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THE INCLUSION-EXCLUSION PRINCIPLE WITHOUT DISTRIBUTIVITY

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ABSTRACT. Inspired by the article of P. Grzegorzewski [*The inclusion-exclusion principle for* IF*-events,* Inform. Sci. **181** (2011), 536–546], who has worked two generalizations of the inclusion-exclusion principle for IF-events, a generalization of the inclusion-exclusion principle for mappings with values in semigroups is presented here. The main idea is in replacing the distributivity and idempotency laws, by one new axiom.

1. Introduction

P. Grzegorzewski gave two generalizations of the inclusion-exclusion principle in [3] for IF-events applying two definitions for the union of IF-events. As a reaction on this paper we have proved the principle for mappings from the set of IF-sets to the semigroup [4]. We have worked on generalizations of his theorem. First, we gave a theorem of the inclusion-exclusion principle on semigroups in [4]. The same we have made for the special case, the mapping from the set of IF-sets to the unit interval. Continuing with this topic, another method of the proof of the inclusion-exclusion principle for mappings with values in semigroups is presented in this paper. The conditions of distributivity and idempotency were replaced by the new axiom.

The classical inclusion-exclusion principle states

$$\begin{aligned}\mathcal{P}\left(A_1 \cup \ldots \cup A_n\right) = &\sum_{i=1}^{n} \mathcal{P}\left(A_i\right) - \sum_{i<j}^{n} \mathcal{P}\left(A_i \cap A_j\right) \\ &+ \sum_{i<j<k}^{n} \mathcal{P}\left(A_i \cap A_j \cap A_k\right) - \cdots + (-1)^{n+1} \mathcal{P}\left(\bigcap_{i=1}^{n} A_i\right)\end{aligned}$$

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for any sequence $A_1, \ldots, A_n$ from the domain of $\mathcal{P}\colon \mathcal{R} \to R$ (where $\mathcal{P}$ is non--negative, additive function in probability theory).

P. Grzegorzewski has defined the probability on IF-event A by the formula

$$\mathcal{P}(A) = \left[\int_\Omega \mu_A dP, 1 - \int_\Omega \nu_A dP \right],$$

where $\mathcal{P}$ is the probability measure through Ω.

This inclusion-exclusion principle has the form [3]:

$$\mathcal{P}\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n \mathcal{P}(A_i) - \sum_{i<j} \mathcal{P}(A_i \cap A_j) + \sum_{i<j<k} \mathcal{P}(A_i \cap A_j \cap A_k) - \cdots$$
$$\cdots + (-1)^{n+1} \mathcal{P}(A_1 \cap A_2 \cap \ldots \cap A_n);$$

$$\mathcal{P}\left(\oplus_{i=1}^n A_i\right) = \sum_{i=1}^n \mathcal{P}(A_i) - \sum_{i<j} \mathcal{P}(A_i \odot A_j) + \sum_{i<j<k} \mathcal{P}(A_i \odot A_j \odot A_k) - \cdots$$
$$\cdots + (-1)^{n+1} \mathcal{P}(A_1 \odot A_2 \cap \ldots \odot A_n).$$

Other authors (Lavinia Ciungu, Beloslav Riečan and Mária Kuková) have worked this topic on IF-sets and IF-events by different methods and using different operators. We mention a brief summary of their results. The axiomatic definition of probability on IF-sets given in [6] is the following.

A mapping $m\colon \mathcal{F} \longrightarrow [0,1]$ is an IF-state if the following properties are satisfied:

1. $m(1_\Omega, 0_\Omega) = 1$, $m(0_\Omega, 1_\Omega) = 0$;
2. $m(A \oplus B) = m(A) + m(B) - m(A \odot B)$ for all $A, B \in \mathcal{F}$;
3. $A_n \nearrow A$ implies $m(A_n) \nearrow m(A)$.

Let $m\colon \mathcal{F} \longrightarrow [0,1]$ be an IF-state. Then there are probability measures

$$P, Q\colon \mathcal{S} \longrightarrow [0,1] \quad \text{and} \quad \alpha \in [0,1]$$

such that

$$m(A) = \int_\Omega \mu_A dP + \alpha \left(1 - \int_\Omega (\mu_A + \nu_A)\, dQ\right) \quad \text{for all } A \in \mathcal{F}.$$

The result of L. Ciungu [2] using Łukasziewicz operators is following. Let A_i be IF-events, $A_i = (\mu_{A_i}, \nu_{A_i})$, $i = 1, \ldots, n$. Let m be an IF-state and

$$m(A) = \int_\Omega \mu_A dP + \alpha \left(1 - \int_\Omega (\mu_A + \nu_A)\, dQ\right).$$

Then m satisfies the inclusion-exclusion principle

$$m\left(\bigoplus_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} m\left(A_i\right) - \sum_{i<j}^{n} m\left(A_i \odot A_j\right) + \cdots + (-1)^{n+1} m\left(\bigodot_{i=1}^{n} A_i\right).$$

In the proof of L. Ciungu representation theorem from [1] was used.

Another representation theorem by Riečan and Ciungu [1], corresponding to axiomatic definition of probability from Riečan [7] was used in the proof of M. Kuková [5]:

Any probability $\overline{\mathcal{P}}\colon \mathcal{F} \to \mathcal{J}$ can be expressed by the formulas:

$$\overline{\mathcal{P}}(A) = \left[\overline{\mathcal{P}^{\flat}}(A), \overline{\mathcal{P}^{\sharp}}(A)\right]$$

and

$$\overline{\mathcal{P}^{\flat}}(A) = \int_X \mu_A dP + \alpha \int_X (1 - \mu_A - \nu_A)\, dQ,$$

$$\overline{\mathcal{P}^{\sharp}}(A) = \int_X \mu_A dR + \beta \int_X (1 - \mu_A - \nu_A)\, dS.$$

Kuková [5] proved:

$$\overline{\mathcal{P}}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \overline{\mathcal{P}}\left(A_i\right) - \sum_{i<j} \overline{\mathcal{P}}\left(A_i \cap A_j\right)$$
$$+ \sum_{i<j<k} \overline{\mathcal{P}}\left(A_i \cap A_j \cap A_k\right) - \cdots + (-1)^{n+1} \overline{\mathcal{P}}\left(A_1 \cap A_2 \cap \ldots \cap A_n\right).$$

Kuková used also the Łukasiewicz connectives:

$$A \oplus B = \big((\mu_A + \mu_B) \wedge 1, (\nu_A + \nu_B - 1) \vee 0\big),$$
$$A \odot B = \big((\mu_A + \mu_B - 1) \vee 0, (\nu_A + \nu_B) \wedge 1\big)$$

to prove another form of the principle [5], that is

$$\overline{\mathcal{P}}\left(\bigoplus_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \overline{\mathcal{P}}(A_i) - \sum_{i<j} \overline{\mathcal{P}}\left(A_i \odot A_j\right)$$
$$+ \sum_{i<j<k} \overline{\mathcal{P}}\left(A_i \odot A_j \odot A_k\right) - \cdots + (-1)^{n+1} \overline{\mathcal{P}}\left(A_1 \odot A_2 \cap \ldots \odot A_n\right).$$

2. Inclusion-exclusion principle without distributivity for mapping with values in semigroups

As it is mentioned in the introduction, we skip the conditions of distributivity and idempotency in the algebraic system $(\mathcal{G},+,\cdot)$, where " $\cdot$ " is commutative binary operation, and $(\mathcal{G},+)$ is a commutative semigroup.

The only required condition is the axiom

$$m\big((a+b)\cdot c\big)+m\,(a\cdot b\cdot c)=m\,(a\cdot c)+m\,(b\cdot c)\,.$$

Example 2.1. Let $\mathcal{S}$ be a σ-algebra of subsets of a set X. Let H be a linear vector Banach Riesz space. Then, any vector measure $m:\mathcal{S}\to H$ satisfies the above condition.

Assumptions 2.2.

- *$(\mathcal{G},+,\cdot)$ is an algebraic system, where $(\mathcal{G},+)$ is a commutative semigroup and " $\cdot$ " is commutative binary operation,*
- *$m:\mathcal{G}\to H$ is a mapping from the algebraic system $(\mathcal{G},+,\cdot)$ to the commutative semigroup $(H,+)$, satisfying the valuation property*

$$m(a+b)+m(a\cdot b)=m(a)+m(b), \tag{I}$$

- *there holds the axiom*

$$m\big((a+b)\cdot c\big)+m\,(a\cdot b\cdot c)=m\,(a\cdot c)+m\,(b\cdot c)\,. \tag{II}$$

Theorem 2.3. *For n even*

$$m\left(\sum_{k=1}^{n}a_k\right)+\sum_{k=1}^{n/2}\sum_{1\le i_1<i_2<\cdots<i_{2k}\le n}m\big(a_{i_1}\cdot a_{i_2}\cdots\ a_{i_{2k}}\big)$$

$$=\sum_{k=1}^{n/2}\sum_{1\le i_1<i_2<\cdots<i_{2k-1}\le n}m\left(a_{i_1}\cdot a_{i_2}\cdots\ a_{i_{2k-1}}\right). \tag{III}$$

For n odd there holds

$$m\left(\sum_{k=1}^{n}a_k\right)+\sum_{k=1}^{((n+1)/2)-1}\sum_{1\le i_1<i_2<\cdots<i_{2k}\le n}m\big(a_{i_1}\cdot a_{i_2}\cdots\ a_{i_{2k}}\big)$$

$$=\sum_{k=1}^{(n+1)/2}\sum_{1\le i_1<i_2<\cdots<i_{2k-1}\le n}m\left(a_{i_1}\cdot a_{i_2}\cdots\ a_{i_{2k-1}}\right). \tag{IV}$$

In the following examples on the sum of even and odd number of elements from $(\mathcal{G},+,\cdot)$ the process used in the proof for n general is presented. We add the same terms to both sides of the equation and use the valuation property.

Example 2.4. For $n = 3$. Let $a, b, c \in \mathcal{G}$. Using the valuation property (I) we can write

$$m\big((a+b)+c\big) + m\big((a+b)\cdot c\big) = m\,(a+b) + m\,(c)\,,$$

from where we have:

$$m\,(a+b+c) + m\big((a+b)\cdot c\big) + m\,(a\cdot b) + m\,(a\cdot b\cdot c)$$
$$= m(a+b) + m(c) + m(a\cdot b) + m(a\cdot b\cdot c),$$

$$m(a+b+c) + m(a\cdot c) + m(b\cdot c) + m(a\cdot b)$$
$$= m(a+b) + m(c) + m(a\cdot b) + m(a\cdot b\cdot c),$$

$$m(a+b+c) + m(a\cdot b) + m(a\cdot c) + m(b\cdot c)$$
$$= m(a) + m(b) + m(c) + m(a\cdot b\cdot c).$$

A little bit complicated is the example for n even.

Example 2.5. For $n = 4$

$$m\big((a+b+c)+d\big) + m\big((a+b+c)\cdot d\big)$$
$$= m(a+b+c) + m(d),$$

$$m(a+b+c+d) + m(a\cdot c) + m(b\cdot c) + m(a\cdot b) + m(a\cdot d + b\cdot d + c\cdot d)$$
$$= m(a+b+c) + m(d) + m(a\cdot c) + m(b\cdot c) + m(a\cdot b),$$

$$m(a+b+c+d) + m(a\cdot c) + m(b\cdot c) + m(a\cdot b) + m(a\cdot d + b\cdot d + c\cdot d)$$
$$= m(a) + m(b) + m(c) + m(d) + m(a\cdot b\cdot c),$$

$$m(a+b+c+d) + m(a\cdot c) + m(b\cdot c) + m(a\cdot b) + m(a\cdot d + b\cdot d + c\cdot d)$$
$$+ m(a\cdot d\cdot b\cdot d) + m(a\cdot d\cdot c\cdot d) + m(b\cdot d\cdot c\cdot d)$$
$$= m(a) + m(b) + m(c) + m(d) + m(a\cdot b\cdot c) + m(a\cdot b\cdot d)$$
$$+ m(a\cdot c\cdot d) + m(b\cdot c\cdot d),$$

$$m(a+b+c+d) + m(a\cdot c) + m(b\cdot c) + m(a\cdot b) + m(a\cdot d) + m(b\cdot d) + m(c\cdot d)$$
$$+ m\,(a\cdot b\cdot c\cdot d)$$
$$= m(a) + m(b) + m(c) + m(d)$$
$$+ m(a\cdot b\cdot c) + m(a\cdot b\cdot d) + m(a\cdot c\cdot d) + m(b\cdot c\cdot d).$$

P r o o f o f t h e T h e o r e m 2.3. We use the method of mathematical induction here. The assumption is that the principle for n elements of $(\mathcal{G},+,\cdot)$ holds and we will prove, that it holds also for $n+1$ elements.

At first, we will prove the principle for n even. From the induction assumption we have

$$m\left(\sum_{k=1}^{n} a_k\right)+\sum_{k=1}^{n/2}\sum_{1\le i_1<i_2<\cdots<i_{2k}\le n} m\left(a_{i_1}\cdot a_{i_2}\cdots\ a_{i_{2k}}\right)$$
$$=\sum_{k=1}^{n/2}\sum_{1\le i_1<i_2<\cdots<i_{2k-1}\le n} m\left(a_{i_1}\cdot a_{i_2}\cdots\ a_{i_{2k-1}}\right).$$

The unique condition used here is the axiom (II) in its general form:

$$m\left(\left(\sum_{k=1}^{n} a_k\right)\cdot a_{n+1}\right)+\sum_{k=1}^{n/2}\sum_{1\le i_1<i_2<\cdots<i_{2k}\le n} m\left(a_{i_1}\cdot a_{i_2}\cdots\ a_{i_{2k}}\cdot a_{n+1}\right)$$
$$=\sum_{k=1}^{n/2}\sum_{1\le i_1<i_2<\cdots<i_{2k-1}\le n} m\left(a_{i_1}\cdot a_{i_2}\cdots\ a_{i_{2k-1}}\cdot a_{n+1}\right). \tag{V}$$

By the help of (I) and by adding the same terms to both sides of the equation we get

$$m\left(\sum_{k=1}^{n+1} a_k\right)+m\left(\left(\sum_{k=1}^{n} a_k\right)\cdot a_{n+1}\right)+\sum_{k=1}^{n/2} S_{2k}^{(n)}$$
$$+\sum_{k=1}^{n/2}\sum_{1\le i_1<i_2<\cdots<i_{2k}\le n} m\left(a_{i_1}\cdot a_{i_2}\cdots\ a_{i_{2k}}\cdot a_{n+1}\right)$$
$$=m\left(\sum_{k=1}^{n} a_k\right)+m\left(a_{n+1}\right)$$
$$+\sum_{k=1}^{n/2} S_{2k}^{(n)}+\sum_{k=1}^{n/2}\sum_{1\le i_1<i_2<\cdots<i_{2k}\le n} m\left(a_{i_1}\cdot a_{i_2}\cdots\ a_{i_{2k}}\cdot a_{n+1}\right). \tag{VI}$$

We use the induction assumption (III) on the right side of the equation

$$
\begin{aligned}
&m\left(\sum_{k=1}^{n+1} a_k\right) + \sum_{k=1}^{n/2} S_{2k}^{(n)} \\
&\quad + \sum_{k=1}^{n/2} \sum_{1 \le i_1 < i_2 < \cdots < i_{2k} \le n} m\left(a_{i_1} \cdot a_{i_2} \cdots\ a_{i_{2k}} \cdot a_{n+1}\right) + m\left(\left(\sum_{k=1}^{n} a_k\right) \cdot a_{n+1}\right) \\
&= \sum_{k=1}^{n/2} S_{2k-1}^{(n)} + m\left(a_{n+1}\right) + \sum_{k=1}^{n/2} \sum_{1 \le i_1 < i_2 < \cdots < i_{2k} \le n} m\left(a_{i_1} \cdot a_{i_2} \cdots\ a_{i_{2k}} \cdot a_{n+1}\right).
\end{aligned} \tag{VII}
$$

There can be used the result (V) for the left side of the equation

$$
\begin{aligned}
&m\left(\sum_{k=1}^{n+1} a_k\right) + \sum_{k=1}^{n/2} S_{2k}^{(n)} \\
&\quad + \sum_{k=1}^{n/2} \sum_{1 \le i_1 < i_2 < \cdots < i_{2k-1} \le n} m\left(a_{i_1} \cdot a_{i_2} \cdots\ a_{i_{2k-1}} \cdot a_{n+1}\right) \\
&= \sum_{k=1}^{n/2} S_{2k-1}^{(n)} + m(a_{n+1}) \\
&\quad + \sum_{k=1}^{n/2} \sum_{1 \le i_1 < i_2 < \cdots < i_{2k} \le n} m\left(a_{i_1} \cdot a_{i_2} \cdots\ a_{i_{2k}} \cdot a_{n+1}\right).
\end{aligned} \tag{VIII}
$$

This yields the final formula of the inclusion-exclusion principle for $n+1$ odd number of elements from $(G, +, \cdot)$. Neither distributivity nor idempotency law were used.

$$
\begin{aligned}
&m\left(\sum_{k=1}^{n+1} a_k\right) + \sum_{k=1}^{\left((n+2)/2\right)-1} \sum_{1 \le i_1 < i_2 < \cdots < i_{2k} \le n+1} m\left(a_{i_1} \cdot a_{i_2} \cdots\ a_{i_{2k}}\right) \\
&= \sum_{k=1}^{(n+2)/2} \sum_{1 \le i_1 < i_2 < \cdots < i_{2k-1} \le n+1} m\left(a_{i_1} \cdot a_{i_2} \cdots\ a_{i_{2k-1}}\right).
\end{aligned}
$$

The proof for n odd is analogical, and concludes the proof. □

3. Conclusions

The further research could be focused on the relevance of this axiom for the family of IF-sets, and then to prove the inclusion-exclusion principle for IF-sets by the help of the axiom.

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