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CHAOS AND STABILITY IN SOME RANDOM DYNAMICAL SYSTEMS

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ABSTRACT. Nonchaotic behavior in the sense of Li and Yorke chaos in discrete dynamical systems generated by a continuous selfmapping of a real compact interval means that every trajectory can be approximated by a periodic one. Stability of this behavior was analyzed also for dynamical systems with small random perturbations. In this paper we study similar properties for nonautonomous periodic dynamical systems with random perturbations and for random dynamical systems generated by two continuous maps and their perturbations.

1. Introduction

One dimensional discrete dynamical system is given by a pair (I, f), where I is real compact interval, $f : I \to I$ is continuous, and by difference equation

$$x_{n+1} = f(x_n),$$

where $x_0 \in I$. Li and Yorke chaos in these systems can be characterized in several ways. The following characterization is from [6].

Theorem 1.1. *Let $f\colon I \to I$ be continuous, I real compact interval. Dynamical system (I, f) is nonchaotic if and only if for any $x \in I$ and $\varepsilon > 0$ there is a periodic point $p \in I$ such that*

$$\lim_{n\to\infty} \sup |f^n(x) - f^n(p)| < \varepsilon.$$

Recall that point $p \in I$ is periodic if $f^m(p) = p$ for some m, f^m denotes the mth iterate of f. The behavior from Theorem 1.1 need not be stable with respect to small perturbations: simple example is a map f with an interval of fixed (or periodic) points. Perturbed trajectories $\{x_n\}$ can be dense in this interval (and cause chaotic behavior). There are also conditions which guarantee stability of nonchaotic behavior with respect to small perturbations or in the case

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of a small random perturbations of the system. The original definition of nonchaotic stable behavior can be found in [9], the following one is from [7] and considers random perturbations of the system.

Definition 1. Dynamical system (I, f) is nonchaotic stable with respect to random perturbations ε_n if for any $\varepsilon > 0$ and $x \in I$ there is a periodic point p of f such that

$$P\Big(\lim_{n\to\infty} \sup |X_n(x) - f^n(p)| < \varepsilon\Big) = 1,$$

where

$$X_n(x) = f(X_{n-1}) + \varepsilon_n \tag{1}$$

with $X_0 = x$ and ε_n random perturbations.

Remark 1. For precise definition of (1) see [7] or Section 2.

When trajectories $\{f^n(x)\}$ are subjected to identically distributed independent small random perturbation with uniform distribution, there will always be some intervals where the system behaves chaotically. However, sometimes lengths of these intervals may be arbitrarily small when perturbations are sufficiently small. In [5] it was shown that this property is generic in the space $C(I, I)$ of all continuous functions $I \to I$ with uniform metric. In Section 2 we show that some results from [5] are preserved also for nonautonomous periodic dynamical systems with small random perturbations. In Section 3 we look at these systems as at special random dynamical systems and formulate some conditions when these random dynamical systems exhibit properties similar to the nonchaotic behavior.

2. Nonautonomous periodic dynamical systems with random perturbations

If we consider a sequence of continuous maps $\{f_n\}$, $f_n\colon I \to I$, I real compact interval, a generalization of discrete dynamical system is given by $\big(I, \{f_n\}\big)$ and equation

$$x_{n+1} = f_n(x_n),$$

where $x_0 \in I$. This is called nonautonomous discrete dynamical system and of course, in special case $f_n = f$ we obtain one dimensional discrete dynamical system. A survey of results on nonautonomous systems can be found in [3], such systems have been studied also in, e.g., [8], [10]. In general, nonchaotic properties of nonautonomous discrete dynamical systems cannot be derived from nonchaotic properties of generating functions. However, in several papers these problems were studied for special classes of nonautonomous systems, e.g., periodic ones or such that the sequence of generating functions converges to a given function. In this section we consider periodic nonautonomous systems.

Definition 2. Let $I = [a, b]$ be real compact interval and consider a nonautonomous periodic dynamical system generated by continuous maps $f_1, \dots, f_p : I \to I$, i.e.,

$$x_{n+1} = f_{(n \bmod p)+1}(x_n) \tag{2}$$

for $x_0 \in I$.

For any $S \subset R$ let $O_\delta(S)$ denote the set

$$O_\delta(S) = \{x \in R : d(x, S) < \delta\},$$

where

$$d(x, S) = \inf_{y \in S} \mid x - y \mid .$$

To consider random perturbations of such systems take $\delta > 0$ and the following extensions of $f_i, i = 1, \dots, p$ to the interval $O_\delta(I)$

$$f_i(x) = \begin{cases} f_i(a) & \text{if } x \in [a - \delta, a], \\ f_i(x) & \text{if } x \in [a, b], \\ f_i(b) & \text{if } x \in [b, b + \delta]. \end{cases} \tag{3}$$

Now let $\varepsilon_n, n = 1, 2, \dots$ be independent identically distributed random variables with support in $[-\delta, \delta]$. To describe a nonautonomous dynamical system generated by continuous maps $f_1, \dots, f_p \colon I \to I$, I real compact interval, we follow a technique from, e.g., [4], where the map $\mathcal{F} \colon \{1, \dots, p\} \times I \to \{1, \dots, p\} \times I$ was considered

$$\mathcal{F}(i, x) = \big((i + 1) \mod p, f_i(x)\big).$$

In the following we shall study nonautonomous discrete dynamical system with small random perturbations

$$X_{n+1} = f_{(n \bmod p)+1}(X_n) + \varepsilon_n, \tag{4}$$

where $X_0 = x_0 \in I$. To work with these systems we take $\mathcal{F}^*$ defined by

$$\mathcal{F}^*(i, x) = \Big((i + 1) \bmod p, O_\delta\big(f_i(x)\big)\Big).$$

Consequently, for $S \subset \{1, \dots, p\} \times I$,

$$\mathcal{F}^*(S) = \bigcup_{(i,x) \in S} \Big\{(i + 1) \bmod p, y), y \in O_\delta\big(f_i(x)\big)\Big\}.$$

Consider the set

$$\Gamma(i, J) = \bigcap_{m=1}^{\infty} \overline{\bigcup_{n=m}^{\infty} \mathcal{F}^{*n}(i, J)}.$$

The set $\Gamma(i, J)$ is closed. To show that $\mathcal{F}^*\big(\Gamma(i, J)\big) = \Gamma(i, J)$, we prove the following lemma.

Lemma 2.1. *For any nonincreasing sequence of closed sets $\{S_k\}_{k=1}^{\infty}$, where $S_k \subset \{1,\dots,p\} \times I$ we have*

$$\mathcal{F}^*\left(\bigcap_{k=1}^{\infty} S_k\right) = \bigcap_{k=1}^{\infty} \mathcal{F}^*(S_k).$$

P r o o f. Let $(i,x) \in \bigcap_{k=1}^{\infty} \mathcal{F}^*(S_k)$. Since for any k, $(i,x) \in \mathcal{F}^*(S_k)$, there is a sequence y_k, δ_k, $k = 1,2,\dots$ such that

$$(i,x) = (i, y_k + \delta_k),$$

where

$$y_k = f_{i-1}(x_k)$$

and $\delta_k < \delta$ such that $(i-1, x_k) \in S_k$. We can choose k_n such that $x_{k_n} \to x_0$ and $\delta_{k_n} \to \delta$ for $n \to \infty$. We have $(i, x_0) \in S_k$ for any k. Using continuity of f_i: $f_{i-1}(x_{k_n}) \to f_{i-1}(x_0) = y_0$ for $n \to \infty$ and

$$(i,x) = (i, y_0 + \delta).$$

Since $(i, y_0) \in \mathcal{F}(\bigcap_{k=1}^{\infty} S_k)$, we have $(i,x) \in \mathcal{F}^*(\bigcap_{k=1}^{\infty} S_k)$. Now let $(i,x) \in \mathcal{F}^*(\bigcap_{k=1}^{\infty} S_k)$. We have

$$(i,x) = (i, y + \delta),$$

where $(i,y) \in \mathcal{F}(\bigcap_{k=1}^{\infty} S_k)$. Then $y = f_{i-1}(z) + \delta$ for $(i-1, z) \in \bigcap_{k=1}^{\infty} S_k$ and $(i,x) \in \mathcal{F}^*(S_k)$ for any k. $\square$

Lemma 2.2. *For any $i = 1,\dots,p$ and $J \subset I$: $\mathcal{F}^*\big(\Gamma(i,J)\big) = \Gamma(i,J)$.*

P r o o f. Using Lemma 2.1 for $S_k = B_k(i,J) = \overline{\bigcup_{m=k}^{\infty} \mathcal{F}^{*n}(i,J)}$ we have

$$\begin{aligned}
\mathcal{F}^*\big(\Gamma(i,J)\big) &= \mathcal{F}^*\left(\bigcap_{k=1}^{\infty} B_k(i,J)\right) \\
&= \bigcap_{k=1}^{\infty} \mathcal{F}^*\big(B_k(i,J)\big) \\
&= \bigcap_{k=1}^{\infty} B_{k+1}(i,J) \\
&= \Gamma(i,J).
\end{aligned}$$

$\square$

Since $\{X_n\}_{n=1}^{\infty}$ is a Markov process, the following lemma from [5] can be applied.

Lemma 2.3. *Let $\{X_n\}$ be a Markov process with values in S. Let $B \subset S$ be a Borel set. Suppose that there exist $r \in N$ and $\lambda \in (0,1]$ with the property that for any j the probability $P(X_{j+i} \in B$ for some $i \le r) \ge \lambda$. Then $P(X_n \in B$ for infinitely many $n) = 1$.*

The system $\{\Gamma(i,J),\ i = 1,\dots,p,\ J \subset I\}$ is partially ordered by inclusion and contains at least one minimal element. Let $\mathcal{Z}$ be the system of all minimal elements of this system, for $S \in \mathcal{Z}$ denote

$$S^0 = \{x \in I; (i,x) \in S \text{ for some } i\}.$$

Let $\{X_n\}$ be given by (4) and such that $x_0 \in S^0$. Since S is minimal, $\Gamma(i,J) = S$ for any $(i,J) \subset S$. We show that there is $r \in N$ and $\lambda \in (0,1]$ with the property that for any interval K, $K \subset S^0$ the probability $P(X_n \in K$ for some $n \le r) \ge \lambda$. Denote K_0 the middle third of the interval K and ν its length. We can assume that $(i,K_0) \in S$ and consequently $(\mathcal{F}^*)^{k(J)}(i,J) \cap (i,K_0) \ne \emptyset$ for some $k(J)$ and any $(i,J) \subset S$. Let

$$r = \max\{k(J), J \text{ contains an interval of length } 2\delta\}.$$

Since $f_1,\dots,f_p$ are continuous, for $\nu > 0$ there is $\delta_0 > 0$ such that any two realizations of the process (4) for $n \le r$ and $\varepsilon_n < \delta_0$ differ less than $\frac{\nu}{3}$. Since some realization of the process (4) of length $\le r$ belongs to K_0, there is $\lambda > 0$ such that $P(X_n \in K$ for some $k \le r) \ge \lambda > 0$. Using Lemma 2.3 we obtain that the behavior of the system can now be described by the following two theorems.

Theorem 2.4. *Let $S \in \mathcal{Z}$ and $(i,x_0) \in S$ and $K \subset S^0$ be an interval. Then for any process $\{X_n\}$ defined by (4) we have:*

- *$P(X_n \in K$ for infinitely many $n) = 1$,*
- *$P(X_n \in S^0$ for all $n) = 1$.*

Theorem 2.5. *For any $x_0 \in I$ and the process $\{X_n\}$ defined by (2) there is $S \in \mathcal{Z}$ and $n_0 \ge 0$ such that $P(X_n \in S^0$ for all $n \ge n_0) = 1$.*

Any connected component of $S \in \mathcal{Z}$ has length greater than 2δ. Let $m(X_n) = m(f_1,\dots,f_p,\delta)$ be the length of maximal connected component of $S \in \mathcal{Z}$. Consider the set

$$\mathcal{F}(\varepsilon) = \{(f_1,\dots,f_p); \text{there is } \delta > 0 \text{ such that } m(f_1,\dots,f_p,\delta) < \varepsilon\}.$$

Similarly as in [5] it is possible to obtain the following result.

Theorem 2.6. *For any $\varepsilon > 0$ the set $\mathcal{F}(\varepsilon)$ is open and dense in $C(I,I) \times C(I,I) \times \dots \times C(I,I)$.*

P r o o f. Consider the following system of intervals

$$S_k = \Big\{\big[(2i-1)/2^k, (2i)/2^k\big];\ i = 1,\dots,2^{k-1}\Big\}$$

and take the set of functions $\mathcal{F}_k \subset C(I,I) \times \cdots \times C(I,I)$ with the property that for $(f_1, \dots f_p) \in \mathcal{F}_k$: f_i, $i = 1, \dots, p$ is constant on any interval from S_k and this constant is equal to center of some interval in S_k. If $x_0 \in \bigcup S_k$ and $(f_1, \dots, f_p) \in \mathcal{F}_k$, then X_n given by (4) is in $\bigcup S_k$ for any n. Further, let $\mathcal{G}_k \supset \mathcal{F}_k$ be such that for any $(g_1, \dots, g_p) \in \mathcal{G}_k$ there is $(f_1, \dots, f_p) \in \mathcal{F}_k$ with $\|f_i - g_i\| < \frac{1}{2^{k+2}}$; $i = 1, \dots, p$. Then for any $(g_1, \dots, g_p) \in \mathcal{G}_k$ and X_n given by (4), where $x_0 \in \bigcup S_k$, $\delta < \frac{1}{2^{k+2}}$, X_n is in $\bigcup S_k$ for any n. If $x_0 \in I$ is arbitrary, then either $X_n \in \bigcup S_k$ for some n or $X_n \notin \bigcup S_k$ for any n. In both cases we obtain $m(X_n) < \frac{1}{2^k}$. Let $\varepsilon > 0$. For any $(f_1, \dots, f_p) \in C(I,I) \times \cdots \times C(I,I)$ and $\xi > 0$ there is $(g_1, \dots, g_p) \in \mathcal{G}_k$, where $\varepsilon < \frac{1}{2^{k+2}}$ and $\|g_i - f_i\| < \xi$. Hence for any X_n generated by $(g_1, \dots, g_p)$, $\delta < \frac{1}{2^{k+2}}$ we have $m(X_n) < \varepsilon$ which proves the density of $\mathcal{F}(\varepsilon)$. □

Remark 2. Systems generated by $(f_1, \dots, f_p) \in \bigcap_{\varepsilon > 0} \mathcal{F}_\varepsilon$ exhibit the following property: There are intervals where trajectories are dense but the lengths of these may be arbitrarily small when perturbations are sufficiently small.

3. Random dynamical system generated by two nonchaotic maps

Similar properties may be considered in a random dynamical systems. For definition of random dynamical system see, e.g., [1] or [2]. We consider random dynamical systems generated by small random perturbations of two continuous maps f and g defined by

$$X_{n+1} = F_n(X_n) + \varepsilon_n, \tag{5}$$

where $X_0 = x_0 \in I$ and $F_1, F_2, \dots$, $\epsilon_1, \epsilon_2, \dots$ are mutually independent with distribution

$$Q(F_n = f_i, \varepsilon_n \in K) = p_i \nu(K) \tag{6}$$

for $i = 1, 2$, where $f_1 = f$, $f_2 = g$, $p_1 + p_2 = 1$, $p_1, p_2 \geq 0$ and ν is a probability measure, $\nu(-\delta, \delta) = 1$.

Example. If $f(x) = \frac{x}{2}$ and $g(x) = \frac{x+1}{2}$, then every trajectory of discrete dynamical system generated by f or g converges to fixed point of f, g, respectively. However, for $\{X_n\}$ given by (5) and any interval $K \subset I$ we have $P(X_n \in K \text{ for infinitely many } n) = 1$ for $p_1 = p_2$, uniform distribution of ε_n and any $\delta > 0$.

To get a simpler behavior of the system, the following conditions can be used. Here $S_{f,x}$ denotes the set S^0 from Theorem 2.5 for X_n given by (4) with $f_n = f$ and $x_0 = x$.

Theorem 3.1. *Assume that for all $x \in I$, $S_{f,x} = M_2$, $S_{g,x} = M_1$, $f(M_1 \cup M_2) \subset M_1 \cup M_2$ and also $g(M_1 \cup M_2) \subset M_1 \cup M_2$ and let X_n be defined by* (5)*. Then there is n_0 such that $P(X_n \in M_1 \cup M_2$ for all $n \geq n_0) = 1$ and for any interval $K \subset M_1 \cup M_2$ we have $P\big(\bigcup_{n=1}^{\infty}(X_n \in K) = 1\big)$.*

P r o o f. We show that $P(X_n \in M_1 \cup M_2$ for all sufficiently large $n) = 1$. Denote Y_n corresponding system generated by perturbations of f, i.e., the system (4), where $f_n = f$ and similarly Z_n the system obtained considering only perturbations of g. Then $P(X_n = Y_n$ for $n \leq m) = p_1^m$. By Theorem 2.5 there is m such that $P(Y_n \in M_2$ for all $n \geq m) = 1$. Using Lemma 2.3 since there is m and $\lambda > 0$ such that $P(X_n \in M_2$ for $n \leq m) \geq \lambda$ we obtain that $P(X_n \in M_2$ for infinitely many $n) = 1$. Since $M_1 \cup M_2$ is invariant for the process we obtain that $P(X_n \in M_1 \cup M_2$ for all sufficiently large $n) = 1$. Let $K \subset M_1 \cup M_2$ be any interval and let $x_0 \in M_1 \cup M_2$. Without loss of generality we may assume that $K \subset M_1$ and $x_0 \in M_2$. From the proof of Theorem 2.5 we have there are m and $\lambda > 0$ such that $P(Z_n \in K$ for some $n \leq m) \geq \lambda$. Further $P(X_n = Z_n$ for all $n \leq m) = p_2^m$. Consequently, there is $\lambda_1 = \lambda p_2^m$ such that $P(X_n \in K$ for some $n \leq m) \geq \lambda_1$ and using Lemma 2.3 we obtain $P(X_n \in K$for infinitely many $n) = 1$. $\square$

Remark 3. Using this result it is possible to give classes of systems which behave nonchaotically in the sense of Remark 2. It can also be seen that the condition from Theorem 3.1 is necessary to obtain a system with such behavior.

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