P. Stanica (2014): There exists infinite many n such that Fibonacci number F_n starts with digits K_1 and $\phi(F_n)$ starts with digits K_2 in the base b representation. Here K_1 and K_2 are arbitrary and $\varphi(x)$ is the Euler function.

We see that (XI) is equivalent to the sequence

$$\big(\log_b F_n, \log_b \varphi(F_n)\big) \bmod 1, \qquad n = 1,2,\dots,$$

is everywhere dense in $[0,1]^2$, but the authors use the following method:

(i) By the first author $\varphi(F_n)/F_n$ is dense in $[0,1]$. Thus, for an interval I with arbitrary small length which containing K_2/K_1, there exists $\varphi(F_a)/F_a \in I$.

(ii) Then $\varphi(F_{ap})/F_{ap} \in I$ for all sufficiently large primes p.

(iii) There exists infinitely many primes p such that F_{ap} starts with K_1.

(iv) Finally, multiplying I by F_{ap} they find $\varphi(F_{ap})$ which starts with K_2.

(XII) **Definition.** The sequence (x_n, y_n), $x_n > 0$, $y_n > 0$, $n = 1, 2, \dots$, satisfies 2-dimensional B.L. in base b, if for every K_1, K_2 we have

$$\lim_{N\to\infty} \frac{\#\{n \le N;\ \text{the first } r \text{ digits of } x_n = K_1 \text{ and the first } l \text{ digits of } y_n = K_2\}}{N}$$
$$= \log_b\left(1 + \frac{1}{K_1}\right).\log_b\left(1 + \frac{1}{K_2}\right).$$

(XIII) From definition follows: The sequence (x_n, y_n) satisfies 2-dimensional Benford law (B.L.) if and only if $(\log_b x_n, \log_b y_n) \bmod 1$ is u.d. in $[0,1)^2$.

(XVI) **Open problem:** Prove that the sequence (n^{n^2}, n^n) satisfies 2-dimensional B.L. in any base b. Motivation is that by [SP, 3.13.4.] the sequence $(n^2 \log n, n \log n) \bmod 1$ is u.d. in $[0,1]^2$.

(XV) **Open problem:** Prove that the sequence $(\log_b n, \log_b \log n) \bmod 1$ is dense in $[0,1]^2$. If this is true, then there exists infinite many n such that n starts with digits K_1 and $\log n$ starts with digits K_2 in the base b representation, where K_1 and K_2 are arbitrary positive integers. By [SP, 3.13.5.] the sequence $(\log n, \log\log n) \bmod 1$ is dense in $[0,1]^2$ but not u.d.

Proposed by O. Strauch.

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2. Open theories

2.1. Uniform distribution theories

There are some different ways for generalizing of the classical u.d. theory, see [KN, Chap. 3 and 4], [H, Chap. 2], [DT, Chap. 2] and [SP, p. 1–5, 1.5]. For example:

- Points of investigated sequences x_n are elements from a general space.
- For basic sets $A_x = \big\{n \in \mathbb{N}; x_n \in [0,x)\big\}$ in the definition of u.d. as $d(A_x) = x$, the asymptotic density d is exchanged by another types of densities.
- The asymptotic density d is preserved but in A_x the relation $x_n \in [0,x)$ is exchanged by more complicated relations (cf. O. Strauch (1998)).

Here we start with a main theorem of u.d. theory due to H. Weyl (cf. [KN, p. 2, Th. 1.1], [SP, p. 1–4, Th. 1.4.0.1]):

Weyl's limit relation. *The sequence $u(n)$, $n = 1, 2, \dots$ from the unit interval $[0,1]$ is u.d. if and only if for every continuous $f\colon [0,1] \to \mathbb{R}$ we have*

$$\int_0^1 f(x)\,\mathrm{d}x = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f\big(u(n)\big).$$

This relation can be used as a definition of u.d. of $u(n)$ and also for definition of u.d. in abstract spaces, see [KN, p. 171, Def. 1.1]: Let $\mathbf{X}$ be a compact Hausdorff space and $C(\mathbf{X})$ consists of all real-valued continuous functions on $\mathbf{X}$. Let $\mathrm{d}X$ be a nonnegative regular normed Borel measure in $\mathbf{X}$. The sequence $u(n) \in \mathbf{X}$, $n = 1, 2, \dots$ is called u.d. in $\mathbf{X}$ with respect to $\mathrm{d}X$ if

$$\forall\big(f \in \mathcal{C}(\mathbf{X})\big)\int_{\mathbf{X}} f(X)\,\mathrm{d}X = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f\big(u(n)\big).$$

The basic application of Weyl's limit relation is a possibility computing the Riemann integral $\int_0^1 f(x)\,\mathrm{d}x$ on $[0,1]$ of a continuous function $f(x)$ as the limit $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f\big(u(n)\big)$ of arithmetic means of $f(x)$ (quasi-Monte Carlo method) and vice-versa, the limit of arithmetic means by integral. Looking at $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f\big(u(n)\big)$ as an integral defined on $\mathbb{Z}^+$, then the classical u.d. theory is the theory of coherence between two types of integrals. Thus the concept of u.d. theory can be generalized (see O. Strauch [1999, Chap. 4]) to a theory of the integral equation

$$\int_{\mathbf{X}} f(X)\,\mathrm{d}X = \int_{\mathbf{Y}} f\big(u(Y)\big)\,\mathrm{d}Y, \tag{1}$$

of two types of integrals, where $\mathbf{X}, \mathbf{Y}$ are arbitrary spaces equipped with integrals or measures $\mathrm{d}X$ and $\mathrm{d}Y$, respectively, or more generally, equipped with functionals which in the following we also call integrals. Here $f\colon \mathbf{X} \to \mathbb{R}$ and $u\colon \mathbf{Y} \to \mathbf{X}$. The main problem is to compute integral of the first type on the left-hand (1) by the integral of the second type on the right-hand side. This is our aim in these new u.d. theories. Here are a few selected spaces with theories of integration.

$\mathbf{X}_1 = [0,1]$, equipped with the integral $\int_0^1 f(x)\,\mathrm{d}x$;

$\mathbf{X}_2 = \{1,2,\dots\}$, equipped with the integral $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N f(n)$;

$\mathbf{X}_3 = [0,+\infty)$, equipped with the integral $\lim_{T\to\infty} \frac{1}{T}\int_0^T f(x)\,\mathrm{d}x$;

$\mathbf{X}_4 = [0,1]$, equipped with the integral $\max_{x\in[0,1]} f(x)$;

$\mathbf{X}_5 = \{1,2,\dots\}$, equipped with the integral $\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^N f(n)$;

$\mathbf{X}_6 = [0,+\infty)$, equipped with the integral $\int_0^{+\infty} f(x)\,\mathrm{d}x$;

$\mathbf{X}_7 = \{1,2,\dots\}$, equipped with the integral $\lim_{N\to\infty} \frac{1}{N^\alpha}\sum_{n=1}^N f(n)$;

$\mathbf{X}_8 = [0,1]^s$, equipped with the integral $\int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x}$;

$\mathbf{X}_9 = \{1,2,\dots\} \times [0,+\infty)$, equipped with the integral $\lim_{N,T\to\infty} \frac{1}{NT}\sum_{n=1}^N \int_0^T f(n,x)\,\mathrm{d}x$.

Varying couples $(\mathbf{X}_i, \mathbf{X}_j)$ we find the following known u.d. theories

- $\int_0^1 f(x)\,\mathrm{d}x = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N f\bigl(u(n)\bigr)$ – classical u.d. theory;
- $\int_0^1 f(x)\,\mathrm{d}g(x) = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N f\bigl(u(n)\bigr)$ – theory of g-distributed sequences, see [KN, pp. 54–57] and [SP, p. 1–11, 1.8.1.];
- $\int_0^1 f(x)\,\mathrm{d}x = \lim_{T\to\infty} \frac{1}{T}\int_0^T f\bigl(u(x)\bigr)\,\mathrm{d}x$ – c.u.d. theory, see [KN, pp. 78–87] and [DT, pp. 277–300];
- $\int_0^1 f(x)\,\mathrm{d}x = \int_0^1 f\bigl(u(x)\bigr)\,\mathrm{d}x$ – theory of u.d. preserving functions, it was introduced by Š. Porubský, T. Šalát and O. Strauch (1998), see [SP, p. 2–45, 2.5.1];
- $\max_{x\in[0,1]} f(x) = \limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^N f\bigl(u(n)\bigr)$ – theory of maldistributed sequences, it was introduced by G. Myerson (1993), see [SP, p. 1–19, 1.8.10];
- $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N f(n) = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N f\bigl(u(n)\bigr)$ – u.d. theory in $\mathbb{Z}$, see [KN, pp. 305–319].

These examples lead to the following common definition.

Definition. Let $\mathcal{C}$ be a set of functions $f\colon \mathbf{X}\to\mathbb{R}$. The u.d. theory $(\mathcal{C},\mathbf{X},\mathbf{Y})$-u.d. is a theory of functions $u\colon \mathbf{Y}\to\mathbf{X}$ satisfying

$$\forall\,(f\in\mathcal{C})\int\limits_{\mathbf{X}} f(X)\,\mathrm{d}X=\int\limits_{\mathbf{Y}} f\big(u(Y)\big)\mathrm{d}Y. \tag{2}$$

These functions u are called u.d. in $(\mathcal{C},\mathbf{X},\mathbf{Y})$-u.d. theory.

- The $(\mathbf{X},\mathbf{Y})$-u.d. theory is a theory of the integral equation (1), in which the set $\mathcal{C}$ is not strictly specified.
- The $(\mathbf{Y},\mathbf{X})$-u.d. is the inverse theory to the $(\mathbf{X},\mathbf{Y})$-u.d.
- The $(\mathbf{X},\mathbf{Y})$-u.d. theory is empty if there does not exist any u for some class f such that (1) is valid.
- The $(\mathbf{X},\mathbf{X})$-u.d. theory of $\int_{\mathbf{X}} f(X)\,\mathrm{d}X=\int_{\mathbf{X}} f\big(u(X)\big)\,\mathrm{d}X$ is a theory of integration, where the inside part $u(X)$ can be omitted. We shall call it u.d. preserving theory (abbreviating u.d.p. theory), because the equation

$$\int\limits_{\mathbf{X}} f(X)\,\mathrm{d}X=\int\limits_{\mathbf{X}} f\big(u(X)\big)\,\mathrm{d}X=\int\limits_{\mathbf{Z}} f\Big(u\big(v(Z)\big)\Big)\mathrm{d}Z \tag{3}$$

gives

Theorem. *Let u be u.d. in $(\mathcal{C},\mathbf{X},\mathbf{X})$-u.d. theory and $\mathcal{C}\circ u=\mathcal{C}$, where $\mathcal{C}\circ u=\big\{f(u(X);f\in\mathcal{C}\big\}$. Then*

$$v \text{ is u.d. in } (\mathcal{C},\mathbf{X},\mathbf{Z}) \iff u\circ v \text{ is u.d. in } (\mathcal{C},\mathbf{X},\mathbf{Z}).$$

- Using the equation

$$\int\limits_{\mathbf{X}} f(X)\,\mathrm{d}X=\int\limits_{\mathbf{Y}} f\big(u(Y)\big)\,\mathrm{d}Y=\int\limits_{\mathbf{Z}} f\Big(u\big(v(Z)\big)\Big)\,\mathrm{d}Zs \tag{4}$$

a new $(\mathbf{Y},\mathbf{Z})$-u.d. theory can be found by means of known $(\mathbf{X},\mathbf{Y})$-u.d. and $(\mathbf{X},\mathbf{Z})$-u.d. theory.

In the following we list some **new u.d theories:**

(I) $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(n)=\int_0^1 f\big(u(x)\big)\,\mathrm{d}x$ is an inverse theory to the classical u.d. one.

(II) $\lim_{T\to\infty}\frac{1}{T}\int_0^T f(x)\,\mathrm{d}x=\int_0^1 f\big(u(x)\big)\,\mathrm{d}x$ is an inverse theory to the c.u.d. theory.

(III) $\int_0^{+\infty} f(x)\,\mathrm{d}x=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f\big(u(n)\big)$ is an empty u.d. theory.

(IV) In $\int_0^1\int_0^1 f(x,y)\,\mathrm{d}x\,\mathrm{d}y=\int_0^1 f\big(u(x),v(x)\big)\,\mathrm{d}x$, the curve $\big(u(x),v(x)\big)$ must be Peano. The equation (4) in the form

$$\int\limits_0^1\!\!\int\limits_0^1 f(x,y)\,\mathrm{d}x\,\mathrm{d}y=\int\limits_0^1 f\big(u(x),v(x)\big)\,\mathrm{d}x=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f\Big(u\big(w(n)\big),v\big(w(n)\big)\Big)$$

gives that $\big(u(x),v(x)\big)$ is u.d. in this theory if and only if for every classical u.d.sequence $w(n)$, $n=1,2,\dots$, in $[0,1]$, the sequence $\big(u(w(n)),v(w(n))\big)$ is u.d. in $[0,1]^2$.

(V) In $\lim_{T\to\infty}\frac{1}{T}\int_0^T f(x)\,\mathrm{d}x = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N f\big(u(n)\big)$, assuming that $\mathcal{C}$ contains only bounded continuous functions f which have bounded variation on every interval $[0,n]$ such that $V\big(f/[0,n]\big)=\mathcal{O}(n)$, then we can construct u.d. sequence $u(n)$ directly: $u(i)$, $i=1,2,\dots,n^2$ is composed with n parts that are contained in the intervals $[0,1),[1,2),\dots,[n-1,n)$, all are congruent $\bmod 1$ and having discrepancy $D_n^*\to 0$. By applying Koksma inequality we have

$$\left|\frac{1}{n}\int_0^n f(x)\,\mathrm{d}x - \frac{1}{n^2}\sum_{i=1}^{n^2} f\big(u(i)\big)\right| \le D_n^*\frac{V(f/[0,n))}{n}.$$

(VI) In $\int_0^1 f(x)\,\mathrm{d}x = \lim_{N,T\to\infty}\frac{1}{NT}\sum_{n=1}^N\int_0^T f\big(u(n,x)\big)\,\mathrm{d}x$ we can used the following L^2 discrepancy

$$D_{N,T}^{(2)}(u) = \frac{1}{3} + \frac{1}{NT}\sum_{n=1}^N\int_0^T\big(u(n,x)\big)^2\,\mathrm{d}x - \frac{1}{NT}\sum_{n=1}^N\int_0^T u(n,x)\,\mathrm{d}x$$
$$-\frac{1}{2(NT)^2}\sum_{m,n=1}^N\int_0^T\int_0^T|u(m,x_1)-u(n,x_2)|\,\mathrm{d}x_1\,\mathrm{d}x_2,$$

which characterizes u.d. of $u(n,x)$.

(VII) The $\int_0^1 f(x)\,\mathrm{d}x = \int_0^1 f\big(u(x)\big)\,\mathrm{d}x$ is known u.d.p. theory, where the (3) has the form

$$\int_0^1 f(x)\,\mathrm{d}x = \int_0^1 f\big(u(x)\big)\,\mathrm{d}x = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N f\Big(u\big(v(n)\big)\Big)$$

which gives

$$(*)\qquad u(x) \text{ is u.d.p.} \iff u\big(v(n)\big) \text{ is u.d. in } [0,1].$$

In [SP, p. 2–45, 2.5.1] the result (*) was used as definition: The map $u\colon[0,1]\to[0,1]$ is called **uniform distribution preserving** (abbreviated u.d.p.) if for any u.d. sequence x_n, $n=1,2,\dots$, in $[0,1]$ the sequence $u(x_n)$ is also u.d. In this u.d.p. theory we register the following progress:

A Riemann integrable function $u\colon[0,1]\to[0,1]$ is a u.d.p. transformation if and only if one of the following conditions is satisfied:

(i) $\int_0^1 h(x)\,\mathrm{d}x = \int_0^1 h\big(u(x)\big)\,\mathrm{d}x$ for every continuous $h\colon[0,1]\to\mathbb{R}$.

(ii) $\int_0^1\big(u(x)\big)^k\,\mathrm{d}x = \frac{1}{k+1}$ for every $k=1,2,\dots$

(iii) $\int_0^1 e^{2\pi iku(x)}\mathrm{d}x = 0$ for every $k=\pm1,\pm2,\dots$

(iv) There exists an increasing sequence of positive integers N_k and an N_k–almost u.d. sequence x_n for which the sequence $u(x_n)$ is also N_k–almost u.d.

(v) There exists an almost u.d. sequence x_n in $[0,1)$ such that the sequence $u(x_n) - x_n$ converges to a finite limit.

(vi) There exists at least one $x \in [0,1]$ of which orbit $x, u(x), u\big(u(x)\big), \dots$ is almost u.d.

(vii) u is measurable in the Jordan sense and $|u^{-1}(I)| = |I|$ for every subinterval $I \subset [0,1]$.

(viii) $\int_0^1 u(x)\,\mathrm{d}x = \int_0^1 x\,\mathrm{d}x = \frac{1}{2}$,
$\int_0^1 \big(u(x)\big)^2\,\mathrm{d}x = \int_0^1 x^2\,\mathrm{d}x = \frac{1}{3}$,
$\int_0^1\int_0^1 |u(x) - u(y)|\,\mathrm{d}x\,\mathrm{d}y = \int_0^1\int_0^1 |x - y|\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{3}$.

From the other properties of u.d.p. transformations let us mention:

(ix) Let u_1, u_2 be u.d.p. transformations and α a real number. Then $u_1\big(u_2(x)\big)$, $1 - u_1(x)$ and $u_1(x) + \alpha \bmod 1$ are again u.d.p. transformations.

(x) Let u_n be a sequence of u.d.p. transformations uniformly converging to u. Then u is u.d.p.

(xi) Let $u\colon [0,1] \to [0,1]$ be piecewise differentiable. Then u is u.d.p. if and only if $\sum_{x\in u^{-1}(y)} \frac{1}{|u'(x)|} = 1$ for all but a finite number of points $y \in [0,1]$.

(xii) A piecewise linear transformation $u\colon [0,1] \to [0,1]$ is u.d.p. if and only if $|J_j| = |I_{j,1}| + \dots + |J_{j,n_j}|$ for every $J_j = (y_{j-1}, y_j)$, where $0 = y_0 < y_1 < \dots < y_m = 1$ is the sequence of ordinates of the ends of line segment components of the graph of f and $u^{-1}(J_j) = I_{j,1} \cup \dots \cup J_{j,n_j}$.

(xiii) $u(x)$ is u.d.p. if and only if

$$\int_0^1\int_0^1 F\big(u(x), u(y)\big)\,\mathrm{d}x\,\mathrm{d}y = 0,$$

where

$$F(x,y) = (1/2)\big(|x - u(y)| + |y - u(x)| - |x - y| - |u(x) - u(y)|\big).$$

NOTES.

The problem to find all continuous u.d.p. is formulated in J a.- I. R i v k i d (1973). The results (i)-(vii), (ix)-(xii) are proved in Š. P o r u b s k ý, T. Š a l á t and O. S t r a u c h (1988). The criterion (viii) and (xiii) are given in O. S t r a u c h [1999, p. 116, 67]. Some parts of these results are also proved independently in W. Bosch (1988). R. F. Tichy and R. Winkler (1991) gave a generalization for compact metric spaces. Some related results can be found in: M. Paštéka (1987), Y. Sun (1993, 1995), P. Schatte (1993), S. H. Molnár (1994) and J. Schmeling and R. Winkler (1995).

Multidimensional u.d.p. map $\Phi : [0,1]^s \to [0,1]^s$ is called uniformly distribution preserving (u.d.p.) map if for every uniformly distributed (u.d.) sequence $\mathbf{x}_n$, $n=1,2,\dots$, the image $\Phi(\mathbf{x}_n)$ is again u.d. The main criterion of u.d.p. map is

Theorem. *A map $\Phi(\mathbf{x})$ is u.d.p. if and only if for every continuous $f : [0,1]^s \to \mathbb{R}$ we have*

$$\int_{[0,1]^s} f\big(\Phi(\mathbf{x})\big)\,\mathrm{d}\mathbf{x} = \int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

The multi-dimensional u.d.p. functions are:

(i) $\Phi(\mathbf{x}) = \mathbf{x} \oplus \boldsymbol{\sigma}$, where
$x \oplus \sigma = \frac{x_0+\sigma_0 \pmod b}{b} + \frac{x_1+\sigma_1 \pmod b}{b^2} + \cdots$ and
$\mathbf{x} \oplus \boldsymbol{\sigma} = (x_1 \oplus \sigma_1, x_2 \oplus \sigma_2, \dots, x_s \oplus \sigma_s)$;

(ii) $\Phi(\mathbf{x}) = \big(\Phi_1(x_1), \dots, \Phi_s(x_s)\big)$, where $\Phi_n(x)$ are one-dimensional u.d.p. maps, especially

(iii) $\Phi(\mathbf{x}) = \mathbf{b}^{\boldsymbol{\alpha}}\mathbf{x} \bmod 1 = \big(b_1^{\alpha_1}x_1, \dots, b_s^{\alpha_s}x_s\big) \bmod 1$;

(iv) $\Phi(\mathbf{x}) = \mathbf{x} + \boldsymbol{\sigma} \bmod 1 = (x_1+\sigma_1, \dots, x_s+\sigma_s) \bmod 1$;

(v) $\Phi(\mathbf{x}) = (A\mathbf{x})^T \bmod 1$, where A is an $s \times s$ nonsingular integer matrix, cf. S. Steinerberger [Th. 2, 2009];

(vi) $\Phi(\mathbf{x}) = \pi(\mathbf{x})$, where $\pi(\mathbf{x}) = \big(x_{\pi(1)}, \dots, x_{\pi(n)}\big)$ is a permutation.

Open question: Find another multidimensional u.d.p.

Proposed by O. Strauch.

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2.2. Distribution functions of sequences

For a multi-dimensional sequence $\mathbf{x}_n$, $n = 1, 2, \dots$, in $[0, 1)^s$, the theory of the set $G(\mathbf{x}_n)$ of all d.f.s of $\mathbf{x}_n$, $n = 1, 2, \dots$, is **open**. A motivation to study of $G(\mathbf{x}_n)$ is the deterministic analysis of sequences in 2.3.[3] The set $G(\mathbf{x}_n)$ has the following fundamental properties for every sequence $\mathbf{x}_n$ in $[0, 1)^s$:
(I) $G(\mathbf{x}_n)$ is non-empty, and it is either a singleton or has infinitely many elements. Precisely, $G(\mathbf{x}_n)$ is non-empty, closed and connected set in the weak topology, and these properties are characteristic for $G(\mathbf{x}_n)$, i.e., given a non-empty set H of distribution functions, there exists a sequence $\mathbf{x}_n$ in $[0, 1)^s$ such that $G(\mathbf{x}_n) = H$ if and only if H is closed and connected.
(II) There are no general methods for computing $G(\mathbf{x}_n)$, without the following one: Let $F(\mathbf{x}, \mathbf{y})$ be a continuous function defined on $[0, 1]^s \times [0, 1]^s$ and let $G(F)$ be the set of all d.f.'s $g(\mathbf{x})$ satisfying

$$\int\limits_{[0,1]^s\times[0,1]^s} F(\mathbf{x}, \mathbf{y})\, \mathrm{d}g(\mathbf{x})\, \mathrm{d}g(\mathbf{y}) = 0. \tag{1}$$

If the sequence $\mathbf{x}_n$, $n = 1, 2, \dots$, satisfies

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{n,m=1}^{N} F(\mathbf{x}_m, \mathbf{x}_n) = 0,$$

then $G(\mathbf{x}_n) \subset G(F)$.

Open problem 1. Find a method for solving the moment problem (1).

[3]We shall identify the notion of the **distribution** of a sequence $\mathbf{x}_n \bmod 1$, $n = 1, 2, \dots$, with the set $G(\mathbf{x}_n \bmod 1)$, i.e., the distribution of $\mathbf{x}_n \bmod 1$ is known if we know the set $G(\mathbf{x}_n \bmod 1)$.

(III) The definition of d.f.s of sequences in the multi-dimensional case is different as in the one-dimensional one.

• If $\mathbf{x} = (x_1, \dots, x_s) \in \mathbb{R}^s$ is given, then $\mathbf{x} \bmod 1$ denotes the sequence $\big(\{x_1\}, \dots$ $\dots, \{x_s\}\big)$. If $\mathbf{x}_n = (x_{n,1}, \dots, x_{n,s})$ is the sequence of points in $\mathbb{R}^s$, then we define

• the s**-dimensional counting function** by

$$A\big([u_1, v_1) \times \cdots \times [u_s, v_s); N; \mathbf{x}_n \bmod 1\big) = \\ \#\big\{n \le N \{x_{n,1}\} \in [u_1, v_1), \dots, \{x_{n,s}\} \in [u_s, v_s)\big\}.$$

• the s**-dimensional step d.f.** also called the **empirical distribution** by

(i) $F_N(\mathbf{x}) = \frac{1}{N} A\big([0, x_1) \times \cdots \times [0, x_s); N; \mathbf{x}_n \bmod 1\big)$ if $\mathbf{x} \in [0,1)^s$,

(ii) $F_N(\mathbf{x}) = 0$ for every $\mathbf{x}$ having a vanishing coordinate,

(iii) $F_N(\mathbf{1}) = \mathbf{1}$,

(iv) $F_N(1, \dots, 1, x_{i_1}, 1, \dots, 1, x_{i_2}, 1, \dots, 1, x_{i_l}, 1 \dots, 1) = F_N(x_{i_1}, x_{i_2}, \dots, x_{i_l})$ for every restricted l-dimensional face sequence $(x_{n,i_1}, x_{n,i_2}, \dots, x_{n,i_l})$ of $\mathbf{x}_n$ for $l = 1, 2, \dots, s$.

Then

• If $f\colon [0,1]^s \to \mathbb{R}$ is continuous, again

$$\frac{1}{N} \sum_{n=1}^{N} f(\mathbf{x}_n \bmod 1) = \int_{[0,1]^s} f(\mathbf{x})\, \mathrm{d}F_N(\mathbf{x}).$$

• A function $g : [0,1]^s \to [0,1]$ is called a **d.f.** if

(i) $g(\mathbf{1}) = 1$,

(ii) $g(\mathbf{0}) = 0$, and moreover $g(\mathbf{x}) = 0$ for any $\mathbf{x}$ with a vanishing coordinate,

(iii) $g(\mathbf{x})$ is non-decreasing, i.e., $\Delta^{(s)}_{h_s}\big(\dots(\Delta^{(1)}_{h_1} g(x_1, \dots, x_s))\big) \ge 0$ for any $h_i \ge 0$, $x_i + h_i \le 1$, where
$\Delta^{(i)}_{h_i} g(x_1, \dots, x_s) = g(x_1, \dots, x_i + h_i, \dots, x_s) - g(x_1, \dots, x_i, \dots, x_s)$.

• If g is such d.f. then $\int_{[0,1]^2} \mathrm{d}g(\mathbf{x}) = 1$.

• If $\mathrm{d}g(\mathbf{x}) = \Delta^{(s)}_{\mathrm{d}x_s} \dots \Delta^{(1)}_{\mathrm{d}x_1} g(x_1, \dots, x_s)$ is the differential of $g(\mathbf{x})$ at the point $\mathbf{x} = (x_1, \dots, x_s)$, then also $\mathrm{d}g(\mathbf{x}) = \Delta(g, J)$, where $J = [x_1, x_1 + \mathrm{d}x_1] \times \dots$ $\dots \times [x_s, x_s + \mathrm{d}x_s]$, and $\Delta(g, J)$ is an alternating sum of the values of g at the vertices of J (function values at the adjacent vertices have opposite signs), i.e.,

$$\Delta(g, J) = \sum_{\varepsilon_1=1}^{2} \cdots \sum_{\varepsilon_k=1}^{2} (-1)^{\varepsilon_1 + \cdots + \varepsilon_k} g\Big(x^{(1)}_{\varepsilon_1}, \dots, x^{(k)}_{\varepsilon_k}\Big)$$

for an interval $J = \big[x^{(1)}_1, x^{(1)}_2\big] \times \big[x^{(2)}_1, x^{(2)}_2\big] \times \cdots \times \big[x^{(k)}_1, x^{(k)}_2\big] \subset [0,1]^k$. Moreover, $g(\mathbf{x})$ is non-decreasing if and only if $\mathrm{d}g(\mathbf{x}) \ge 0$ for every $\mathbf{x} \in [0,1]^s$ and $\mathrm{d}\mathbf{x} \ge \mathbf{0}$.

• The d.f. $g(1,\dots,1,x_{i_1},1,\dots,1,x_{i_2},1,\dots,1,x_{i_l},1\dots,1)$ is called an **l-dimensional face d.f. of g in variables** $(x_{i_1},x_{i_2},\dots,x_{i_l})\in(0,1)^l$, $0\le l\le s$.
• We shall identify two d.f.'s $g(\mathbf{x})$ and $\widetilde{g}(\mathbf{x})$ if:

(i) $g(\mathbf{x})=\widetilde{g}(\mathbf{x})$ at every common point $\mathbf{x}\in(0,1)^s$ of continuity, and

(ii) $g(1,\dots,1,x_{i_1},1,\dots,1,x_{i_2},1,\dots,1,x_{i_l},1\dots,1)=$
$=\widetilde{g}(1,\dots,1,x_{i_1},1,\dots,1,x_{i_2},1,\dots,1,x_{i_l},1\dots,1)$
at every common point $(x_{i_1},x_{i_2},\dots,x_{i_l})\in(0,1)^l$ of continuity in every l-dimensional face d.f. of g and $\widetilde{g}$, $l=1,2,\dots,s$.

• The s-dimensional d.f. $g(\mathbf{x})$ is a d.f. of the sequence $\mathbf{x}_n \bmod 1$ if

(i) $g(\mathbf{x})=\lim_{k\to\infty}F_{N_k}(\mathbf{x})$ for all continuity points $\mathbf{x}\in(0,1)^s$ of g (so-called the **weak limit**) and,

(ii) $g(1,\dots,1,x_{i_1},1,\dots,1,x_{i_2},1,\dots,1,x_{i_l},1\dots,1)=$
$=\lim_{k\to\infty}F_{N_k}(x_{i_1},x_{i_2},\dots,x_{i_l})$
weakly over $(0,1)^l$ and every l-dimensional face sequence of $\mathbf{x}_n$ for $l=1,2,\dots,s$, and for a suitable sequence of indices $N_1<N_2<\cdots$

• The Second Helly theorem shows that the weak limit[4] $F_{N_k}(\mathbf{x})\to g(\mathbf{x})$ implies

$$\int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}F_{N_k}(\mathbf{x})\to\int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}g(\mathbf{x})$$

for every continuous $f\colon[0,1]^s\to\mathbb{R}$.
• $G(\mathbf{x}_n \bmod 1)$ denotes the set of all d.f.'s of $\mathbf{x}_n \bmod 1$.

(IV) **For one-dimensional case** $s=1$ we have:

(1) The continuity of all d.f.s in $G(x_n \bmod 1)$ follows from the limit
$\lim_{K\to\infty}\frac{1}{K}\sum_{k=1}^K\beta_k=0$, where $\beta_k=\limsup_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^N e^{2\pi ikx_n}\right|^2$.

(2) The lower and upper d.f. $\underline{g}$, $\overline{g}$ of x_n belong to $G(x_n \bmod 1)$ if and only if $\int_0^1\big(\overline{g}(x)-\underline{g}(x)\big)\mathrm{d}x=\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^N\{x_n\}-\liminf_{N\to\infty}\frac{1}{N}\sum_{n=1}^N\{x_n\}$.

(3) Let H be non-empty, closed, and connected set of d.f.'s. Denote $\underline{g}_H(x)=\inf_{g\in H}g(x)$ and $\overline{g}_H(x)=\sup_{g\in H}g(x)$. Further, if $g\in H$ let Graph(g) be the continuous curve formed by all the points $\big(x,g(x)\big)$ for $x\in[0,1]$, and the all line segments connecting the points of discontinuity
$\big(x,\liminf_{x'\to x}g(x')\big)$ and $\big(x,\limsup_{x'\to x}g(x')\big)$.
Assume that for every $g\in H$ there exists a point $(x,y)\in$ Graph(g) such that $(x,y)\notin$ Graph$(\widetilde{g})$ for any $\widetilde{g}\in H$ with $\widetilde{g}\ne g$. If moreover $\underline{g}=\underline{g}_H$ and $\overline{g}=\overline{g}_H$ for the lower d.f. $\underline{g}$ and the upper d.f. $\overline{g}$ of the sequence $x_n\in[0,1)$ and $G(x_n)\subset H$, then $G(x_n)=H$.

[4] that is (i), and (ii) above are fulfilled

(4) For given two different d.f.s $g_1(x)$, and $g_2(x)$, we define
$F_{g_2}(x,y) = \int_0^x g_2(t)\,\mathrm{d}t + \int_0^y g_2(t)\,\mathrm{d}t - \max(x,y) + \int_0^1 \big(1 - g_2(t)\big)^2 \mathrm{d}t,$
$F_{g_1,g_2}(x) = \frac{\int_0^x (g_2(t)-g_1(t))\,\mathrm{d}t - \int_0^1 (1-g_2(t))(g_2(t)-g_1(t))\,\mathrm{d}t}{\int_0^1 (g_2(t)-g_1(t))^2\,\mathrm{d}t},$
$F_{g_1,g_2}(x,y) = F_{g_2}(x,y) - F_{g_1,g_2}(x)F_{g_1,g_2}(y)\int_0^1 \big(g_2(t) - g_1(t)\big)^2 \mathrm{d}t.$
Let $g_1(x) \neq g_2(x)$ be two d.f.'s. Then the set of d.f.s $G(x_n)$ of x_n in $[0,1)$ satisfies

$$G(x_n) = \big\{tg_1(x) + (1-t)g_2(x);\ t \in [0,1]\big\}$$

if and only if

(i) $\lim_{N\to\infty} \frac{1}{N^2}\sum_{m,n=1}^N F_{g_1,g_2}(x_m,x_n) = 0,$

(ii) $\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^N F_{g_1,g_2}(x_n) = 0,$

(iii) $\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^N F_{g_1,g_2}(x_n) = 1.$

(5) A symmetric continuous $F(x,y)$ defined on $[0,1]^2$ is called **copositive** if $\int_0^1\int_0^1 F(x,y)\,\mathrm{d}g(x)\,\mathrm{d}g(y) \geq 0$ for all distribution functions $g\colon [0,1] \to [0,1]$. Let $F(x,y)$ be a copositive function having continuous $F'_x(x,1)$ a.e. and let d.f. $g_1(x)$ be a strictly increasing solution of the moment problem (1). Then for every strictly increasing d.f. $g(x)$ we have
$\int_0^1\int_0^1 F(x,y)\,\mathrm{d}g(x)\,\mathrm{d}g(y) = 0 \Longleftrightarrow F'_x(x,1) = \int_0^1 g(y)\,\mathrm{d}_y F'_x(x,y)$ a.e. on $[0,1]$.

(6) Let $F(x,y)$ be a continuous, symmetric, copositive and $F''_{xy} = 0$ a.e. such that the set $H(F)$ of jumps of $F'_x(x,y)$ is covered by
$H(F) \subset \bigcup_{i,j=1}^M \big\{(x_i(t), x_j(t)); t \in [\alpha,\beta)\big\}$ with pairwise disjoint sets
$\big\{x_1(t); t \in [\alpha,\beta)\big\}, \dots, \big\{x_M(t); t \in [\alpha,\beta)\big\}.$
Assume that the derivatives $x'_i(t)$, $i = 1,2,\dots,k$, are continuous and let $\mathbf{A}(t)$ denote the associated matrix defined by
$\mathbf{A}(t) = \frac{1}{2}\big(\mathrm{d}_y F'_x(x_i(t),x_j(t))|x'_i(t)| + \mathrm{d}_y F'_x(x_j(t),x_i(t))|x'_j(t)|\big)$ and
$\mathbf{g}(t) = \big(g(x_1(t)), g(x_2(t)), \dots, g(x_M(t))\big)$
is the vector associated with $g\colon [0,1] \to [0,1]$. Finally, let g_1 be a strictly increasing solution of the moment problem (1). Then we have

$$\int_0^1\!\!\int_0^1 F(x,y)\,\mathrm{d}g(x)\,\mathrm{d}g(y) = \int_\alpha^\beta \big(\mathbf{g}(t) - \mathbf{g}_1(t)\big)\mathbf{A}(t)\big(\mathbf{g}(t) - \mathbf{g}_1(t)\big)^T \mathrm{d}t$$

for all distribution functions $g\colon [0,1] \to [0,1]$.

(7) Directly by definition $G(x_n)$ we showed: Assume
- $f(x)$ be a real-valued function defined for $x \geq 1$ such that $f(x)$ is strictly increasing with its inverse function $f^{-1}(x)$.
- $\lim_{k\to\infty} \frac{f^{-1}(k+x) - f^{-1}(k)}{f^{-1}(k+1) - f^{-1}(k)} = \tilde{g}(x)$ for each $x \in [0,1]$, point of continuity of $\tilde{g}(x)$;

- $\lim_{k\to\infty}\frac{f^{-1}(k+u)}{f^{-1}(k)}=\psi(u)$ for each $u\in[0,1]$, point of continuity of $\psi(u)$, or $\psi(u)=\infty$ for $u>0$;
- $\lim_{k\to\infty} f^{-1}(k+1)-f^{-1}(k)=\infty$. Then we have: If $1<\psi(1)<\infty$ and $f'(x)\to 0$ as $x\to\infty$, then
  $G\big(f(n) \bmod 1\big)=\big\{g_u(x)=\frac{\min(\psi(x),\psi(u))-1}{\psi(u)}+\frac{1}{\psi(u)}\tilde{g}(x); u\in[0,1]\big\}$,
  where $\tilde{g}(x)=\frac{\psi(x)-1}{\psi(1)-1}$ and $F_{N_i}(x)\to g_u(x)$ as $i\to\infty$ if and only if $f(N_i) \bmod 1 \to u$. The lower d.f. $\underline{g}(x)$ and the upper d.f. $\overline{g}(x)$ of $f(n) \bmod 1$ are $\underline{g}(x)=\tilde{g}(x)$, $\overline{g}(x)=1-\frac{1}{\psi(x)}\big(1-\tilde{g}(x)\big)$. Furthermore $\underline{g}(x)=g_0(x)=g_1(x)$ belongs to $G\big(f(n) \bmod 1\big)$ but $\overline{g}(x)=g_x(x)$ does not.

(8) Let x_n and y_n be two sequences in $[0,1)$ and $G((x_n,y_n))$ denote the set of all d.f.s of the two-dimensional sequence (x_n,y_n). If $z_n=x_n+y_n \bmod 1$, then the set $G(z_n)$ of all d.f.s of z_n has the form

$$G(z_n)=\left\{g(t)=\int_{0\le x+y<t}1.\mathrm{d}g(x,y)+\int_{1\le x+y<1+t}1.\mathrm{d}g(x,y); g(x,y)\in G((x_n,y_n))\right\}$$

assuming that all the used Riemann-Stieltjes integrals exist.

NOTES.

(I) A purely topological characterization of $G(\mathbf{x}_n)$ with a short history can be found in R. Winkler (1997).

(II) O. Strauch (1994).

(III) For definitions, cf. [SP, 1.11, p. 1–60].

(IV) (1) is a generalization of the Wiener–Schoenberg theorem given by P. Kostyrko, M. Mačaj, T. Šalát and O. Strauch (2001). (2), (3) and (4) are from O. Strauch (1997). (5) and (6) are proved in O. Strauch (2000), (7) in R. Giuliano Antonini and O. Strauch (2008) and (8) in O. Strauch and O. Blažeková (2006).

Proposed by O. Strauch.

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2.3. Deterministic analysis of sequences

Assume that the s-dimensional sequence

$$\mathbf{x}_n = (x_{n,1}, \dots, x_{n,s}) \in [0,1)^s, \qquad n = 1, 2, \dots, N$$

is a result of an N often repeated measurement of s physical variables $X_1, \dots, X_s$. If $g(x_1, \dots, x_s)$ is an a.d.f. of $\mathbf{x}_n$, $n = 1, 2, \dots$ (assuming in the moment that $N \to \infty$), then $g(x_1, \dots, x_s)$ contains some informations about relations between variables $X_1, \dots, X_s$. For example

(i) $X_1, \dots, X_s$ are independent if and only if every d.f. $g(\mathbf{x}) \in G(\mathbf{x}_n)$ can be written as a product $g(\mathbf{x}) = g_1(x_1) \dots g_s(x_s)$ of one-dimensional d.f.s. Here g_i, $i = 1, \dots, s$ depend on $g \in G(\mathbf{x}_n)$.

(ii) If X_s depends on $X_1, \dots, X_{s-1}$, and $I_1, \dots, I_s$ are subintervals in $[0,1)$, then the implication

$$(X_1 \in I_1 \wedge \dots \wedge X_{s-1} \in I_{s-1}) \Longrightarrow (X_s \in I_s),$$

can be evaluated by $\int_{I_1 \times \dots \times I_s} h(\mathbf{x})\,\mathrm{d}\mathbf{x}$, where $h(\mathbf{x})$ is the density of $g(\mathbf{x})$ (if it exists).

The studying of $\mathbf{x}_n$, $n = 1, 2, \dots$ via $G(\mathbf{x}_n)$ we shall call **deterministic analysis** of the sequence $\mathbf{x}_n$, since for approximate computation of $g(\mathbf{x}) \in G(\mathbf{x}_n)$ can be used discrepancies of $\mathbf{x}_n$, $n = 1, 2, \dots, N$. We do not use probabilities and statistical methods.

For approximate computation of $G(\mathbf{x}_n)$ we need there solve the following problem.

Open problem: For a big dimension s there exists no real employing N for a good approximation of a d.f. $g(x_1, \dots, x_s)$ of $\mathbf{x}_n$ by the step d.f.

$$F_N(x_1, \dots, x_s) = \frac{\#\{n \le N;\ x_{n,1} < x_1, \dots, x_{n,s} < x_s\}}{N}.$$

But using partial sequences

$$(x_{n,i_1},\dots,x_{n,i_k}), \qquad n=1,2,\dots,N$$

with small dimension k it can be found N such that the corresponding step d.f. well approximates the marginal d.f. $g(1,\dots,1,x_{i_1},1,\dots,1,x_{i_2},1,\dots)$. Problem is to reconstruct $g(x_1,\dots,x_s)$ by using marginals

$$g(1,\dots,1,x_{i_1},1,\dots,1,x_{i_2},1,\dots)$$

with small dimensions.

In the following we shall formulate above problem more elementary.

Open problem 1. Let $\mathbf{x}_n=(x_{n,1},x_{n,2},\dots,x_{n,s})$, $n=1,2,\dots$, be an infinite s-dimensional sequence in the unit cube $[0,1)^s$. Assume that, for fixed $k<s$, all k-dimensional marginal sequences $(x_{n,i_1},\dots,x_{n,i_k})$ are u.d.

(I) Find all possible d.f.s of $\mathbf{x}_n$.

(II) Find some (possible "minimal") criterions which imply u.d. of the original sequence $\mathbf{x}_n$, $n=1,2,\dots$

• In connection with (I) we denote by $G_{s,k}$ the set of all d.f.s $g(\mathbf{x})$ on $[0,1]^s$ for which all k-dimensional marginals (i.e., faces) of d.f.'s satisfy

$$g(1,\dots,1,x_{i_1},1,\dots,1,x_{i_2},1,\dots,1,x_{i_k},1,\dots,1)=x_{i_1}x_{i_2}\dots x_{i_k}.$$

• For $k=1$, these d.f.'s are called *copulas*, which were introduced by M. Sklar (1959). All basic properties of copulas can be found in the monograph R. B. Nelsen (1999).

• Thus, by definition $G_{s,k}$, the $G_{2,1}$ is the set of all two-dimensional d.f.s $g(x,y)$ defined on $[0,1]^2$ such that their marginals d.f.'s satisfy $g(x,1)=x$ and $g(1,y)=y$.

$G_{2,1}$ contains:

- $g_1(x,y)=xy$,
- $g_2(x,y)=\min(x,y)$,
- $g_3(x,y)=\max(x+y-1,0)$,
- $g_\theta(x,y)=\big(\min(x,y)\big)^\theta(xy)^{1-\theta}$, where $\theta\in[0,1]$ (Cuadras-Augé family, cf. R. B. Nelsen [1999, p. 12, Ex. 2.5],
- $g_4(x,y)=\frac{xy}{x+y-xy}$ (see R. B. Nelsen [1999, p. 19, 2.3.4],
- $\tilde{g}(x,y)=x+y-1+g(1-x,1-y)$ for every $g(x,y)\in G_{2,1}$ (Survival copula, see R. B. Nelsen [1999, p. 28, 2.6.1],
- $g_5(x,y)=\min\big(ya(x),x(b(y)\big)$, where $a(0)=b(0)=0$, $a(1)=b(1)=1$ and $a(x)/x$, $b(y)/y$ are both decreasing on $(0,1]$ (Marshall copula, cf. R. B. Nelsen [1999, p. 51, Exerc. 3.3].

Here are some new copulas:

– $g_6(x,y) = \frac{1}{z_0}\min(xy, xz_0, yz_0)$ for fixed z_0, $0 < z_0 \le 1$.
– $g_7(x,y) = \frac{1}{z_0u_0}\min(xyz_0, xyu_0, xz_0u_0, yz_0u_0)$ for fixed $z_0, u_0 \in [0,1]^2$.
– Shuffle of M is a copula defined in R. B. Nelsen [1999, p. 59, 3.2.3.], cf. the Problem 1.37.
– Generalized shuffle of M: Let $f : [0,1] \to [0,1]$ be an arbitrary uniform distribution preserving function (called u.d.p., see Problem 2.1 (VII)) and graph $f = \{(x, f(x)); x \in [0,1]\}$. Then the generalized shuffle of M is the copula

$$g(x,y) = \left|\text{Project}_x\big(\text{graph } f \cap [0,x) \times [0,y)\big)\right|.$$

There are some basic properties of $G_{2,1}$:

– $G_{2,1}$ is closed under point-wise limit and convex linear combinations.
– For every $g(x,y) \in G_{2,1}$ and every $(x_1,y_1),(x_2,y_2) \in [0,1]^2$ we have $|g(x_2,y_2) - g(x_1,y_1)| \le |x_2 - x_1| + |y_2 - y_1|$.
– For every $g(x,y) \in G_{2,1}$ we have $g_3(x,y) = \max(x+y-1,0) \le g(x,y) \le \min(x,y) = g_2(x,y)$ (Fréchet–Hoeffding bounds, see R. B. Nelsen [1999, p. 9].
– M. Sklar (1959) proved that for every d.f. $g(x,y)$ on $[0,1]^2$ there exists $\tilde{g}(x,y) \in G_{2,1}$ such that

$$g(x,y) = \tilde{g}\big(g(x,1), g(1,y)\big) \qquad \text{for every} \quad (x,y) \in [0,1]^2.$$

If $g(x,1)$ and $g(1,y)$ are continuous, then $\tilde{g}(x,y)$ is unique (cf. R. B. Nelsen [1999, p. 15, Th. 2.3.3].

• Let (x_n, y_n), $n = 1,2,\dots$, be a sequence in $[0,1)^2$ such that both coordinate sequences x_n, $n = 1,2,\dots$, and y_n, $n = 1,2,\dots$ are u.d. Then the set $G((x_n,y_n))$ of all d.f. of (x_n,y_n), $n = 1,2,\dots$ satisfies

– $G((x_n,y_n)) \subset G_{2,1}$,
– $G((x_n,y_n))$ is nonempty, closed and connected, and vice-versa
– for every nonempty, closed and connected $H \subset G_{2,1}$, there exists a sequence $(x_n,y_n) \in [0,1)^2$ such that $G((x_n,y_n)) = H$.
– Let $F(x,y,u,v,)$ be a continuous function defined on $[0,1]^4$ and assume that

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{m,n=1}^{N} F(x_m, y_m, x_n, y_n) = 0.$$

Then every d.f. $g(x,y)\in G((x_n,y_n))$ satisfies the following equation:

$$\int_0^1\int_0^1 g(u,v)\,\mathrm{d}_u\,\mathrm{d}_vF(1,1,u,v)+\int_0^1\int_0^1 g(x,y)\,\mathrm{d}_x\,\mathrm{d}_yF(x,y,1,1)$$

$$-\int_0^1\int_0^1\int_0^1 g(u,v)y\,\mathrm{d}_y\,\mathrm{d}_u\,\mathrm{d}_vF(1,y,u,v)-\int_0^1\int_0^1\int_0^1 g(u,v)x\,\mathrm{d}_x\,\mathrm{d}_u\,\mathrm{d}_vF(x,1,u,v)$$

$$-\int_0^1\int_0^1\int_0^1 g(x,y)v\,\mathrm{d}_v\,\mathrm{d}_x\,\mathrm{d}_yF(x,y,1,v)-\int_0^1\int_0^1\int_0^1 g(x,y)u\,\mathrm{d}_u\,\mathrm{d}_x\,\mathrm{d}_yF(x,y,u,1)$$

$$+\int_0^1\int_0^1\int_0^1\int_0^1 g(x,y)g(u,v)\,\mathrm{d}_u\,\mathrm{d}_v\,\mathrm{d}_x\,\mathrm{d}_yF(x,y,u,v)$$

$$=-F(1,1,1,1)+\int_0^1 v\,\mathrm{d}_vF(1,1,1,v)+\int_0^1 u\,\mathrm{d}_uF(1,1,u,1)$$

$$+\int_0^1 x\,\mathrm{d}_xF(x,1,1,1)+\int_0^1 y\,\mathrm{d}_yF(1,y,1,1)$$

$$-\int_0^1\int_0^1 yv\,\mathrm{d}_y\,\mathrm{d}_vF(1,y,1,v)-\int_0^1\int_0^1 yu\,\mathrm{d}_y\,\mathrm{d}_uF(1,y,u,1)$$

$$-\int_0^1\int_0^1 xv\,\mathrm{d}_x\,\mathrm{d}_vF(x,1,1,v)-\int_0^1\int_0^1 xu\,\mathrm{d}_x\,dd_uF(x,1,u,1).$$

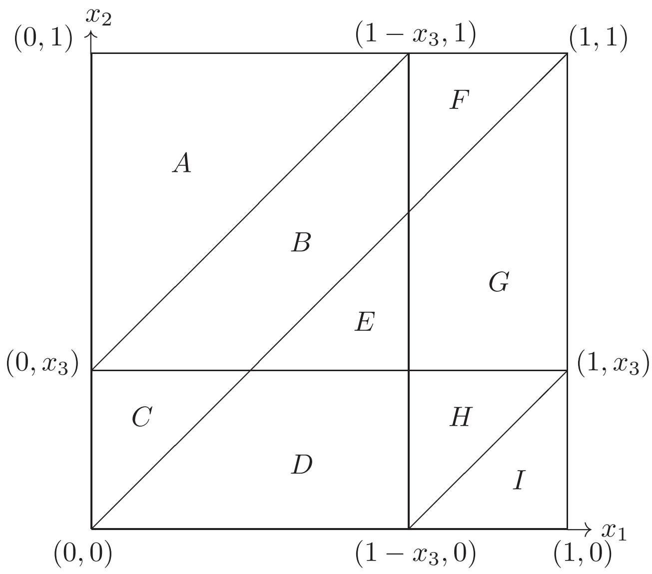
• By definition of $G_{s,k}$, the $G_{3,2}$ is the set of all three-dimensional d.f.s $g(x,y,z)$ defined on $[0,1]^3$ such that their two-dimensional marginals (or faces) d.f.'s satisfy $g(x,y,1)=xy$, $g(1,y,z)=yz$ and $g(x,1,z)=xz$.

The $G_{3,2}$ contains

- $g_1(x,y,z)=xyz$,
- $g_2(x,y,z)=\min(xy,xz,yz)$,
- $g_3(x,y,z)=\frac{1}{u_0}\min(xyz,xyu_0,xzu_0,yzu_0)$, for fixed u_0, $0<u_0\le 1$,
- $g_4(x,y,z)$ is a.d.f. of a three-dimensional sequence $\big(u_n,v_n,\{u_n-v_n\}\big)$, where two-dimensional (u_n,v_n) is u.d. in $[0,1]^2$. Applying Weyl's criterion we see that also $\big(u_n,\{u_n-v_n\}\big)$ and $\big(v_n,\{u_n-v_n\}\big)$ are u.d. and the d.f. $g_4(x,y,z)$ has the following explicit form (cf. O. Strauch (2003).)

$$g_4(x_1,x_2,x_3)=\begin{cases} x_1x_2 & \text{if } (x_1,x_2)\in A,\\ -\frac12(x_1^2+x_2^2+x_3^2)+x_1x_2+x_2x_3 & \text{if } (x_1,x_2)\in B,\\ -\frac12 x_1^2+x_1x_2 & \text{if } (x_1,x_2)\in C,\\ \frac12 x_2^2 & \text{if } (x_1,x_2)\in D,\\ -\frac12 x_3^2+x_2x_3 & \text{if } (x_1,x_2)\in E,\\ -\frac12 x_2^2+x_1x_2+x_1x_3+x_2x_3-x_1-x_3+\frac12 & \text{if } (x_1,x_2)\in F,\\ \frac12 x_1^2+x_1x_3+x_2x_3-x_1-x_3+\frac12 & \text{if } (x_1,x_2)\in G,\\ \frac12(x_1^2+x_2^2+x_3^2)+x_1x_3-x_1-x_3+\frac12 & \text{if } (x_1,x_2)\in H,\\ x_1x_2+x_2x_3-x_2 & \text{if } (x_1,x_2)\in I. \end{cases}$$

where the regions A,B,C,D,E,F,G,H,I are shown on the following figure



– For every $g(x,y,z)\in G_{3,2}$ and fixed z_0, $0<z_0\le 1$ we have $\frac{1}{z_0}g(x,y,z_0)\in G_{2,1}$. Vice versa, if $g_z(x,y)$, $z\in[0,1]$ is a system of d.f.s in $G_{2,1}$ such that $g_1(x,y)=xy$ and for every $z'\le z$, we have $z'\mathrm{d}_x\mathrm{d}_y g_{z'}(x,y)\le z\mathrm{d}_x\mathrm{d}_y g_z(x,y)$ on $[0,1]^2$, then $g(x,y,z)=zg_z(x,y)\in G_{3,2}$.

– Multi-dimensional case: Let $g(x_1,x_2,\dots,x_s)$ be an s-dimensional d.f. and let

$$g_1(x_1)=g(x_1,1,\dots,1),\quad g_2(x_2)=g(1,x_2,1,\dots,1),\dots$$

be margins of $g(x_1,x_2,\dots,x_s)$. By Sklar's theorem there exists s-dimensional copula $c(x_1,x_2,\dots,x_s)$ such that

$$g(x_1,x_2,\dots,x_s)=c\big(g_1(x_1),g_2(x_2),\dots,g_s(x_s)\big).$$

Furthermore, for arbitrary continuous $F(x_1,x_2,\dots,x_s)$ we have

$$\int_{[0,1]^s} F(x_1,x_2,\dots,x_s)\,\mathrm{d}g(x_1,x_2,\dots,x_s) = \int_{[0,1]^s} F\big(g_1^{-1}(x_1),g_2^{-1}(x_2),\dots,g_s^{-1}(x_s)\big)\,\mathrm{d}c(x_1,x_2,\dots,x_s).$$

• In the direction (II) of Open problem for testing of u.d. of $\mathbf{x}_n$ it can be used statistical independence of marginal sequences, but formulas for L^2 discrepancy of statistical independence and classical L^2 discrepancy have the following similar structures: For $\mathbf{x}=(x_1,\dots,x_s)$ and $\mathbf{y}=(y_1,\dots,y_s)$ denote $\big(\mathbf{1}-\max(\mathbf{x},\mathbf{y})\big) = \big(1-\max(x_1,y_1)\big)\dots\big(1-\max(x_s,y_s)\big)$, $\mathbf{0}=(0,\dots,0)$ and $\mathbf{1}=(1,\dots,1)$. For every two d.f.'s $g_1(\mathbf{x})$ and $g_2(\mathbf{x})$ defined in $[0,1]^s$ we have (see O. Strauch (2003))

$$\int_{\mathbf{0}}^{\mathbf{1}} \big(g_1(\mathbf{x})-g_2(\mathbf{x})\big)^2 \mathrm{d}\mathbf{x} = \int_{\mathbf{0}}^{\mathbf{1}}\int_{\mathbf{0}}^{\mathbf{1}} \big(\mathbf{1}-\max(\mathbf{x},\mathbf{y})\big)\mathrm{d}\big(g_1(\mathbf{x})-g_2(\mathbf{x})\big)\mathrm{d}\big(g_1(\mathbf{y})-g_2(\mathbf{y})\big) \tag{1}$$

Now, divide the vector $\mathbf{x}=(x_1,\dots,x_s)$ into two face vectors $\mathbf{x}^{(1)}=(x_{i_1},\dots,x_{i_l})$ and $\mathbf{x}^{(2)}=(x_{j_1},\dots,x_{j_k})$, $l+k=s$. Similarly, divide the s-dimensional sequence $\mathbf{x}_n$, $n=1,2,\dots$ in $[0,1)^s$ with step d.f. $F_N(\mathbf{x})=F_N\big(\mathbf{x}^{(1)},\mathbf{x}^{(2)}\big)$ into two face sequences

l-dimensional $\mathbf{x}_n^{(1)}$, $n=1,2,\dots$, with step d.f. $F_N\big(\mathbf{x}^{(1)},\mathbf{1}\big)$, and

k-dimensional $\mathbf{x}_n^{(2)}$, $n=1,2,\dots$, with step d.f. $F_N\big(\mathbf{1},\mathbf{x}^{(2)}\big)$.

Using (1) we see that the L^2 discrepancy (with respect to $g(\mathbf{x})$) and statistical L^2 discrepancy have the following similar structures

$$\int_{\mathbf{0}}^{\mathbf{1}} \big(F_N(\mathbf{x})-g(\mathbf{x})\big)^2 \mathrm{d}\mathbf{x} = \int_{\mathbf{0}}^{\mathbf{1}}\int_{\mathbf{0}}^{\mathbf{1}} \big(\mathbf{1}-\max(\mathbf{x},\mathbf{y})\big)\cdot\mathrm{d}\,\big(F_N(\mathbf{x})-g(\mathbf{x})\big)\mathrm{d}\,\big(F_N(\mathbf{y})-g(\mathbf{y})\big),$$

$$\begin{aligned}
&\int_{\mathbf{0}}^{\mathbf{1}} \Big(F_N(\mathbf{x}) - F_N\big(\mathbf{x}^{(1)},\mathbf{1}\big)F_N\big(\mathbf{1},\mathbf{x}^{(2)}\big)\Big)^2 \mathrm{d}\mathbf{x} \\
&= \int_{\mathbf{0}}^{\mathbf{1}}\int_{\mathbf{0}}^{\mathbf{1}}\int_{\mathbf{0}}^{\mathbf{1}}\int_{\mathbf{0}}^{\mathbf{1}} \Big(\mathbf{1}-\max\big(\mathbf{x}^{(1)},\mathbf{y}^{(1)}\big)\Big)\Big(\mathbf{1}-\max\big(\mathbf{x}^{(2)},\mathbf{y}^{(2)}\big)\Big) \\
&\quad\cdot\mathrm{d}\Big(F_N(\mathbf{x}^{(1)},\mathbf{x}^{(2)}) - F_N(\mathbf{x}^{(1)},\mathbf{1},)F_N\big(\mathbf{1},\mathbf{x}^{(2)}\big)\Big) \\
&\quad\cdot\mathrm{d}\Big(F_N\big(\mathbf{y}^{(1)},\mathbf{y}^{(2)}\big) - F_N\big(\mathbf{y}^{(1)},\mathbf{1},\big)F_N\big(\mathbf{1},\mathbf{y}^{(2)}\big)\Big).
\end{aligned}$$

Expressing L^2 discrepancy as

$$
\begin{aligned}
&\int_{\mathbf{0}}^{\mathbf{1}} \big(F_N(\mathbf{x}) - g(\mathbf{x})\big)^2 \mathrm{d}\mathbf{x} \\
&= \int_{\mathbf{0}}^{\mathbf{1}}\int_{\mathbf{0}}^{\mathbf{1}} \Big[\Big(F_N\big(\mathbf{x}^{(1)},\mathbf{x}^{(2)}\big) - F_N\big(\mathbf{x}^{(1)},\mathbf{1}\big)F_N\big(\mathbf{1},\mathbf{x}^{(2)}\big)\Big) \\
&\quad + \Big(F_N\big(\mathbf{x}^{(1)},\mathbf{1}\big) - g\big(\mathbf{x}^{(1)},\mathbf{1}\big)\Big)F_N\big(\mathbf{1},\mathbf{x}^{(2)}\big) \\
&\quad + g\big(\mathbf{x}^{(1)},\mathbf{1}\big)\Big(F_N\big(\mathbf{1},\mathbf{x}^{(2)}\big) - g\big(\mathbf{1},\mathbf{x}^{(2)}\big)\Big) \\
&\quad + \Big(g\big(\mathbf{x}^{(1)},\mathbf{1}\big)g\big(\mathbf{1},\mathbf{x}^{(2)}\big) - g\big(\mathbf{x}^{(1)},\mathbf{x}^{(2)}\big)\Big)\Big]^2 \mathrm{d}\mathbf{x}^{(1)}\mathrm{d}\mathbf{x}^{(2)}
\end{aligned}
$$

the Cauchy inequality implies

$$
\begin{aligned}
&\sqrt{\int_{\mathbf{0}}^{\mathbf{1}} \big(F_N(\mathbf{x}) - g(\mathbf{x})\big)^2 \mathrm{d}\mathbf{x}} \\
&\le \sqrt{\int_{\mathbf{0}}^{\mathbf{1}}\int_{\mathbf{0}}^{\mathbf{1}} \Big(F_N\big(\mathbf{x}^{(1)},\mathbf{x}^{(2)}\big) - F_N\big(\mathbf{x}^{(1)},\mathbf{1}\big)F_N\big(\mathbf{1},\mathbf{x}^{(2)}\big)\Big)^2 \mathrm{d}\mathbf{x}^{(1)}\,\mathrm{d}\mathbf{x}^{(2)}} \\
&+ \sqrt{\int_{\mathbf{0}}^{\mathbf{1}}\int_{\mathbf{0}}^{\mathbf{1}} \Big(g\big(\mathbf{x}^{(1)},\mathbf{x}^{(2)}\big) - g\big(\mathbf{x}^{(1)},\mathbf{1}\big)g\big(\mathbf{1},\mathbf{x}^{(2)}\big)\Big)^2 \mathrm{d}\mathbf{x}^{(1)}\,\mathrm{d}\mathbf{x}^{(2)}} \\
&+ \sqrt{\int_{\mathbf{0}}^{\mathbf{1}} \Big(F_N\big(\mathbf{x}^{(1)},\mathbf{1}\big) - g\big(\mathbf{x}^{(1)},\mathbf{1}\big)\Big)^2 \mathrm{d}\mathbf{x}^{(1)} \int_{\mathbf{0}}^{\mathbf{1}} F_N^2\big(\mathbf{1},\mathbf{x}^{(2)}\big)\,\mathrm{d}\mathbf{x}^{(2)}} \\
&+ \sqrt{\int_{\mathbf{0}}^{\mathbf{1}} \Big(F_N\big(\mathbf{1},\mathbf{x}^{(2)}\big) - g\big(\mathbf{1},\mathbf{x}^{(2)}\big)\Big)^2 \mathrm{d}\mathbf{x}^{(2)} \int_{\mathbf{0}}^{\mathbf{1}} g^2\big(\mathbf{x}^{(1)},\mathbf{1}\big)\,\mathrm{d}\mathbf{x}^{(1)}}.
\end{aligned}
$$

Thus we have an upper bound of the classical L^2 discrepancy of $\mathbf{x}_1,\dots,\mathbf{x}_N$ which contains the L^2 discrepancy of statistical independence of partial sequences $\mathbf{x}_1^{(1)},\dots,\mathbf{x}_N^{(1)}$ and $\mathbf{x}_1^{(2)},\dots,\mathbf{x}_N^{(2)}$. Note that the infinite partial sequences $\mathbf{x}_n^{(1)}$, $n=1,2,\dots$, and $\mathbf{x}_n^{(2)}$, $n=1,2,\dots$ of the sequence $\mathbf{x}_n$, $n=1,2,\dots$ are statistically independent if and only if for every d.f. $g(\mathbf{x})\in G(\mathbf{x}_n)$ we have $g(\mathbf{x}) = g\big(\mathbf{x}^{(1)},\mathbf{1}\big)\cdot g\big(\mathbf{1},\mathbf{x}^{(2)}\big)$ in common points of continuity od d.f.s. It can be used as a definition of independence.

Proposed by O. Strauch.

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2.4. Exponential sequences

The theory of the sequences $\lambda\theta^n \bmod 1$, $n = 1, 2, \dots$, $\theta > 1$ is not satisfactory. Characterization of distribution of such sequences is a well-known and largely **unsolved problem**, see [SP, p. 2–149]. In the following we listed some conjectures and some positive results.

NOTES.

(1) J. F. Koksma (1935) proved that the sequence $\lambda\theta^n \bmod 1$ with $\lambda \neq 0$ fixed is u.d. for almost all real $\theta > 1$. If we take $\lambda = 1$ then we get that the sequence $\theta^n \bmod 1$ is u.d. for almost all real numbers $\theta > 1$. However, no explicit example of a real number θ is known for which this sequence is u.d.

(2) If $\theta > 1$ is fixed then H. Weyl (1916) proved that the sequence $\lambda\theta^n \bmod 1$ is u.d. for almost all real λ.

(3) A. D. Pollington (1983) proved that the Hausdorff dimension of the set of all $\lambda \in \mathbb{R}$ for which the sequence $\lambda\theta^n \bmod 1$ is nowhere dense is $\geq \frac{1}{2}$.

(4) $(3/2)^n \bmod 1$ is u.d. in $[0, 1]$ (**conjecture**).

(5) $(3/2)^n \bmod 1$ is dense in $[0, 1]$ (**conjecture**).

(6) $\limsup_{n\to\infty}\{(3/2)^n\} - \liminf_{n\to\infty}\{(3/2)^n\} > 1/2$ (T. Vijayaraghavan's (1940) **conjecture**).

(7) K. Mahler's (1968) **conjecture**: There is no $0 \neq \xi \in \mathbb{R}$ such that $0 \leq \{\xi(3/2)^n\} < 1/2$ for all $n = 0, 1, 2, \dots$. Such ξ does not exists if for each $\xi > 0$ the sequence of integer parts $[\xi(3/2)^n]$, $n = 1, 2, \dots$, contains infinitely many odd numbers.

(8) There is no $0 \neq \xi \in \mathbb{R}$ such that the closure of $\{\{\xi(3/2)^n\}\,;\, n = 0, 1, 2, \dots\}$ is nowhere dense in $[0, 1]$ (**conjecture**).

(9) L. Flatto, J. C. Lagarias and A. D. Pollington (1995) showed that for every $\xi > 0$ we have $\limsup_{n\to\infty}\{\xi(3/2)^n\} - \liminf_{n\to\infty}\{\xi(3/2)^n\} \geq 1/3$.

(10) G. Choquet (1980) proved the existence of infinitely many $\xi \in \mathbb{R}$ for which $1/19 \leq \{\xi(3/2)^n\} \leq 18/19$ for $n = 0, 1, 2, \dots$. Him is ascribed the conjecture (v).

(11) A. Dubickas (2006[a]) proved that for any $\xi \neq 0$ the sequence of fractional part $\{\xi(3/2)^n\}$, $n = 1, 2, \dots$, contains at least one limit point in the

interval $[0.238117\dots, 0.761882\dots]$ of length $0.523764\dots$. This immediately follows from:

(12) A. Dubickas (2006[a]): Set $T(x) = \prod_{n=0}^{\infty}\left(1 - x^{2^n}\right)$. If $\xi \neq 0$ then the sequence $||\xi(3/2)^n||$, $n = 1, 2, \dots$, has a limit point $\geq \left(3 - T(2/3)\right)/12 = 0.238117\dots$ and a limit point $\leq \left(1 + T(2/3)\right)/4 = 0.285647\dots$

(12') A. Dubickas (2007) from (22') derived: $\left\{\xi(-3/2)^n\right\}$ has a limit point ≤ 0.533547 and a limit point ≥ 0.466452.

(13) S. Akiyama, C. Frougny and J. Sakarovitch (2006): There is $\xi \neq 0$ such that $||\xi(3/2)^n|| < 1/3$ for $n = 1, 2, \dots$

(14) A. Pollington: There is $\xi \neq 0$ such that $||\xi(3/2)^n|| > 4/65$ for $n = 1, 2, \dots$

(15) R. Tijdeman (1972) showed that for every pair of integers k, m with $k \geq 2$ and $m \geq 1$ there exists $\xi \in [m, m+1)$ such that $0 \leq \left\{\xi((2k+1)/2)^n\right\} \leq 1/(2k-1)$ for $n = 0, 1, 2, \dots$

(16) O. Strauch (1997) proved that every d.f. $g(x)$ of $\xi(3/2)^n \bmod 1$ satisfies the functional equation

$$g(x/2)+g\big((x+1)/2\big)-g(1/2) = g(x/3)+g\big((x+1)/3\big)+g\big((x+2)/3\big)-g(1/3)-g(2/3).$$

A non-trivial solution (cf. O. Strauch (1999, p. 126)) is

$$g(x) = \begin{cases} 0 & \text{if } x \in [0, 1/6], \\ 2x - 1/3 & \text{if } x \in [1/6, 3/12], \\ 4x - 5/6 & \text{if } x \in [3/12, 5/18], \\ 2x - 5/18 & \text{if } x \in [5/18, 2/6], \\ 7/18 & \text{if } x \in [2/6, 8/18], \\ x - 1/18 & \text{if } x \in [8/18, 3/6], \\ 8/18 & \text{if } x \in [3/6, 7/9], \\ 2x - 20/18 & \text{if } x \in [7/9, 5/6], \\ 4x - 50/18 & \text{if } x \in [5/6, 11/12], \\ 2x - 17/18 & \text{if } x \in [11/12, 17/18], \\ x & \text{if } x \in [17/18, 1]. \end{cases}$$

(17) O. Strauch (1997) introduced: The set $X \subset [0, 1]$ is said to be a set of uniqueness of d.f.s of $\xi(3/2)^n \bmod 1$, if for every two d.f.s $g_1(x)$, $g_2(x)$ of $\xi(3/2)^n \bmod 1$ with $g_1(x) = g_2(x)$ for $x \in X$ then $g_1(x) = g_2(x)$ for every $x \in [0, 1]$. He gives the following sets of uniqueness: $X = [0, 2/3]$, $X = [1/3, 1]$, $X = [0, 1/3] \cup [2/3, 1]$, $X = [2/9, 1/3] \cup [1/2, 1]$ or $X = [0, 1/2] \cup [2/3, 7/9]$.

(18) The elements of the sequence $(3/2)^n$ appear in the Waring problem. Let

$$g(k) = \min\left\{s; a = n_1^k + \cdots + n_s^k \text{ for all } a \in \mathbb{N} \text{ and suitable } n_i \in \mathbb{N}_0\right\}.$$

S. Pillai (1936) proved that if $k \geq 5$ and if we write $3^k = q2^k + r$ with $0 < r < 2^k$, then $g(k) = 2^k + \left[(3/2)^k\right] - 2$, provided that $r + q < 2^k$, i.e., $3^k - 2^k\left[(3/2)^k\right] < 2^k - \left[(3/2)^k\right]$.

(19) **Open problem** is to characterize distribution of $e^n \bmod 1$ and $\pi^n \bmod 1$.

(20) If $p > q > 1$ are integers and $\gcd(p, q) = 1$ then the sequence $(p/q)^n \bmod 1$, $n = 1, 2, \dots$, has an infinite number of points of accumulation. This was firstly proved by Ch. Pisot (1938), then by T. Vijayaraghavan (1940) and L. Rédei (1942). The density of $(p/q)^n \bmod 1$ in $[0, 1]$ is an **open problem** posed by Ch. Pisot and T. Vijayaraghavan.

(21) L. Flatto, J.C. Lagarias and A.D. Pollington (1995) proved that if $\xi > 0$, then $\limsup_{n\to\infty}\left\{\xi(p/q)^n\right\} - \liminf_{n\to\infty}\left\{\xi(p/q)^n\right\} \geq 1/p$.

(22) A. Dubickas (2006[a]): Denote $T(x) = \prod_{n=0}^{\infty}(1 - x^{2^n})$, $E(x) = \frac{1-(1-x)T(x)}{2x}$. If $\xi \neq 0$ and $p > q > 1$, $\gcd(p, q) = 1$, then the sequence $||\xi(p/q)^n||$, $n = 1, 2, \dots$ has a limit point $\geq E(q/p)/p$ and a limit point $\leq 1/2 - \big(1 - e(q/p)\big)T(q/p)/2q$, where $e(q/p) = 1 - (q/p)$ if $p + q$ is even and $e(q/p) = 1$ if $p + q$ is odd.

(22') A. Dubickas (2007): Set $F(x) = \prod_{k=1}^{\infty}\big(1 - x^{(2^k+(-1)^{k-1})/3}\big)$. For two coprime positive integers $p > q > 1$ and any real number $\xi \neq 0$, the sequence of fractional part $\left\{\xi(-p/q)^n\right\}$, $n = 0, 1, 2, \dots$, has a limit point $\leq 1 - \big(1 - F(q/p)\big)/q$ and a limit point $\geq \big(1 - F(q/p)\big)/q$.

(22") A. Dubickas (2006[a]): Set

$$T(x) = \prod_{n=0}^{\infty}\big(1 - x^{2^n}\big) \quad \text{and} \quad E(x) = \tfrac{1-(1-x)T(x)}{2x}.$$

Let ξ be an irrational number and let $p > 1$ be an integer. Then the sequence $||\xi p^n||$, $n = 1, 2, \dots$ has a limit point $\geq \xi_p = E(1/p)/p$, and a limit point $\leq \hat{\xi}_p = e(1/p))T(1/p)/2$, where $e(1/p) = 1 - (1/p)$ if p is odd and $e(1/p) = 1$ if p is even. Furthermore, both bounds are best possible: in particular, ξ_p, $\hat{\xi}_p$ are irrational and $||\xi_p p^n|| < \xi_p$, $||\hat{\xi}_p p^n|| > \hat{\xi}_p$ for every $n = 1, 2, \dots$

(23) S.D. Adhikari, P. Rath and N. Saradha (2005) prove that evey d.f. $g(x)$ of $\{\xi(p/q)^n\}$ satisfies the functional equation

$$\sum_{i=0}^{q-1} g\big(\tfrac{x+i}{q}\big) - \sum_{i=0}^{q-1} g\big(\tfrac{i}{q}\big) = \sum_{i=0}^{p-1} g\big(\tfrac{x+i}{p}\big) - \sum_{i=0}^{p-1} g\big(\tfrac{i}{p}\big).$$

(24) S. D. Adhikari, P. Rath and N. Saradha (2005) prove that every interval $I \subset [0, 1]$ of the length $|I| = (p - 1)/q$ and every complement $[0, 1] - \big[(i-1)/p, i/p\big]$, $i = 1, 2, \dots, p$, are sets of uniquenes of d.f.s of $\left\{\xi(p/q)^n\right\}$, for definition see (17). In the second case, if $j/q \in \big[(i-1)/p, i/p\big]$ for some $1 \leq j < q$ they assume $p \geq q^2 - q$.

(25) T. Vijayaraghavan (1940a): Let $\theta = q^{\frac{1}{k}}$ be irrational, where k and $q \geq 2$ are integers. Then the set of limit points of the sequence $\theta^n \bmod 1$ is infinite.

(26) H. Helson and J.-P. Kahane (1965): Let $\theta > 1$ be a real number. There exists uncountably many ξ such that the sequence $\xi\theta^n \bmod 1$ does not have the a.d.f.

(27) A. Zame (1967): For an arbitrary d.f. $g(x)$ and for any sequence u_n of real numbers which satisfies $\lim_{n\to\infty}(u_{n+1} - u_n) = \infty$, there exists a real number θ such that the sequence $\theta^{u_n} \bmod 1$ has $g(x)$ as its a.d.f.

• A real algebraic integer $\theta > 1$ is called a P.V. number (Pisot–Vijayaraghavan number) if all its conjugates $\neq \theta$ lie strictly inside the unit circle.

(28) Let θ be a P.V. number. Then $\theta^n \bmod 1 \to 0$ as $n \to \infty$.

(29) A. Thue (1912) proved that θ is a P.V. number if and only if $\{\theta^n\} = \mathcal{O}(c^n)$ for some $0 < c < 1$.

(30) G. H. Hardy (1919) proved that if $\theta > 1$ is any algebraic number and $\lambda > 0$ is a real number so that $\{\lambda\theta^n\} = \mathcal{O}(c^n)$, $(0 < c < 1)$, then θ is a P.V. number. Hardy posed an interesting and still **unanswered question** whether there is a transcendental numbers $\theta > 1$ for which a $\lambda > 0$ exists such that $\{\lambda\theta^n\} \to 0$.

(31) T. Vijayaraghavan (1941) proved that if $\theta > 1$ is an algebraic and if θ^n, $n = 1, 3, \ldots$, has only a finite set of limit points, then θ is a P.V. number.

(32) Ch. Pisot (1937, [a]1937) proved that if $\theta > 1$ and $\lambda > 0$ are real numbers such that $\sum_{n=1}^{\infty}\{\lambda\theta^n\} < +\infty$, then θ is a P.V. number.

(33) The set S of all P.V. numbers is closed (R. Salem (1944)). Two smallest elements of S are $1.324717\ldots$, and $1.380277\ldots$, the real roots of $x^3 - x - 1$, and $x^4 - x^3 - 1$, respectively. Both are isolated points of S and S contains no other point in the interval $(1, \sqrt{2}]$ (C.L. Siegel (1944)). The next one is $1.443269\ldots$, the real root of $x^5 - x^4 - x^3 + x^2 - 1$ and $1.465571\ldots$, the real root of $x^3 - x^2 - 1$. The smallest limit point of S is the root $\frac{(1+\sqrt{5})}{2} = 1.618033\ldots$ of $x^2 - x - 1$, an isolated point of the derived set S' of S (J. Dufresnoy and Ch. Pisot (1952), (1953)). The smallest number S'' is 2.

• The real algebraic integer $\theta > 1$ is called a Salem number if all its conjugates lie inside or on the circumference of the unit circle and at least one of conjugates of θ lies on the circumference of the unit circle. It is well known that if θ is a Salem number of degree d, then d is even, $d \geq 4$ and $1/\theta$ is the only conjugate of θ with modulus less than 1, all the other conjugates are of modulus 1.

(34) Let θ be a Salem number. the sequence $\theta^n \bmod 1$ is dense in $[0, 1]$, but not u.d. (Ch. Pisot and R. Salem (1964)) Salem numbers are the only known concrete numbers whose powers are dense mod 1 in $[0, 1]$, see the monograph of M. J. Bertin, A. Decomps-Guilloux, M. Grandet-Hugot, M. Pathiaux-Delefosse, and J. P. Schreiber [1992, pp. 87–89]. The survey paper of E. Ghate and E. Hironaka (2001) deals with the following **open problem:** Is the set of Salem numbers bounded away from 1?

D. H. Lehmer (1933) found the monic polynomial

$$L(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1,$$

where its real root $\theta = 1.17628\ldots$ is both the smallest known Salem number.
(35) Let θ be the Salem numbers of degree greater than or equal to 8. Then the sequence $x_n = \theta^n \bmod 1$, $n = 1, 2, \ldots$, has a.d.f. $g(x) \neq x$ which satisfies $\left|\left(g(y) - g(x)\right) - (y - x)\right| \leq 2\zeta\left(\frac{\deg(\theta)-2}{4}\right)(2\pi)^{1-\frac{\deg(\theta)}{2}}(y - x)$, where $\zeta(z)$ is the Riemann zeta function, $\deg(\theta)$ is the degree of θ over $\mathbb{Q}$ and $0 \leq x < y \leq 1$. This was proved by S. Akiyama and Y. Tanigawa (2004).
(35') Toufik Zaïmi (2006): Let θ be a Salem number and let λ be a nonzero element of the field $\mathbb{Q}(\theta)$ and denote $\Delta = \limsup_{n\to\infty}\{\lambda\theta^n\} - \liminf_{n\to\infty}\{\lambda\theta^n\}$. Then (i) $\Delta > 0$. (ii) If λ is an algebraic integer, then $\Delta = 1$. Furthermore, for any $0 < t < 1$ there is an algebraic integer λ and a subinterval $I \subset [0, 1]$ with the length t such that the sequence $\{\lambda\theta^n\}$, $n = 1, 2, \ldots$ has no limit point in I. (iii) If $\theta - 1$ is a unit, then $\Delta \geq 1/L$, where L is the sum of the absolute values of the coefficients of the minimal polynomial of θ. (iv) If $\theta - 1$ is not a unit, then $\inf_\lambda \Delta = 0$.
(35") A. Dubickas (2006[b]): If θ is either a P.V. or Salem number and $\lambda \neq 0$ and $\lambda \notin \mathbb{Q}(\theta)$, then $\Delta \geq 1/L$, where Δ and λ are defined as in (35').
(35"') A. Dubickas (2006[b]): Let $d \geq 2$ be a positive integer. Suppose that $\alpha > 1$ is a root of the polynomial $x^d - x - 1$. Let ξ be an arbitrary positive number that lies outside the field $\mathbb{Q}(\alpha)$ if $d = 2$ or $d = 3$. Then the sequence $[\xi\alpha^n]$, $n = 1, 2, \ldots$, contains infinitely many even numbers and infinitely many odd numbers. Thus α satisfies Mahler's conjecture (7), i.e., $0 \leq \{\xi\alpha^n\} < 1/2$, does not holds for all $n = 1, 2, \ldots$
(35"") D. Berend and G. Kolesnik (2011): If λ is a Salem number of degree 4, then the sequence $n\lambda^n \bmod 1$, $n = 1, 2, \ldots$ is u.d. Precisely they proved: Let λ be a Salem number of degree 4 and $P(x)$ a nonconstant polynomial with integer coefficients. Then the sequence

$$\left(P(n)\lambda^n, P(n+1)\lambda^{n+1}, P(n+2)\lambda^{n+2}, P(n+3)\lambda^{n+3}\right) \bmod 1, \ \ n = 1, 2, \ldots \text{ is u.d.}$$

(36) I. I. Pjateckiĭ–Šapiro (1951) proved that every distribution function $g(x)$ of the sequence $\alpha q^n \bmod 1$ with integer $q > 1$ satisfies the functional equation

$$g(x) = \sum_{i=0}^{n-1}\left(g((x+i)/q) - g(i/q)\right).$$

(37) If α is a non-zero real number and $q \geq 2$ an integer then the sequence $\alpha q^n \bmod 1$ has a.d.f. $g(x)$ if and only if $\int_0^1 f(x)\mathrm{d}g(x) = \int_0^1 f(qx)\mathrm{d}g(x)$ for every continuous $f(x)$ which is defined on $[0, 1]$. (I. I. Pjateckiĭ–Šapiro (1951)).
(38) Let α be a non-zero real and $q \geq 2$ be an integer. If the sequence

$x_n = \alpha q^n \bmod 1$ has absolutely continuous a.d.f. $g(x)$, then $g(x) = x$ and thus the sequence x_n is u.d.

(39) If α is irrational, then for any integer $q \ge 2$ the set of all limit points of the sequence $\alpha q^n \bmod 1$ is infinite (T. Vijayaraghavan (1940a)).

(39') A. Dubickas (2007): For an integer $b \le -2$ and any irrational ξ we have $\liminf_{n\to\infty}\{\xi b^n\} \le F(-1/b))/q$ and $\limsup_{n\to\infty}\{\xi b^n\} \ge \big(1 - F(q/p)\big)/q$, where $F(x) = \prod_{k=1}^{\infty}\big(1-x^{(2^k+(-1)^{k-1})/3}\big)$. From it he derives: (i) $\liminf_{n\to\infty}\{\xi(-2)^n\} < 0.211811$ and $\limsup_{n\to\infty}\{\xi(-2)^n\} > 0.788189$; (ii) The sequence of integer parts $[\xi(-2)^n]$, $n = 0, 1, 2, \dots$, contains infinitely many numbers divisible by 3 and infinitely many numbers divisible by 4.

• The number α is normal in the base q if and only if $\alpha q^n \bmod 1$ is u.d. The number α is called absolutely normal if it is normal in the base q for all integers $q \ge 2$. The number α is called simply normal to base q if each digit from 0 to $q - 1$ appears with the asymptotic frequency $\frac{1}{q}$.

(40) **It is not known** whether the following constants of general interest e, π, $\sqrt{2}$, $\log 2$, $\zeta(3)$, $\zeta(5)$, … are normal in the base 10. All are, conjecturally, absolutely normal.

(41) The first classical example $\alpha_0 = 0.123456789101112\dots$ of a simple normal number in base $q = 10$ is given by Champernowne (1933). It is also normal in $q = 10$.

(42) Let $f(x) = \alpha_0 x_0^{\beta} + \alpha_1 x^{\beta_1} + \dots + \alpha_k x^{\beta_k}$ be a generalized polynomial, where α's and β's are real numbers such that $\beta_0 > \beta_1 > \dots > \beta_k \ge 0$. Assume that $f(x) \ge 1$ for $x \ge 1$ and that $q \ge 2$ is a fixed integer. Put $\alpha = 0.\big[f(1)\big]\big[f(2)\big]\dots$, where the integer part $\big[f(n)\big]$ is represented in the q-adic digit expansion. Then α is normal in the base q. This was proved by Y.-N. Nakai and I. Shiokawa in the series of papers (1990, [a]1990, 1992). They give the following examples $\alpha = 0.1247912151822\dots$ with $f(x) = x^{\sqrt{2}}$, and $\alpha = 0.151222355069\dots$ with $f(x) = \sqrt{2}x^2$.

(43) If $f(x)$ is a non-constant polynomial with rational coefficients all of whose values at $x = 1, 2, \dots,$ are positive integers, then the normality of α in base 10 was proved by H. Davenport and P. Erdős (1952).

(44) K. Mahler (1937) proved that α defined by an integer polynomial $f(x)$ is a transcendental number of the non-Liouville type.

(45) Y.-N. Nakai and I. Shiokawa (1997): Let $f(x)$ be a non-constant polynomial which takes positive integral values at all positive integers. The number $\alpha = 0.f(2)f(3)f(5)f(7)f(11)\dots$, where $f(p)$ is represented in the q-adic digit expansion and p runs through the primes, is normal in the integral base q. The normality of $\alpha = 0.235711\dots$ with respect to base $q = 10$ was conjectured by D. G. Champernowne (1933) and proved by A. H. Copeland and P. Erdős (1946).

(45') M. G. Madritsch, J. M. Thuswaldner and R. F. Tichy (2008) extended the results of Nakai and Shiokawa by showing that, if f is an entire function of logarithmic order, then the numbers $0.\big[f(1)\big]\big[f(2)\big]\big[f(3)\big]\dots$ and $0.\big[f(2)\big]\big[f(3)\big]\big[f(5)\big]\big[f(7)\big]\dots$, where $\big[f(n)\big]$ stands for the base q expansion of the integer part of $f(n)$, are normal.

(46) H. Furstenberg (1967) proved that if $p, q > 1$ are integers not both integer powers of the same integer (i.e., p and q are *multiplicatively independent*), then for every irrational α the sequence $p^m q^n \alpha \bmod 1$, $m, n = 1, 2, \dots$ is everywhere dense in $[0, 1]$.

(47) B. Kra (1999) extended (46) to the following: Let p_i and q_i be integers and α_i real, $i = 1, 2, \dots, k$. If $p_1, q_1 > 1$ are multiplicatively independent, α_1 is irrational, and $(p_i, q_i) \neq (p_1, q_1)$ for $i > 1$ then the sequence $\sum_{i=1}^k p_i^m q_i^n \alpha_i \bmod 1$, $m, n = 1, 2, \dots$ is dense in $[0, 1]$. He also gave the following:

(48) Let $p, q > 1$ be multiplicatively independent integers and let x_n, $n = 1, 2, \dots$, be any sequence of real numbers. Then for any irrational α the sequence $p^m q^n \alpha + x_n \bmod 1$, $m, = 1, 2, \dots$ is dense in $[0, 1]$.

(49) **Conjecture:** Let λ_i, μ_i, for $i = 1, 2, \dots, k$ be real algebraic numbers, $|\lambda_i|, |\mu_i| > 1$, λ_i, μ_i are multiplicatively independent, and $(\lambda_i, \mu_i) \neq (\lambda_j, \mu_j)$ for $i \neq j$. Then for any real numbers $\alpha_1, \dots, \alpha_k$ with at least one $\alpha_i \notin \mathbb{Q}\big(\cup_{i=1}^k \{\lambda_i, \mu_i\}\big)$ the sequence $\sum_{i=1}^k \lambda_i^m \mu_i^n \alpha_i \bmod 1$, $m, n = 1, 2, \dots$ is dense in $[0, 1]$.

(50) Conjecture (49) was stated by R. Urban (2007). He proved it for special algebraic integers of degree 2, see 1.28. As illustrating examples he gave:
For any α_1, α_2 with at least one non-zero, the sequence $\big\{(\sqrt{23}+1)^m(\sqrt{23}+2)^n \alpha_1 + (\sqrt{61}+1)^m(\sqrt{61}-6)^n \alpha_2\big\}$, $m, n = 1, 2, \dots$ is everywhere dense and also for irrational α_2 the sequence $\big\{(3+\sqrt{3})^m 2^n + 5^m 7^n \alpha_2 \sqrt{2}\big\}$, $m, n = 1, 2, \dots$, is everywhere dense in $[0, 1]$. For more information see Problem 1.28.

Submitted by O. Strauch

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2.5. Duffin-Scheaffer conjecture and related sequences

D.S.C.: Let $f(q)$ be a function defined on the positive integers and let $\varphi(q)$ be the Euler totient function. The Duffin and Schaeffer conjecture (D.S.C.) says that for an arbitrary function $f \geq 0$ defined on positive integers (zero values are also allowed for f) the diophantine inequality

$$\left| x - \frac{p}{q} \right| < f(q), \qquad \gcd(p,q) = 1, \quad q > 0 \tag{1}$$

has infinitely many integer solutions p and q for almost all $x \in [0,1]$ (in the sense of Lebesgue measure) if and only if the following series diverges

$$\sum_{q=1}^{\infty} \varphi(q) f(q) = \infty.$$

Notes:

(I) The D.S.C. is one of the most important unsolved problems in metric number theory, cf. Encyclopaedia of Mathematics 2000 (M. Hazewinkel, ed.).

(II) It was inspired by A. Khintchine (1924) result: If $q^2 f(q)$ is nonincreasing and $\sum_{q=1}^{\infty} q f(q)$ diverges, then (1) has infinitely many integer solutions for almost all x. Originally, he did not assume $\gcd(p,q) = 1$.

(III) By the Borel-Cantelli lemma, (1) has only finitely many solutions for almost all x if $\sum_{q=1}^{\infty} \varphi(q) f(q)$ converges.

(IV) By the Gallagher ergodic theorem (P. Gallagher (1965)) the set of all $x \in [0,1]$ for which (1) has infinitely many integer solutions has measure either 0 or 1.

(V) R. J. Duffin and A. C. Schaeffer (1941) improved Khintchine's theorem in (II) for $f(q)$ satisfying $\sum_{q=1}^{Q} q f(q) \leq c \sum_{q=1}^{Q} \varphi(q) f(q)$ for infinitely many Q and some positive constant c. They also have given an example of $f(q)$ such that $\sum_{q=1}^{\infty} qf(q)$ diverges, $\sum_{q=1}^{\infty} \varphi(q)f(q)$ converges and (1) has for almost all $x \in [0,1]$ only finitely many solutions p and q, where the $\gcd(p,q) = 1$ is omitted (cf. (VI')). This naturally leads to D.S.C. with $\sum_{q=1}^{\infty} \varphi(q)f(q)$ replaced of $\sum_{q=1}^{\infty} qf(q)$.

(VI) In the following a class of sequences q_n $n = 1, 2, \dots,$ distinct positive integers and a class of functions f is said to satisfy D.S.C. if the divergence $\sum_{n=1}^{\infty} \varphi(q_n)f(q_n)$ implies that for almost all $x \in [0,1]$ there exist infinitely many n such that the diophantine inequality

$$\left| x - \frac{p}{q_n} \right| < f(q_n), \qquad \gcd(p, q_n) = 1 \tag{2}$$

has an integer solution p. There are tree types of results of q_n, f satisfying D.S.C.:
(a) any one-to-one sequence q_n and special f;
(b) any $f \geq 0$ and a special q_n (e.g., $q_n = n^k$);
(c) special q_n, f.
For example:
(VIa) Following f's satisfy D.S.C. with every one-to-one sequence q_n:

(i) $f(n) = \frac{c}{n^2}$, where $c > 0$ is a constant.
(ii) $f(n) = O(n^{-2})$.
(iii) $f(n) = O\big(\frac{\exp(g(n))^{\gamma}}{n^2}\big)$, where $\gamma = e^{\frac{1}{2}} - \varepsilon$, $\varepsilon > 0$ and $g(n)$ is the first positive integer for which $\sum_{\substack{p|n, p>g(n) \\ p-\text{prime}}} \frac{1}{p} < 1$.

Note that (i) was proved by P. Erdős (1970), (ii) by J. D. Vaaler (1978) and (iii) by V. T. Vil'chinskiĭ (1979).

(VIb) Following one-to-one sequences q_n satisfy D.S.C. with every $f \geq 0$ (zero values are also allowed):

(i) $\frac{\varphi(q_n)}{q_n} \geq c > 0$ for every n.
(ii) $\frac{\varphi(q_n)}{\varphi(q_{n+1})} \leq c < 1$ for all sufficiently large n.
(iii) $\sum_{i \neq j = 1}^{\infty} \frac{4^{\omega(q_{ij})}}{\varphi(q_{ij})} < +\infty$,
where $q_{ij} = \frac{q_i q_j}{\gcd(q_i, q_j)^2}$ and $\omega(n) = \#\{p - \text{prime}, p|n\}$.
(iv) $\sum_{n=1}^{\infty} \frac{\varphi(q_n)}{q_n} < +\infty$.
(v) $(q_m, q_n) = 1$ for every $m \neq n$.
(vi) $\sum_{i,j=1}^{\infty} \frac{(\log q_{ij})^2}{q_{ij}} \frac{\varphi(q_i)}{q_i} \frac{\varphi(q_j)}{q_j} < +\infty$.
(vii) $\sum_{n=1}^{\infty} \frac{(\log q_n)^2}{q_n^{2\varepsilon}} < +\infty$ and $d_{ij} \leq (q_i q_j)^{\frac{1}{2} - \varepsilon}$ for some $\varepsilon > 0$ and every $i \neq j$, where $d_{ij} = \gcd(q_i, q_j)$.

(viii) The sequence $d_{ij} = \gcd(q_i, q_j)$, $i, j = 1, 2, \dots$, has only finitely many different terms.

(ix) $\frac{q_n}{q_{n+1}} \le c < 1$ for every n.

(x) $\frac{\varphi(q_n)}{q_n} < Kn^{-\delta}$ for some $K, \delta > 0$ and $n = 1, 2, \dots$

(xi) $q_n = n^k$, for $k \ge 2$.

(xii) $q_n = q^n$, $q_n = n!$, $q_n = 2^{2^n} + 1$-Fermat numbers, $q_n = F_n$-Fibonacci numbers, $q_n = q^n - 1$, $q_n = q^n + 1$ (for every positive integer $q \ge 2$).

(xiii) q_n is a one-to-one sequence of primes.

Note that (i) and (ii) can be found in R. J. Duffin and A. C. Schaeffer (1941).
(iii) and (iv) was proved by O. Strauch [1982, Th. 14 and 15].
(v) and (vi) by O. Strauch [1983, Th. 7 and 2].
(vii) by O. Strauch [1984, Th. 6].
(viii) by O. Strauch [1986, Th. 8].
(ix), (x), (xi) by G. Harman [1990, Th. 1].
(xii) by O. Strauch (1986).
(xiii) is an example in R. J. Duffin and A. C. Schaeffer [1941, p. 245]. Also G. Herman [1998, p. 27, Cor. 2] notes that Duffin-Schaeffer criterion in (VIc(i)) (also in (VIb(i))) holds for one-to-one sequence q_n of primes, since it satisfies D.S.C. with every $f \ge 0$.

(VIc) The following special sequences q_n and functions f satisfy D.S.C.:

(i) $\sum_{n=1}^{N} q_n f(q_n) \le c \sum_{n=1}^{N} \varphi(q_n) f(q_n)$ for $N = 1, 2, \dots$

(i') $q_n = n$ and $q_n^c f(q_n)$ is non-increasing. Here c may be any real constant.

(ii) $\sum_{n=1}^{N} \varphi(q_n) f(q_n) > cN^{\delta}$ for infinitely many N, where $c, \delta > 0$ are constants.

(iii) $f(q_n) > cq_n(\varphi(q_n)/q_n)^R$ for $n = 1, 2, \dots$, where $c, R > 0$ are constants.

(iv) q_n strictly increase, $q_n f(q_n)$ does not increase and the lower asymptotic density $\underline{d}(q_n) > 0$.

(v) $f(q_n)\varphi(q_n) = O\big((n \log n \log\log n)^{-1}\big)$ and $c_1 n^A \le q_n \le c_2 n^B$, where c_1,c_2,A,B are positive constants and $B \ge 1$.

(vi) $\frac{\max_{i \le n} \varphi(q_i)}{\sum_{i=1}^{n} \varphi(q_i)} \ge c > 0$ for $n = 1, 2, \dots$ and $f(q_n)$ is nonincreasing.

(vii) q_n is strictly increasing with $\underline{d}(q_n) > 0$ and $f(q) \ge cf(s)$ for all $q = 1, 2, \dots$, every $s \in \{q+1, q+2, \dots, 2q\}$ and some constant $c > 0$.

Note that (i) and (i') was proved by R. J. Duffin and A. C. Schaeffer (1942); (ii) and (iii) G. Harman (1990, 1998, p. 66, Th. 3.7); (iv) G. Harman [1998, p. 41, Cor. 3]; (v) G. Harman [1998, p. 57, Th. 2.10]; (vi) O. Strauch (1982); and (vii) E. Zoli (2008).

(VI') O. Strauch (1982): Let p_n, $n=1,2,\dots$ be the increasing sequence of all primes and put $q_n=p_1p_2\dots p_n$ and $f(q_n)=(q_n n\log n)^{-1}$. Then $\sum_{n=1}^{\infty}\varphi(q_n)f(q_n)$ converges, $\sum_{n=1}^{\infty}q_nf(q_n)$ diverges and for almost all $x\in[0,1]$ the inequality (2) has only finitely many solutions p,q_n, but infinitely many if the assumption $\gcd(p,q_n)=1$ is omitted.
(VI") P. A. Caltin (1976) **conjectured** that the divergence

$$\sum_{q=1}^{\infty}\varphi(q)\max_{m\ge1}f(m.q)$$

is the necessary and sufficient condition for (1) to have infinitely many solutions. The D.S.C. implies this conjecture, cf. G. Harman [1998, pp. 28–29].
(VI*) It is interesting that one-dimensional D.S.C. is open, but multidimensional D.S.C. was proved by A. D. Pollington and R. C. Vaughan (1999) in the following form: For every $k=2,3,\dots$, for every one-to-one sequence q_n, $n=1,2,\dots$, of positive integers and for any nonnegative function f, if the series $\sum_{n=1}^{\infty}\big(\varphi(q_n)f(q_n)\big)^k$ diverges, then for almost all $\mathbf{x}=(x_1,\dots,x_k)$ and for infinitely many n there exists an integer vector $\mathbf{p}=(p_1,\dots,p_k)$ such that the inequalities

$$\left|x-\frac{p_1}{q_n}\right|<f(q_n),\dots,\left|x-\frac{p_k}{q_n}\right|<f(q_n),$$

hold, where $\gcd(p_1p_2\dots p_k,q_n)=1$.
(VI*b) G. Harman [1998, p. 65, Th. 3.6] proved another multidimensional D.S.C.: Let $f_1(n),\dots,f_k(n)$ be functions of n taking values in $[0,c)$ for some $c>0$. Write $\theta(n)=\prod_{j=1}^{k}\big(nf_j(n)\big)$ and suppose, for some positive reals ε and K, that for each n for which $\theta(n)\ne0$ we have $\max_{1\le j\le k}\frac{\theta(n)}{nf_j(n)}\le K\big(\theta(n)\big)^{\varepsilon}$. Let q_n be a sequence of distinct positive integers for which

$$\sum_{n=1}^{\infty}\big(\varphi(q_n)f_1(q_n)\big)\dots\big(\varphi(q_n)f_k(q_n)\big)=\infty.$$

Then for almost all $\mathbf{x}=(x_1,\dots,x_k)\in[0,1]^k$ there are infinitely many solutions

$$\left|x-\frac{p_1}{q_n}\right|<f_1(q_n),\dots,\left|x-\frac{p_k}{q_n}\right|<f_k(q_n),\qquad \gcd(p_1p_2\dots p_k,q_n)=1.$$

There are two following types of sequences inspired by D.S.C., namely eutaxic and quick.
Eutaxic sequences. Let $x_n\in[0,1)$, $z_n\in\mathbb{R}^+$, $n=1,2,\dots$, be two sequences and $x\in[0,1]$. O. Strauch (1994) introduced a new counting function

$$A\big(x;N;(x_n,z_n)\big)=\#\big\{n\le N;|x-x_n|<z_n\big\}.$$

• The sequence x_n is said to be **eutaxic** if for every non-increasing sequence z_n the divergence of $\sum_{n=1}^{\infty}z_n$ implies that

$$\lim_{N\to\infty}A\big(x;N;(x_n,z_n)\big)=\infty$$

for almost all $x \in [0,1]$. If furthermore

$$\lim_{N\to\infty} \frac{A(x;N;(x_n,z_n))}{2\sum_{n=1}^{N} z_n} = 1,$$

then x_n is called **strongly eutaxic**.

NOTES: (VII) Eutaxic sequences were introduced by J. Lesca (1968). He proved that if θ is irrational then the sequence $n\theta \bmod 1$ is eutaxic if and only if θ has bounded partial quotients.

(VIII) M. Reversat proved the same for the strong eutaxicity of $n\theta \bmod 1$, i.e., for sequence $n\theta \bmod 1$ both notions coincide.

(IX) B. de Mathan (1971) defined the counting function

$$A^*(N,x_n) = \#\Big\{0 \le k < N; \exists n \le N\big(x_n \in k/N,(k+1)/N\big)\Big\}$$

and proved that $\liminf_{N\to\infty} A^*(N,x_n)/N = 0$ implies that x_n is not eutaxic. Since for the sequence $x_n = n\theta \bmod 1$ and for θ with unbounded partial quotients we have $\liminf_{N\to\infty} A^*(N,x_n)/N = 0$, B. de Mathan (1971) recovered half of Lesca's result.

(X) A characterization of strong eutaxicity in terms of L^2 discrepancy is an **open problem**, cf. O. Strauch (1994).

Quick sequences. Let $X = \cup_{m=1}^{\infty} I_m$ be a decomposition of an open set $X \subset [0,1]$ into a sequence I_m, $m = 1,2,\dots$, of pairwise disjoint open subintervals of $[0,1]$ (empty intervals are allowed). Let x_n be an infinite sequence in $[0,1)$. Define a new counting function

$$\begin{aligned}\widetilde{A}(X;N;x_n) = &\#\{m \in \mathbb{N}; \exists n \le N \text{ such that } x_n \in I_m\}\\ &+ \#\{n \le N\ ;\ x_n \notin X\},\end{aligned}$$

i.e., if $x_n \in X$ for $n = 1,2,\dots$, then $\widetilde{A}(X;N;x_n)$ is the number of intervals I_m containing at least one element of $x_1, x_2, \dots, x_N$.

• The sequence x_n is said to be **quick** if for any open set $X \subset [0,1]$ with the Lebesgue measure $|X| < 1$, there exists a constant $c = c(X)$ such that

$$\frac{\widetilde{A}(X;N;x_n)}{N} \ge c > 0.$$

(X') Examples of quick sequences:

(i) The sequence x_n of all dyadic rational numbers from $[0,1]$ ordered by $\frac{0}{2}, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \frac{1}{16}, \dots$

(ii) The sequence x_n of all rational numbers (reduced fractions) from $(0,1]$ ordered by $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \dots$

(iii) The sequence x_n of all non-reduced fractions from $(0,1]$ ordered by $\frac{1}{1}, \frac{1}{2}, \frac{2}{2}, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}, \frac{1}{5}, \dots$

(iv) The denominators $1, 2, 3, \dots$ in (iii) can be replaced by any arithmetic subsequence of positive integers.

• The sequence x_n is said to be **uniformly quick** (abbreviated u.q.) if for any open set $X \subset [0,1]$ we have

$$\lim_{N\to\infty} \frac{\widetilde{A}(X;N;x_n)}{N} = 1 - |X|.$$

• If this limit holds for a special sequence of indices $N_1 < N_2 < \dots$, then x_n is said **almost u.q.**

(XI) Quick and u.q. sequences were introduced and studied by O. Strauch (1982, 1983, 1984, [a]1984, 1986) in connection with D.S.C.

(XII) Any quick sequence x_n is eutaxic, i.e., for every non-increasing sequence z_n, $\sum_{n=1}^{\infty} z_n = \infty$, for almost all $x \in [0,1]$, the inequality $|x - y_n| < z_n$ holds for infinitely many n.

(XII') Any u.q. sequence x_n is u.d. in $(0,1]$ and it is also strongly eutaxic.

(XIII) The sequence $x_n = n\theta \bmod 1$ is u.q. if and only if the simple continued fraction expansion of the irrational θ has bounded partial quotients (cf. O. Strauch ([a]1984)).

(XIV) O. Strauch [1982, Th. 3]: The u.d. sequence x_n is u.q. if for infinitely many M there exist c_M, c'_M, and $N_0(M)$ such that $c'_M \to 0$ as $M \to \infty$ and

$$\sum_{\substack{|x_i - x_j| \le t \\ M < i \ne j \le N}} 1 \le c_M t (N-M)^2 + c'_M (N-M)$$

for every $N \ge N_0(M)$ and every $t \ge 0$.

(XV) The u.q. sequences x_n can be used in the numerical evaluation of integrals $\int_X f(x)\,\mathrm{d}x$ over open subsets X of $[0,1]$. Thus also for Jordan non-measurable sets X whose boundaries ∂X are of positive measure $|\partial X| > 0$, cf. O. Strauch (1997).

(XVI) Let q_n, $n = 1, 2, \dots$, be a one-to-one sequence of positive integers and let A_n, $n = 1, 2, \dots$ be a sequence composed from blocks

$$A_n = \left(\frac{1}{q_n}, \frac{a_2}{q_n}, \dots, \frac{a_{\varphi(q_n)}}{q_n} \right),$$

where $1 = a_1 < a_2 < a_3 < \dots < a_{\varphi(q_n)}$ are the integers $< q_n$ coprime to q_n. If A_n, $n = 1, 2, \dots$ is almost u.q. (with respect to the set of indices $N_n = \sum_{i=1}^{n} \varphi(q_i)$), then the D.S.C. holds for q_n and for every non-increasing $f(q_n)$. Thus the D.S.C. follows from the following conjecture immediately:

Conjecture: For every one-to-one sequence of positive integers q_n, the block sequence A_n is almost u.q.

(XVII) For every one-to-one sequence q_n in (VIb) the block sequence A_n is almost u.q., e.g., (v) relatively prime q_n; (xii) q_n = Fibonacci numbers; (ix) lacunary q_n;

(XVIII) From (XIV) it follows immediately: Assume that, for infinitely many m there exist c_m and c'_m such that $c'_m \to 0$ as $m \to \infty$ and for every $0 < x < 1$ and for all sufficiently large n we have the estimation

$$\sum_{\substack{0<\frac{a}{q_i}-\frac{b}{q_j}<x\\(a,q_i)=(b,q_j)=1\\m<i,j\le n}} 1 \le c_m x\left(\sum_{m<i\le n}\phi(q_i)\right)^2 + c'_m\left(\sum_{m<i\le n}\phi(q_i)\right). \tag{3}$$

Then the sequence q_n, $n = 1, 2, \dots$, satisfies D.S.C. with every non-increasing function f. If the estimation (3) holds for every permuted $q_{\pi(n)}$ and for any subsequences $q_{\pi(n_k)}$, $k = 1, 2, \dots$, then the sequence q_n, $n = 1, 2, \dots$, satisfies D.S.C. with every $f \ge 0$, zero values are also allowed. We **conjectured** that (3) holds for any one-to-one integers q_n, where the constants c_m and c'_m depend on q_n. All results in (VIb) can be found by proving (3), where we used the following partial estimations: For given two integers q_i and q_j denote

$$a(x) = \prod_{\substack{p>x,p|a\\p-\text{prime}}} p, \quad d_{ij} = \gcd(q_i, q_j), \quad q_{ij} = \frac{q_i q_j}{d_{ij}^2}, \quad x_{ij} = x d_{ij} q_{ij}.$$

Then the sum

$$\sum_{\substack{0<\frac{a}{q_i}-\frac{b}{q_j}<x\\(a,q_i)=(b,q_j)=1}} 1$$

has the following upper bounds:

(i) $c_0 x\varphi(q_i)\varphi(q_j)\frac{q_{ij}(x_{ij})}{\varphi(q_{ij}(x_{ij}))}$;

(ii) $c_0 x\varphi(q_i)\varphi(q_i)$;

(iii) $c_0 x q_i q_j$;

(iv) $c_0 x\varphi(q_i)\varphi(q_j)\frac{q_{ij}}{\varphi(q_{ij})}$;

(v) $c_0 x\varphi(q_i)\varphi(q_j) + 2^{\omega(q_{ij})}\varphi(d_{ij})$;

where $\omega(n)$ is the number of distinct prime divisors of n and c_0 is an absolute constant.

(XIX) Let $g(x)$ be an integer polynomial. In generally, the D.S.C. for $q_n = g(n)$ is an **open problem** (we know only $g(x) = x^k$, see (VIb(xi))). In connection with (VIb(vii)) there is a **question**: When for polynomial $g(x)$ we have

$$\gcd\big(g(m), g(n)\big) \le c\big(g(m)g(n)\big)^{\frac{1}{2}-\varepsilon} \tag{4}$$

for every sufficiently large integers $m \neq n$, where the constants c and ε depend on $g(x)$.

A. Schinzel (personal communication) has shown that the Masser *abc*-hypothesis implies (4) for $g(x) = x^3 + k$, $k = 1, 2, \dots$, with exponent $\frac{1}{2} - \frac{1}{18} + \varepsilon'$. He conjectured that (4) holds also for $g(x) = x^k + 1$, $k = 3, 4, \dots$, but he proved that (4) does not hold for $g(x) = x^2 + 1$.

(XX) The problem of restricting both numerators p and denominators q in (1) to sets of number-theoretic interest was investigated by G. Harman (1988).

(XX') Firstly, he considers (1), where p, q are both primes. In this case D.S.C. has the following form.

Conjecture: For any function $f \geq 0$ if the sum

$$\sum_{\substack{q=2 \\ q-\text{prime}}}^{\infty} f(q) \frac{q}{\log q} \tag{5}$$

diverges, then for almost all x there are infinitely many primes p, q which satisfy (1).

(XXI) G. Harman (1988) established this D.S.C. for $f(q) \geq 0$ satisfying $0 < \sigma_1 \leq \frac{mf(m)}{nf(n)} \leq \sigma_2$ for all m with $n_0 \leq n < m < 2n$, where σ_1, σ_2, n_0 are positive constants.

(XXII) V. T. Vil'chinskiĭ (1990) replaced (5) by

$$\sum_{\substack{q=2 \\ q-\text{prime}}}^{\infty} f(q) \frac{q^{-m+1+(m/n)}}{\log q} \tag{6}$$

where integers m, n satisfy one from $m = n$, $m > 2n$, or $n > 2m$. He proved that for special $f > 0$, if the series (6) diverges, then for almost all x, there exists infinitely many primes p, q such that $\left|x - \frac{p}{q^m}\right| < f(q)$.

(XXIII) Using the theory of u.q. sequences, the D.S.C. for prime numbers and non-increasing f holds if the block sequence A_n, $n = 1, 2, \dots$,

$$A_n = \left(\frac{q_1}{q_n}, \frac{q_2}{q_n}, \dots, \frac{q_n}{q_n}\right),$$

is u.q., where q_n, $n = 1, 2, \dots$ is the increasing sequence of all primes. Note that the sequence of blocks A_n, $n = 1, 2, \dots$, is u.d., see 1.9 Block sequence, Example (I).

(XXIV) G. Harman [1998, p. 168, Th. 6.2]: Let A and B be sets of positive integers and denote

(i) $\underline{d}(B) > 0$;

(ii) $A(kn)/A(n) > c + 1$ for all n and for some constants $k > 1$, $c > 0$. Here $A(n) = \#\{i \leq n; i \in A\}$;

(iii) $A(2n) - A(n) < C$ for all n, where C depends only on A,

(iv) $\lim_{q\to\infty} \lim_{p\to\infty} \gcd(p,q) = \infty$, where $q \in A$ and $p \in B$;

(v) $f(n)$ is non-increasing;

(vi) $0 < \sigma_1 < \frac{mf(m)}{nf(n)} < \sigma_2$ for all m with $n_0 \le n < m < 2n$, where σ_1, σ_2, n_0 are positive constants;

(vii)
$$\liminf_{n\to\infty} \frac{1}{A(n)} \sum_{\substack{q=1\\ q\in A}}^{n} \frac{1}{q} \sum_{\substack{p\in B\\ p/q\in[x,y]\\ \gcd(p,q)=1}} 1 > c(y-x)$$

for all intervals $[x,y]$, where $c > 0$ is a constant depending on A and B.

Suppose that (i), (vii) and at least one of (ii), (iii), (iv) holds for A and at least one of (v), (vi) holds for f. If the series $\sum_{q=1,q\in A}^{\infty} f(q)$ diverges, then there are infinitely many solutions to

$$\left| x - \frac{p}{q} \right| < f(q), \qquad q \in A, \quad p \in B, \quad \gcd(p,q) = 1 \tag{7}$$

for almost all x.

- $\mathcal{K}(f) = \left\{x \in \mathbb{R}; \left|x - \frac{p}{q}\right| < f(q) \text{ for infinitely many rationals } \frac{p}{q}\right\}$;
- Exact $(f) = \mathcal{K}(f) - \cup_{m\ge 2}\mathcal{K}\big((1-1/m)f\big)$;

(XXV) Y. Bugeaud (2008) collected known results:

(i) If $f(q)$ is non-increasing and $f(q) = o(q^{-2})$ then $\mathcal{K}(f) \neq \emptyset$ (V. Jarník (1931));

(ii) If $q^2 f(q)$ is non-increasing and $\sum_{q=1}^{\infty} f(q)$ converges then for Hausdorff dimension dim Exact (f) = dim $\mathcal{K}(f) = \frac{2}{\lambda}$, where $\lambda = \liminf_{x\to\infty} -\frac{\log f(x)}{\log x}$ (M. M. Dodson (1992)).

Since for convergent $\sum_{q=1}^{\infty} f(q)$ the result (i) is very satisfactory, Y. Bugeaud (2008) proposed two following problems:

Open problem 1. Let $f(q)$ be a non-increasing, $f(q) = o(q^{-2})$ and $\sum_{q=1}^{\infty} f(q)$ diverges then to find Hausdorff dimension of Exact (f).

(iii) Y. Bugeaud (2008): Let $q^2 f(q)$ be a non-increasing, $\sum_{q=1}^{\infty} f(q)$ diverges and $1/(q^{2+\varepsilon}) \le f(q) \le 1/(100q^2 \log q)$ for any $\varepsilon > 0$ and sufficiently large q. Then dim Exact (f) = dim ${}_H\mathcal{K}(f) = 1$.

Open problem 2. Study the set Exact (c/n^2).

(iv) Y. Bugeaud (2008): For any $0 < c < 1/6$ the set Exact (c/n^2) is non-empty.

(XXVI) D. Berend and A. Dubickas (2009) studied diophantine approximation in the form (VII) $|x - x_n| < z_n$. Putting

- $\mathcal{G}(x_n, z_n) = \big\{x \in \mathbb{R}; |x - x_n| < z_n \text{ for infinitely many } n\big\}$ they proved:

(i) Let x_n be an arbitrary dense sequence in $[0,1]$, and z_n be an arbitrary sequence of positive numbers. Then the set $\mathcal{G}(x_n,z_n)$ is an uncountable dense subset of the interval $[0,1]$.
(ii) Let x_n be an arbitrary sequence in $[0,1]$, and z_n be an arbitrary sequence of positive numbers. If $\sum_{n=1}^{\infty} z_n^s<\infty$ for some $0<s<1$, then $\dim_H\mathcal{G}(x_n,z_n)\le s$ (the Hausdorff dimension).
(iii) For any sequence $z_n>0$, $\lim_{n\to\infty} z_n=0$, there exists well distributed $x_n\in[0,1)$ such that $\dim_H\mathcal{G}(x_n,z_n)=0$.
(iv) For every z_n, $\sum_{n=1}^{\infty} z_n=\infty$, there exists $x_n\in[0,1]$ such that $\mathcal{G}(x_n,z_n)=[0,1]$.
(v) Note that, by definition of eutaxic sequence and by (VII), for $x_n=n\theta \bmod 1$, if θ has bounded partial quotients, then the measure $|\mathcal{G}(x_n,z_n)|=1$, for an arbitrary sequence $z_n>0$, $\sum_{n=1}^{\infty} z_n=\infty$.
(XXVII) L. Mišík and O. Strauch (2012) are linked to (XXVI) results:
(i) Let x_n be a sequence in $[0,1)$ such that the set $G(x_n)$ of all d.f.s of x_n contains only continuous d.f.s. Then for every sequence $z_n>0$, $z_n\to 0$, and every $x\in[0,1]$ we have: If $|x-x_{n_k}|<z_{n_k}$, $k=1,2,\dots$, then the asymptotic density $d(n_k)=0$.
(ii) Ex.: The set $G(\log n)$ (was found by A. Wintner (1935)) has only continuous functions. Thus if $|x-\{\log n_k\}|<z_{n_k}$, $k=1,2,\dots$ then $\frac{k}{n_k}\to 0$ for every sequence $z_n>0$, $z_n\to 0$. Note that, recently Y. Ohkubo (2011) proved: Let p_n, $n=1,2,\dots$, be the increasing sequence of all primes. The sequence $\{\log p_n\}$, $n=1,2,\dots$, has the same d.f.s as $\log n \bmod 1$. Thus n_k with $|x-\{\log p_{n_k}\}|<z_{n_k}$, $k=1,2,\dots$ satisfies $\frac{k}{n_k}\to 0$, again.
• G. Myerson (1993) (see [SP, 1.8.10]) introduced the sequence x_n in $[0,1)$ to be *uniformly maldistributed* if for every subinterval $I\subset[0,1)$ with positive length $|I|>0$ we have both

$$\liminf_{n\to\infty}\frac{\#\{i\le n; x_i\in I\}}{n}=0,\quad \limsup_{n\to\infty}\frac{\#\{i\le n; x_i\in I\}}{n}=1.$$

(iii) Let x_n be a uniformly maldistributed sequence in $[0,1)$. Then there exists a decreasing sequence $z_n>0$, $n=1,2,\dots$, $z_n\to 0$, such that for every $x\in[0,1]$, the sequence of all indices n_k, $|x-x_{n_k}|<z_{n_k}$, $k=1,2,\dots$, has the upper asymptotic density $\overline{d}(n_k)=1$.
(iv) Ex.: The sequence $\{\log\log n\}$, $n=2,3,\dots$, is uniformly maldistributed, thus there exists a decreasing sequence $z_n>0$, $n=1,2,\dots$, $z_n\to 0$, such that for every $x\in[0,1]$, $\overline{d}(n_k)=1$ for all possible n_k, $|x-\{\log\log n_k\}|<z_{n_k}$.
(XXVIII) For Hausdorff dimension the analogue of the D.S.C. is true. G. Harman [1998, Th. 10,7] proved that if $\sum_{q=1}^{\infty} qf(q)=\infty$, then the set $X=\big\{x\in[0,1];\text{there exists infinitely many integer solutions}\big|x-\frac{p}{q}\big|, \gcd(p,q)=1, q>0\big\}$ has Hausdorff dimension 1.

(Note that $\sum_{q=1}^{\infty}\varphi(q)f(q)=\infty \Rightarrow \sum_{q=1}^{\infty} qf(q)=\infty$). The dimensional analogue of D.S.C. also proved A. Haynes, A. Pollington and S. Velani (2012).

(XXIX) O. Strauch (1983) also studied the diophantine inequality

$$|x-x_n|<z_n/z. \tag{8}$$

He proved

(i) If x_n is dense in $[0,1]$, $z_n\to 0$, the following sets

$X_1=\{x\in[0,1];\ \text{for every } z>0,\ (8) \text{ and } x_n>x \text{ holds for infinitely many } n\}$,

$X_2=\{x\in[0,1];\ \text{for every } z>0,\ (8) \text{ and } x_n<x \text{ holds for infinitely many } n\}$,

are also dense and they have a power c, and

$X_0=\{x\in[0,1];\ \text{there exists } z>0,\ (8) \text{ holds only for finitely many } n\}$

is of the first category.

(ii) For every $x_n\in[0,1]$, $z_n\to 0$, with the possible exception of a nullset, the unit interval $[0,1]$ can be decomposed into two sets:

$X_3=\{x\in[0,1];\ \text{for every } z>0,\ (8) \text{ holds only for finitely many } n\}$,

$X_4=\{x\in[0,1];\ \text{for every } z>0,\ (8) \text{ and } x_n>x \text{ hold for infinitely many } n$ and also (8) and $x_n<x$ hold for infinitely many $n\}$.

(XXX) A. Haynes, A. Pollington and S. Velani (2012) replace Duffin-Schaeffer series $\sum_{q=1}^{\infty}\varphi(q)f(q)$ by series $\sum_{q=1}^{\infty}\varphi(q)\big(f(q)\big)^{1+\varepsilon}$ and prove that the divergence $\sum_{q=1}^{\infty}\varphi(q)\big(f(q)\big)^{1+\varepsilon}=\infty$ implies that the diophantine inequality (1) has infinitely many integer solutions p and q for almost all $x\in[0,1]$ and for arbitrary $f(q)\ge 0$. Thus D.S.C. holds in this form. Liangpan Li (2013) replace $\sum_{q=1}^{\infty}\varphi\big(q\big)\big(f(q)\big)^{1+\varepsilon}$ by $\sum_{q=1}^{\infty}\varphi(q)q^{\varepsilon}\big(f(q)\big)^{1+\varepsilon}$ in this result.

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Concluding remark. The author would gratefully all comments to problems listed in this paper.

Received November 29, 2013

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