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THE RIEMANN DERANGEMENT THEOREM AND DIVERGENT PERMUTATIONS

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ABSTRACT. In this paper, a strengthening of the Riemann Derangement Theorem, by selecting the appropriate permutation of $\mathbb{N}$ from two families $\mathfrak{DC}$ and $\mathfrak{DD}$ of permutations of $\mathbb{N}$, is presented. The mentioned families are characterized in a natural way; their algebraic properties were investigated by the author in his previous works.

1. Introduction

Subjects referring to the Riemann Derangement Theorem are still popular because the theorem offers a great inspiration. The recent papers [8], [9] (see also [6]) can be set the good examples of this statement. Let us recall its content.

The Riemann Derangement Theorem. *Let $\sum a_n$ be a conditionally convergent real series. Then, for every nonempty and closed interval $I \subset \mathbb{R}^*$ (two points compactification of $\mathbb{R}$), there exists a permutation p of $\mathbb{N}$ such that the set of limit points of series $\sum a_{p(n)}$, denoted by $\sigma a_{p(n)}$, is equal to I.*

In the current paper, we also will return to this classical result by selecting the appropriate permutation p of $\mathbb{N}$ for this theorem from two subfamilies $\mathfrak{DC}$ and $\mathfrak{DD}$ of family $\mathfrak{P}$ of all permutations of $\mathbb{N}$, as being discussed by author (see [1], [2], [7]) earlier.

However, we first need to introduce a few essential concepts. For short, we write $A < B$ for any two nonempty subsets A and B of $\mathbb{N}$ when $a < b$ for each $a \in A$ and $b \in B$.

We call a sequence $\{A_n\}$ of nonempty subsets of $\mathbb{N}$ to be increasing if

$$A_n < A_{n+1} \quad \text{for every } n \in \mathbb{N}.$$

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We say that a nonempty subset A of $\mathbb{N}$ is a union of k mutually separated intervals (abbreviated to **MSI**), if there exist k intervals $I_1, \ldots, I_k$ of $\mathbb{N}$ which form a partition of A and $\operatorname{dist}(I_i, I_j) \geqslant 2$ for any distinct indices $i, j \leqslant k$.

A permutation $p \in \mathfrak{P}$ is said to be divergent permutation if there is a conditionally convergent series $\sum a_n$ of real terms such that the p-rearranged series $\sum a_{p(n)}$ is divergent. The family of all divergent permutations will be denoted by $\mathfrak{D}$. Elements of family $\mathfrak{C} := \mathfrak{P} \setminus \mathfrak{D}$ are called the convergent permutations. We note that if $p \in \mathfrak{C}$ then, for every conditionally convergent series $\sum a_n$, the p-rearranged series $\sum a_{p(n)}$ is also convergent.

Family $\mathfrak{D}$ will be naturally partitioned onto two subfamilies $\mathfrak{DC}$ and $\mathfrak{DD}$, depending on the condition whether, for a given $p \in \mathfrak{D}$, we have $p^{-1} \in \mathfrak{C}$ or $p^{-1} \in \mathfrak{D}$, respectively. Some algebraic properties of those families were investigated by author [1]; among others, the following two relations

$$\mathfrak{DC} \circ \mathfrak{DC} = \mathfrak{DC} \quad \text{and} \quad \mathfrak{DD} \circ \mathfrak{DC} = \mathfrak{DC} \circ \mathfrak{DD} = \mathfrak{DC} \cup \mathfrak{DD}$$

were shown. The sign "$\circ$" denotes the composition of sets of permutations of $\mathbb{N}$ here. In the current paper, we will add some other properties of $\mathfrak{DC}$ and $\mathfrak{DD}$, of analytical nature, connected with the Riemann Derangement Theorem.

2. Main result

The two following dual combinatoric characterizations of convergent and divergent permutations are known [3], [4], [5] and will be explored in the proof of our main result.

Theorem 2.1. *A permutation $p \in \mathfrak{P}$ is a convergent permutation if and only if there exists a positive integer k such that the set $p(I)$ is a union of at most k* **MSI** *for every interval I of $\mathbb{N}$. The minimal positive integer k with this property will be denoted by* $\mathbf{c(p)}$.

Theorem 2.2. *A permutation $p \in \mathfrak{P}$ is a divergent permutation if and only if, for every positive integer n, there exists an interval I of $\mathbb{N}$ such that $p(I)$ is a union of at least n* **MSI**.

Now, we are ready to formulate and to prove the main result—a strengthened version of the Riemann Derangement Theorem (announced by author in [7]).

Theorem 2.3. *Let $\sum a_n$ be a conditionally convergent series and let $I \subset \mathbb{R}^*$ be a nondegenerated closed interval in $\mathbb{R}^*$. Then there exists a permutation $p \in \mathfrak{DD}$ such that $\sigma a_{p(n)} = I$.*

If we assume additionally that $\sum a_n \in I$ or that interval I is of the form $[\alpha, +\infty]$ or $[-\infty, \beta]$, for $\alpha, \beta \in \mathbb{R}^*$, $\alpha < +\infty$ and $\beta > -\infty$, then there exists *a permutation* $q \in \mathfrak{DC}$ *such that* $\mathbf{c}(\mathbf{q^{-1}}) \leqslant 5$ *and* $\sigma a_{q(n)} = I$*. In the case* $\sum a_n \in I$*, the permutation* $q \in \mathfrak{DC}$ *can be selected in such a way that* $\mathbf{c}(\mathbf{q^{-1}}) \leqslant 3$*.*

Proof. Let $\alpha := \sum a_n$, $A^+ := \{i \in \mathbb{N} \colon a_i \geqslant 0\}$ and $A^- := \mathbb{N} \setminus A^+$. Furthermore, let us put $a_n^+ := \max\{0, a_n\}$ and $a_n^- := a_n - a_n^+$ for each $n \in \mathbb{N}$.

In the presented proof, we use the notations: (a, b), $(a, b]$, $[a, b)$ and $[a, b]$ for the intervals of $\mathbb{R}$, as well as for the intervals of $\mathbb{N}$. Which notation is appropriate at a given moment, it will always clearly follow from the considerations.

Let $\beta, \gamma \in \mathbb{R}$, $\beta \leq \alpha \leq \gamma$. First, we define a permutation $q \in \mathfrak{DC}$ such that $\sigma a_{q(n)} := [\beta, \gamma]$ but the initial part of the proof is applicable to the case $\beta < \alpha < \gamma$ only. For this purpose, let us determine $m \in \mathbb{N}$ such that

$$|a_n| < \min\left\{\frac{1}{2}(\gamma - \alpha), \frac{1}{2}(\alpha - \beta)\right\} \tag{1}$$

and

$$\frac{1}{2}(\beta - \alpha) < \alpha - \sum_{k=1}^{n} a_k < \frac{1}{2}(\gamma - \alpha), \tag{2}$$

for each $n \in \mathbb{N}$, $n > m$.

Next, by induction, we find the increasing sequence $\{I_n \colon n \in \mathbb{N}_0\}$ of intervals of $\mathbb{N}$ forming the partition of $\mathbb{N}$ and satisfying, for every $n \in \mathbb{N}$, the following conditions:

$$\sum_{j \in J_{2n-1}} a_j + \sum_{i \in I_{2n-1}} a_i^+ \geqslant \gamma, \quad \text{while} \quad \sum_{j \in J_{2n-1}} a_j + \sum_{\substack{i \in I_{2n-1} \\ i \neq k_{2n-1}}} a_i^+ \leqslant \gamma \tag{3}$$

and

$$\sum_{j \in J_{2n}} a_j + \sum_{i \in I_{2n}} a_i^- \leqslant \beta, \quad \text{while} \quad \sum_{j \in J_{2n}} a_j + \sum_{\substack{i \in I_{2n} \\ i \neq k_{2n}}} a_i^- \geqslant \beta, \tag{4}$$

where $I_0 := [1, m]$, $J_n := \bigcup_{i=0}^{n-1} I_i$ and

$$k_n := \begin{cases} \max\{k \in I_n \colon a_k > 0\} & \text{for} \quad n \in (2\mathbb{N} - 1), \\ \max(I_n \cap A^-) & \text{for} \quad n \in 2\mathbb{N}. \end{cases}$$

Additionally, we set

$$L_{2n-1} := I_{2n-1} \cap A^+, \quad L_{2n} := I_{2n} \cap A^- \quad \text{and} \quad l_n := \operatorname{card} L_n,$$

for each $n \in \mathbb{N}$. We note that $\lim_{n\to\infty} l_n = \infty$. Moreover, if L_n is a union of λ_n **MSI** then, keeping in mind the conditional convergence of series $\sum a_n$ as well as conditions (3) and (4), we easily verify that also $\lim_{n\to\infty} \lambda_n = \infty$.

Permutation q is defined to be an increasing map of sets:

$$I_0 \cup \bigcup_{n\in\mathbb{N}} [\min I_n, l_n + \min I_n) \quad \text{and} \quad \bigcup_{n\in\mathbb{N}} [l_n + \min I_n, \max I_n]$$

onto sets:

$$I_0 \cup \bigcup_{n\in\mathbb{N}} L_n \quad \text{and} \quad \bigcup_{n\in\mathbb{N}} (I_n \setminus L_n),$$

respectively. One can easily verify that we then have

$$q\big([\min I_n, l_n + \min I_n)\big) = L_n, \quad \text{for every} \quad n \in \mathbb{N},$$

and the set $q^{-1}(I)$ is a union of at most three **MSI** for every interval I, i.e., $q \in \mathfrak{DC}$ and $\mathbf{c}(\mathbf{q}^{-1}) \leqslant 3$. In turn, from (3) and (4), it follows that $\sigma a_{q(n)} = [\beta, \gamma]$. However, it is not difficult to see that the above proof readily applies to the cases $\alpha = \beta$ or $\alpha = \gamma$ (for a nondegenerated interval $[\beta, \gamma]$).

Our next goal is to construct a permutation $q \in \mathfrak{DC}$ such that $\sum a_{q(n)} = +\infty$. Let $\{I_n\}$ be an increasing sequence of intervals of $\mathbb{N}$ such that $\bigcup I_n = \mathbb{N}$ and

$$\sum_{i\in I_n} a_i^+ \geqslant n, \tag{5}$$

for each $n \in \mathbb{N}$. Moreover, let

$$\lambda_i := \operatorname{card}(I_i \cap A^+) \quad \text{and} \quad \mu_i := \operatorname{card}(I_i \cap A^-), \qquad i \in \mathbb{N};$$

and

$$J_{n+1} := \left(\sum_{i=1}^{n+2} \lambda_i + \sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n+2} \lambda_i + \sum_{i=1}^{n+1} \mu_i\right] \quad \text{for each } n \in \mathbb{N}.$$

We define the permutation q as an increasing map of sets:

$$[1, \lambda_1 + \lambda_2] \cup \bigcup_{n\in\mathbb{N}} \left(\sum_{i=1}^{n+1} \lambda_i + \sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n+2} \lambda_i + \sum_{i=1}^{n} \mu_i\right]$$

and

$$(\lambda_1 + \lambda_2, \lambda_1 + \lambda_2 + \mu_1] \cup \bigcup_{n=2}^{\infty} J_n$$

onto sets A^+ and A^-, respectively.

One can verify that the permutation q possesses the following properties. First, we note that set $q^{-1}(I)$ is a union of at most five **MSI** for each interval $I \subset \mathbb{N}$, that is $q^{-1} \in \mathfrak{C}$ and $\mathbf{c}(\mathbf{q}^{-1}) \leqslant 5$. Next, from condition (5), we obtain the following estimation

$$\sum_{i=1}^{\max J_n} a_{q(i)} = \sum_{i<\min J_n} a_i + \sum_{i\in J_n} a_i^+ \geqslant n + \sum_{i<\min J_n} a_i, \quad \text{correct for every} \quad n \in \mathbb{N}.$$

Furthermore, it follows from the definition of q that the sequence

$$\left\{\sum_{i=1}^{j} a_{q(i)} \,:\, \min J_n \,<\, j \,\leqslant\, \max J_n\right\} \quad \text{is non-increasing for each } n \in \mathbb{N},$$

whereas the sequence

$$\left\{\sum_{i=1}^{j} a_{q(i)} : \max J_n < j < \min J_{n+1}\right\} \quad \text{is increasing for each } n \in \mathbb{N}.$$

From the last three properties, we receive $\sum a_{q(n)} = +\infty$. In consequence, we have $q \in \mathfrak{D}\mathfrak{C}$.

Let $\beta \in \mathbb{R}$. We want to define a permutation $q \in \mathfrak{D}\mathfrak{C}$ such that $\sigma a_{q(n)} = [\beta, \infty]$. For this purpose, it is enough to choose by induction the increasing sequence $\{I_n\}$ of intervals of $\mathbb{N}$ forming the partition of $\mathbb{N}$ such that for each $n \in \mathbb{N}$ the following conditions are satisfied

$$\sum_{j \in J_{2n-1}} a_{q(j)} \geqslant \beta + n \quad \text{and} \quad \sum_{j \in J_{2n}} a_{q(j)} \leqslant \beta, \tag{6}$$

while

$$\sum_{\substack{j \in J_{2n} \\ j < \max J_{2n}}} a_{q(j)} \geqslant \beta,$$

where $J_n := \bigcup_{i=0}^{n} I_i$, whereas q is defined to be the increasing map of sets $\bigcup_{n \in \mathbb{N}} I_{2n-1}$ and $\bigcup_{n \in \mathbb{N}} I_{2n}$ onto sets A^+ and A^-, respectively. Then, it is easy to verify that $q^{-1} \in \mathfrak{C}$ and $\mathbf{c}(\mathbf{q^{-1}}) \leqslant 5$, and, since from (6) it results that $\sigma a_{q(n)} = [\beta, \infty]$, we have $q \in \mathfrak{D}\mathfrak{C}$.

Definition of permutation $q \in \mathfrak{D}\mathfrak{C}$, such that $\sigma a_{q(n)} = \mathbb{R}^*$, is the same as in the previously discussed case of $\sigma a_{q(n)} = [\beta, \infty]$, $\beta \in \mathbb{R}$, only condition (6) needs to be changed. More precisely, condition (6) must be replaced by the following conditions:

$$\sum_{j \in J_{2n-1}} a_{q(j)} \geqslant n \quad \text{and} \quad \sum_{j \in J_{2n}} a_{q(j)} \leqslant -n,$$

for each $n \in \mathbb{N}$.

Definition of permutation $p \in \mathfrak{D}\mathfrak{D}$, which we will also make dependent on the form of set $\sigma a_{p(n)}$, must be preceded by some essential preparations. The main charge of those preparations lies in the appropriate selection of increasing sequence $\{I_n : n \in \mathbb{N}_0\}$ of the intervals of $\mathbb{N}$ forming the partition of $\mathbb{N}$. We denote as s and t the increasing bijections of $\mathbb{N}$ onto sets

$$\mathbf{D} := \bigcup_{n \in \mathbb{N}} I_{2n-1} \quad \text{and} \quad \mathbf{E} := \bigcup_{n \in \mathbb{N}_0} I_{2n},$$

respectively. We request the intervals I_{2n}, $n \in \mathbb{N}_0$, to be composed of the even number of elements and so that $\operatorname{card} I_{2n} \to \infty$ for $n \to \infty$. Moreover, we demand the series $\sum a_{s(n)} = \sum_{n\in\mathbf{D}} a_n$ and $\sum a_{t(n)} = \sum_{n\in\mathbf{E}} a_n$ both to be convergent, in addition, the first one should be conditionally convergent and the second series $\sum a_{t(n)}$ should be absolutely convergent (and therefore insusceptible to permutations). Let $d := \sum a_{t(n)}$.

Restriction of permutation p to set $\mathbf{E}$ will be defined in the same way in all of the cases discussed below. So, we take that

$$p(2i-2+\min I_{2n}) = i-1+\min I_{2n}$$

and

$$p(2i-1+\min I_{2n}) = i-1+\frac{1}{2}\operatorname{card} I_{2n}+\min I_{2n},$$

for $i = 1,2,\dots,\frac{1}{2}\operatorname{card} I_{2n}$ and for each $n\in\mathbb{N}_0$. It implies that for each case we have $p^{-1}\in\mathfrak{D}$.

First, let us consider the case in which $\sigma a_{p(n)}$ is a nondegenerated closed interval. Let $a,b\in\mathbb{R}$, $a<b$ and $d<b$ (the last condition does not violate the generality of considerations because one can assume d to be any real number). Additionally, we assume that

$$|a_j| < \frac{1}{2}|b-a| \qquad \text{for every} \quad j\in\mathbf{D}. \tag{7}$$

We define by induction an increasing sequence $\{J_n\}$ of intervals of $\mathbb{N}$ forming the partition of $\mathbb{N}$ such that

$$\mu\left(\bigcup_{n\in\mathbb{N}} J_{2n-1}\right) = \mathbf{D}\cap A^+, \quad \mu\left(\bigcup_{n\in\mathbb{N}} J_{2n}\right) = \mathbf{D}\cap A^-,$$

$$\begin{cases} \sum\limits_{j\in K_{2n-1}} a_{\mu(j)} \geqslant b-d, \quad \text{while} \quad \sum\limits_{\substack{j\in K_{2n-1}\\ j<\max K_{2n-1}}} a_{\mu(j)} \leqslant b-d, \\ \text{and} \\ \sum\limits_{j\in K_{2n}} a_{\mu(j)} \leqslant a-d, \quad \text{while} \quad \sum\limits_{\substack{j\in K_{2n}\\ j<\max K_{2n}}} a_{\mu(j)} \geqslant a-d, \end{cases} \tag{8}$$

for each $n\in\mathbb{N}$, where $K_n := \bigcup_{j=1}^n J_j$ and $\mu := ps$. The restriction of p to set $\mathbf{D}$ is defined to be the increasing map of sets

$$s\left(\bigcup_{n\in\mathbb{N}} J_{2n-1}\right) \quad \text{and} \quad s\left(\bigcup_{n\in\mathbb{N}} J_{2n}\right)$$

onto sets $\mathbf{D}\cap A^+$ and $\mathbf{D}\cap A^-$, respectively. Then, what can be easily verified, we have $\sigma a_{p(n)} = [a,b]$ which implies, in particular, that $p\in\mathfrak{D}$.

Let $a \in \mathbb{R}$. For $\sigma a_{p(n)} = [a, \infty]$ or $\sigma a_{p(n)} = \{+\infty\}$, the construction of permutation p is almost analogous as shown before, only conditions (7) and (8) must be modified. So, for $\sigma a_{p(n)} = [a, \infty]$, $a \in \mathbb{R}$, instead of condition (7), we assume that $|a_j| < \frac{1}{2}$, for every $j \in \mathbf{D}$, whereas (8) is replaced, for each $n \in \mathbb{N}$, with the following condition

$$\begin{cases} \sum\limits_{j\in K_{2n-1}} a_{\mu(j)} \geqslant \max\{n, 2a\}, & \text{while} \quad \sum\limits_{\substack{j\in K_{2n-1} \\ j<\max K_{2n-1}}} a_{\mu(j)} \leqslant \max\{n, 2a\}, \\ \text{and} & \\ \sum\limits_{j\in K_{2n}} a_{\mu(j)} \leqslant a, & \text{while} \quad \sum\limits_{\substack{j\in K_{2n} \\ j<\max K_{2n}}} a_{\mu(j)} \geqslant a. \end{cases} \tag{9}$$

For the case $\sigma a_{p(n)} = \{+\infty\}$, we request that $|a_j| < 1$, for each $j \in \mathbf{D}$, whereas (8) is replaced, for each $n \in \mathbb{N}$, with the condition given below

$$\begin{cases} \sum\limits_{j\in K_{2n-1}} a_{\mu(j)} \geqslant 2n, & \text{while} \quad \sum\limits_{\substack{j\in K_{2n-1} \\ j<\max K_{2n-1}}} a_{\mu(j)} \leqslant 2n, \\ \text{and} & \\ \sum\limits_{j\in K_{2n}} a_{\mu(j)} \leqslant n, & \text{while} \quad \sum\limits_{\substack{j\in K_{2n} \\ j<\max K_{2n}}} a_{\mu(j)} \geqslant n. \end{cases} \tag{10}$$

□

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