VERSITA

TATRA MOUNTAINS
Mathematical Publications
DOI: 10.2478/v10127-011-0018-0
Tatra Mt. Math. Publ. **48** (2011), 197–213

THE ESTIMATION OF THE DYNAMICS OF INDIRECT CONTROL SWITCHING SYSTEMS

Miroslava Růžičková — Olena Kuzmych — Denys Ya. Khusainov

Dedicated to the memory of Professor Pavol Marušiak

ABSTRACT. The paper deals with the time switching hybrid systems described by nonlinear control equations with nonlinearity of a sector form. The estimation of the solution of the control system at any moment is established through the method of Lyapunov function. Both cases, stable and unstable control systems are considered. The final estimation of the solution of the hybrid system is obtained from the composition of perturbations of the separate control systems using the condition of solution continuity in the points of switching.

1. Introduction

One of the problems of motion stability is the problem of absolute stability. In technical control systems, the control function is the function of one variable, which has the shape of a curve, located between two lines in the first and third quarters of the coordinate plane. The stability of the control system with a control function, located in this sector, is referred to as absolute stability [1], [9].

Two main approaches to the study of the absolute stability of control systems are known. The first one, so called "frequency method", is connected with the frequency conditions of absolute stability and was developed for example in [5], [7], [8]. The other approach makes use of the Lyapunov function method and is the universal method of research of the nonlinear dynamic systems presented in works [2], [11].

It should be noted that in real engineering systems, as a rule, there are several modes of operation, and the transition from one to another is made when certain

2010 Mathematics Subject Classification: 93C30, 93C10, 93D05, 93D30.
Keywords: nonlinear differential equation, estimation of solution, hybrid system.
The authors would like to thank the following for their support: the Slovak Research and Development Agency (project APPV-0700-07), the Grant Agency of the Slovak republic (VEGA1/0090/09), and the National Scholarship Program of the Slovak Republic (SAIA).

conditions are fulfilled. The systems of this kind, in the most general case, are called hybrid systems [3], [4], [6], [10]. Special cases, dealt with time switching in this article, are those of the switching regulator systems. Time switching linear systems have been described in [13]. For systems of this type there are calculated upper bounds for the final time.

The main goal of the paper is to find the estimation of solution of the time switching hybrid systems which are described by nonlinear control subsystems. Using the estimations for separate subsystems, which are obtained with the method of Lyapunov functions, and using the condition of continuity of solution in the points of switching, we get the estimation of the origin hybrid system solution at a final moment [6] depending on the initial state. The estimations of solution are obtained using the Lyapunov function of Lurie-Postnikov type [1], [5]. The final estimation is obtained from the composition of perturbations for separate subsystems.

The paper is organized as follows. The formulation of the problem is presented and explained in Section 2. Section 3 deals with the case of a stable system of the form (1). A general case, including an unstable system is considered in Section 4. Section 5 summarizes all particular results and gives the final estimation of the main formulated problem. The estimation of the solution of the hybrid system (3) is established on the whole considered interval. Section 6 contains an example. The estimation of the solution of hybrid system is calculated on three time intervals.

2. Statement of the problem

Let us consider a general hybrid system given by a set of nonlinear control subsystems when all of them are in the form

$$\begin{aligned} \dot{x}(t) &= -a\,x(t) + b\,f\big(\sigma(t)\big), \\ \dot{\sigma}(t) &= \quad c\,x(t) - \rho\,f\big(\sigma(t)\big), \qquad t \geq t_0 \geq 0, \end{aligned} \tag{1}$$

where x is the state function, σ is the control, $a > 0$, $\rho > 0$, b, c are constants, $f(\sigma)$ is a continuous Lipschitz function on $[t_0, \infty)$ satisfying there so-called "sector condition". It means, there exist such constants k_1, k_2 that

$$k_1\sigma^2 \leq \sigma f(\sigma) \leq k_2\sigma^2, \qquad 0 < k_1 < k_2 \tag{2}$$

is satisfied. The vector function $(x, \sigma) : [t_0, \infty) \to \mathbb{R}^2$ is said to be the solution of (1) on $[t_0, \infty)$ if x is continuously differentiable on $[t_0, \infty)$ and satisfies the equation (1) on the mentioned interval.

Thus, the considered hybrid system is given by a set of 2-dimensional control subsystems of the type (1) defined on the fixed intervals

$$T_i = [t_{i-1}, t_i], \qquad i = 1, \ldots, N.$$

The moments of time $t_1, t_2, \ldots, t_i, \ldots, t_{N-1}$ are called *switching moments*. Therefore, such a hybrid system can be described by the set of systems

$$\begin{aligned} \dot{x}(t) &= -a_i\, x(t) + b_i\, f_i\big(\sigma(t)\big), \\ \dot{\sigma}(t) &= \quad c_i\, x(t) - \rho_i\, f_i\big(\sigma(t)\big), \qquad t \in [t_{i-1}, t_i]\,, \quad i = 1, \ldots, N, \end{aligned} \tag{3}$$

where $a_i > 0$, $\rho_i > 0$, b_i, c_i are constants, $f_i\,(\sigma)$, $i = 1, \ldots, N$ are continuous Lipschitz functions on $[t_{i-1}, t_i]$ which satisfy conditions

$$k_{1i}\, \sigma^2 \leq \sigma f_i\,(\sigma) \leq k_{2i}\, \sigma_i^2, \qquad 0 < k_{1i} < k_{2i}, \quad i = 1, \ldots, N. \tag{4}$$

The main purpose of the article is to find the estimation of a solution of the hybrid system (3) depending on its initial state $\big(x(t_0), \sigma(t_0)\big)$, whose estimation is

$$|x(t_0)| < \delta_1, \quad |\sigma(t_0)| < \delta_2,$$

where δ_1, δ_2 are positive and sufficiently small constants. We will calculate the estimation of the solution $\big(x(t), \sigma(t)\big)$ at the final moment $t = t_N$, if the condition

$$\begin{aligned} \lim_{s \to +0} x(t_i - s) &= \lim_{s \to +0} x(t_i + s), \\ \lim_{s \to +0} \sigma(t_i - s) &= \lim_{s \to +0} \sigma(t_i + s), \qquad i = 1, \ldots, N-1 \end{aligned}$$

of continuity solution $\big(x(t), \sigma(t)\big)$ at the switching moments holds.

3. The estimation of dynamics of absolutely stable system

In this part we derive an estimation of a solution of the nonlinear control system (1) with continuous Lipschitz function $f(\sigma)$ on $[t_0, \infty)$ satisfying condition (2). We will investigate the estimation using Lyapunov function

$$V\,(x, \sigma) = x^2 + \beta \int_0^\sigma f\,(s)\; ds, \qquad \beta > 0 \tag{5}$$

which is known as a Lurie-Postnikov type of Lyapunov function.

We define a matrix of the form

$$S_1\,[\beta] = \begin{bmatrix} 2\,a & -b - \dfrac{1}{2}\beta c \\ -b - \dfrac{1}{2}\beta c & -\beta \rho \end{bmatrix}, \tag{6}$$

which will be used to prove the next theorem.

Theorem 3.1. *Let the coefficients $a > 0$, $\rho > 0$, b, c, of the system* (1) *satisfy the inequality*

$$a\,\rho - b\,c > 0. \tag{7}$$

Then the system (1) *is absolutely stable. Moreover, the exponential convergence estimation of the solution $\big(x(t), \sigma(t)\big)$, for arbitrary fixed $t \geq t_0$, can be written in the form*

$$|x\,(t)\,| \leq \left[\big|x\,(t_0)\,\big| + \sqrt{\frac{1}{2}\,\beta k_2}\,\big|\sigma\,(t_0)\,\big|\right] e^{-\frac{1}{2}\,\xi_1(\beta_0)(t-t_0)}, \tag{8}$$

$$|\sigma\,(t)\,| \leq \left[\sqrt{\frac{2}{\beta k_1}}\,\big|x\,(t_0)\,\big| + \sqrt{\frac{k_2}{k_1}}\,\big|\,\sigma\,(t_0)\,\big|\right] e^{-\frac{1}{2}\,\xi_1(\beta_0)(t-t_0)}, \tag{9}$$

where

$$\xi_1(\beta_0) = \lambda_{\min}\big(S_1[\beta_0]\big)\,\min\left\{1,\ \frac{k_1^2c^2}{k_2(2a\rho - bc)}\right\}, \tag{10}$$

$$\begin{aligned}\lambda_{\min}\big(S_1[\beta_0]\big) = &\frac{1}{c^2}\Bigg[(a\,c^2 + 2a\rho^2 - b\,c\,\rho)\\ &\quad - \sqrt{\big(a\,c^2 - 2\,a\rho^2 + b\,c\,\rho\big)^2 + 4\,a^2c^2\rho^2}\Bigg].\end{aligned} \tag{11}$$

P r o o f. Regarding (2) it is easy to see that the Lyapunov function (5) satisfies the bilateral estimation

$$x^2\,(t) + \frac{1}{2}\,\beta\,k_1\,\sigma^2(t) \leq V\big(x\,(t)\,, \sigma(t)\big) \leq x^2\,(t) + \frac{1}{2}\,\beta\,k_2\,\sigma^2(t). \tag{12}$$

Let us compute the full derivative of the Lyapunov function (5) along the solution of the system (1). Employing the expressions of $\dot{x}\,(t)$ and $\dot{\sigma}\,(t)$ defined by the system (1) we get

$$\begin{aligned}\frac{d}{dt}V\big(x(t), \sigma(t)\big) = {}& 2x(t)\Big[-ax(t) + b\,f\big(\sigma(t)\big)\Big]\\ &+ \beta f\big(\sigma(t)\big)\Big[c\,x\,(t) - \rho f\big(\sigma(t)\big)\Big].\end{aligned}$$

The last formula can be rewritten in a more convenient quadratic form

$$\frac{d}{dt}\,V\big(x(t), \sigma(t)\big) = -\Big(x\,(t)\,, f\big(\sigma(t)\big)\Big)\begin{pmatrix} 2\,a & -b - \frac{1}{2}\,\beta\,c \\ -b - \frac{1}{2}\,\beta\,c & \beta\rho \end{pmatrix}\begin{pmatrix} x\,(t) \\ f(\sigma) \end{pmatrix}.$$

Let us assume that there exists such positive constant β that the matrix $S_1\,[\beta]$ is positive definite. In accordance with Silvester criterion the mentioned matrix is positive definite if and only if the inequalities

$$a > 0, \qquad 2\,a\,\beta\,\rho - \left[b + \frac{1}{2}\,\beta\,c\right]^2 > 0 \tag{13}$$

hold.

Now, in view of condition (2), the full derivative of the Lyapunov function satisfies the inequality

$$\begin{aligned}\frac{d}{dt}\,V\big(x(t),\sigma(t)\big) &\le -\lambda_{\min}\big(S_1[\beta]\big)\Big[x^2(t) + f^2(\sigma(t))\Big]\\ &\le -\lambda_{\min}\big(S_1[\beta]\big)\left[x^2(t) + k_1^2\,\sigma^2(t)\right],\end{aligned} \tag{14}$$

where

$$\lambda_{\min}\big(S_1[\beta]\big) = \frac{1}{2}\left[(2\,a + \beta\rho) - \sqrt{(2\,a - \beta\rho)^2 + 4\left(b + \frac{1}{2}\,\beta\,c\right)^2}\,\right] \tag{15}$$

is the smaller root of the characteristic equation

$$\det\big(S_1\,[\beta] - \lambda I\big) = \begin{vmatrix} 2\,a - \lambda & -b - \dfrac{1}{2}\,\beta\,c \\ -b - \dfrac{1}{2}\,\beta\,c & \beta\,\rho - \lambda \end{vmatrix} = 0.$$

The convergence estimation of a solution of the system (1) can be obtained from the estimation of $V\big(x(t),\sigma(t)\big)$. Our calculation procedure is split into the following two steps:

1. At first, we rewrite the right part of bilateral inequality (12) to the form

$$-x^2(t) \le -V\big(x(t),\sigma(t)\big) + \frac{1}{2}\,\beta\,k_2\,\sigma^2(t)$$

and we use it to modify the inequality (14). We get

$$\begin{aligned}&\frac{d}{dt}V\big(x(t),\sigma(t)\big)\\ &\le \lambda_{\min}\big(S_1[\beta]\big)\left[-V\big(x(t),\sigma(t)\big) + \frac{1}{2}\,\beta\,k_2\,\sigma^2(t)\right] - \lambda_{\min}\big(S_1[\beta]\big)k_1^2\sigma^2(t)\\ &= -\lambda_{\min}\big(S_1[\beta]\big)V\big(x(t),\sigma(t)\big) - \lambda_{\min}\big(S_1[\beta]\big)\left[k_1^2 - \frac{1}{2}\,\beta\,k_2\right]\sigma^2(t).\end{aligned}$$

From this, under assumption

$$k_1^2 - \frac{1}{2}\,\beta\,k_2 > 0, \tag{16}$$

we get the differential inequality

$$\frac{d}{dt}V\big(x(t),\sigma(t)\big) < -\lambda_{\min}\big(S_1[\beta]\big)V\big(x(t),\sigma(t)\big).$$

Integrating the obtained differential inequality from t_0 to t we have

$$V\big(x(t),\sigma(t)\big) \le V\big(x\,(t_0)\,,\sigma(t_0)\big)\exp\Big\{-\lambda_{\min}\big(S_1[\beta]\big)(t-t_0)\Big\}. \tag{17}$$

2. On the other hand, we rewrite the right part of inequality (12) to the form

$$-\sigma^2\,(t) \le \frac{2}{\beta k_2}\big[\,-V\big(x\,(t)\,,\sigma\,(t)\,\big)+x^2\,(t)\,\big]$$

and using it we also modify the inequality (14). We get

$$\begin{aligned}
&\frac{d}{dt}V\big(x(t),\,\sigma(t)\big)\\
&\le -\lambda_{\min}\big(S_1[\beta]\big)x^2(t)+\lambda_{\min}\big(S_1[\beta]\big)k_1^2\frac{2}{\beta\,k_2}\Big[-V\big(x(t),\,\sigma(t)\big)+x^2(t)\Big]\\
&= -\frac{2\,k_1^2}{\beta k_2}\lambda_{\min}\big(S_1[\beta]\big)V\big(x(t),\,\sigma(t)\big)-\lambda_{\min}\big(S_1[\beta]\big)\left[1-\frac{2\,k_1^2}{\beta\,k_2}\right]x^2(t).
\end{aligned}$$

Hence, if the condition

$$1-\frac{2k_1^2}{\beta\,k_2}\ge 0 \tag{18}$$

holds, we obtain the differential inequality

$$\frac{d}{dt}V\big(x(t),\,\sigma(t)\big) < -\frac{2\,k_1^2}{\beta k_2}\lambda_{\min}\big(S_1[\beta]\big)V\big(x(t),\,\sigma(t)\big).$$

Integrating it, also from t_0 to t, we get

$$V\big(x(t),\sigma(t)\big) \le V\big(x(t_0),\sigma(t_0)\big)\exp\left\{-\frac{2k_1^2}{\beta k_2}\lambda_{\min}\big(S_1[\beta]\big)(t-t_0)\right\}. \tag{19}$$

Both obtained inequalities (17) and (19), can be written in one relationship

$$V\big(x(t),\sigma(t)\big) \le V\big(x(t_0),\sigma(t_0)\big)\exp\big\{-\xi_1\,(\beta)\,(t-t_0)\big\}, \tag{20}$$

where we have denoted

$$\xi_1(\beta)=\begin{cases}\lambda_{\min}\big(S_1[\beta]\big) & \text{if}\quad \beta<\frac{2k_1^2}{k_2},\\[4pt] \frac{2k_1^2}{\beta k_2}\lambda_{\min}\big(S_1[\beta]\big) & \text{if}\quad \beta\ge\frac{2k_1^2}{k_2}.\end{cases}$$

Consequently, substituting the relationship (20) into the bilateral inequality (12) we have

$$\begin{aligned}
x^2(t)+\frac{1}{2}\,\beta\,k_1\,\sigma^2(t) &\le V\big(x(t),\sigma(t)\big)\\
&\le V\big(x(t_0),\sigma(t_0)\big)\exp\big\{-\xi_1(\beta)(t-t_0)\big\}\\
&\le \left[x^2(t_0)+\frac{1}{2}\,\beta\,k_2\,\sigma^2(t_0)\right]\exp\big\{-\xi_1(\beta)(t-t_0)\big\},
\end{aligned}$$

whence we obtain the convergence estimation of solution

$$|x(t)| \le \left[\left|x(t_0)\right| + \sqrt{\frac{1}{2}\,\beta\, k_2}\, \left|\sigma(t_0)\right| \right] e^{-\frac{1}{2}\,\xi_1(\beta)(t-t_0)},$$

$$\left|\sigma(t)\right| \le \left[\sqrt{\frac{2}{\beta k_1}}\, \left|x(t_0)\right| + \sqrt{\frac{k_2}{k_1}}\, \left|\sigma(t_0)\right| \right] e^{-\frac{1}{2}\,\xi_1(\beta)(t-t_0)}.$$

Finally, we are left to discuss the value of the parameter $\beta > 0$. We find such maximum value β that the inequality (13) is satisfied. The problem of finding of the optimal value β is equal to the problem of finding the conditional extremum of the function

$$\Phi(\beta) = 2\,\beta\, a\, \rho - \left(b + \frac{1}{2}\beta\, c\right)^2 \tag{21}$$

with the restricted condition $\beta > 0$. From the necessary condition for extremum of the function $\Phi(\beta)$ we have the value

$$\beta_0 = \frac{2\,(2\,a\,\rho - b\,c)}{c^2} \tag{22}$$

and substituting this optimal value of β to the (21) we get

$$\Phi(\beta_0) = \frac{4\,a\,\rho\,(a\,\rho - b\,c)}{c^2}, \tag{23}$$

which implies the condition of absolute stability (7). The condition (11) is obtained from substituting (22) into (15). □

4. The estimation of dynamics if conditions of absolute stability are not satisfied

Let the system (1) have such coefficients that the condition (7) does not hold. In this case we use the Lyapunov function with the exponential multiplier $e^{\gamma t}$:

$$V(x,\sigma,t) = e^{\gamma t}\left[x^2 + \beta \int_0^{\sigma} f(s)\, ds \right], \qquad \beta > 0 \tag{24}$$

and the matrix $S_2\,[\beta,\gamma]$ of the form

$$S_2\,[\beta,\gamma] = \begin{bmatrix} 2\,a - \gamma & -b - \frac{1}{2}\,\beta\, c \\ -b - \frac{1}{2}\,\beta\, c & \beta\left(\rho - \frac{\gamma k_1}{2\,k_2^2}\right) \end{bmatrix}. \tag{25}$$

Theorem 4.1. *Let the system* (1) *have such coefficients that the condition* (7) *does not hold or* $a \le 0$. *Then the exponential estimation of the solution* $\big(x(t), \sigma(t)\big)$ *of the system* (1) *for arbitrary fixed* $t \ge t_0$ *can be written in the form*

$$\big|x(t)\big| \le \left[\big|x(t_0)\big| + \sqrt{\frac{1}{2}\,\beta\, k_2}\,\big|\sigma(t_0)\big|\right] e^{-\frac{1}{2}\,\xi_2(\beta,\,\gamma)(t-t_0)}, \tag{26}$$

$$\big|\sigma(t)\big| \le \left[\sqrt{\frac{2}{\beta k_1}}\big|x(t_0)\big| + \sqrt{\frac{k_2}{k_1}}\,\big|\sigma(t_0)\big|\right] e^{-\frac{1}{2}\,\xi_2(\beta,\,\gamma)(t-t_0)}, \tag{27}$$

where

$$\xi_2\,(\beta,\gamma) = \min\left\{1, \frac{2k_1^2}{\beta k_2}\right\} \lambda_{\min}\big(S_2[\beta,\gamma]\big), \qquad \gamma < \min\{2a, \gamma_1\}, \tag{28}$$

$$\gamma_1 = \frac{1}{2}\left[\beta\left(\rho + a\,\frac{k_1}{k_2^2}\right) - \sqrt{\beta^2\left(\rho - a\,\frac{k_1}{k_2^2}\right)^2 + 2\beta\,\frac{k_1}{k_2^2}\left(b + \frac{1}{2}\,\beta c\right)^2}\,\right], \tag{29}$$

$$\begin{aligned}\lambda_{\min}\big(S_2[\beta,\gamma]\big) = \frac{1}{2}&\left\{\left[(2a-\gamma) + \beta\Big(\rho - \frac{\gamma k_1}{2\,k_2^2}\Big)\right]\right.\\ &\left.-\sqrt{\left[(2a-\gamma) - \beta\left(\rho - \frac{\gamma k_1}{2k_2^2}\right)\right]^2 + 4\left(b + \frac{1}{2}\,\beta c\right)^2}\,\right\}.\end{aligned} \tag{30}$$

P r o o f. The Lyapunov function (24) with the exponential multiplier, the analogy as the Lyapunov function (5), satisfies the inequality

$$\begin{aligned}e^{\gamma t}\left[x^2(t) + \frac{1}{2}\,\beta k_1 \sigma^2(t)\right] &\le V\big(x(t), \sigma(t), t\big)\\ &\le e^{\gamma t}\left[x^2(t) + \frac{1}{2}\,\beta k_2 \sigma^2(t)\right].\end{aligned} \tag{31}$$

Let us compute the full derivative of the Lyapunov function along the solution of the system (1). It can be written in the form

$$\begin{aligned}&\frac{d}{dt} V\big(x(t),\, \sigma(t), t\big)\\ &= -e^{\gamma t}\Big(x(t),\, f\big(\sigma(t)\big)\Big)\begin{bmatrix} 2\,a & -b - \frac{1}{2}\,\beta c\\ -b - \frac{1}{2}\,\beta c & \beta\rho\end{bmatrix}\begin{pmatrix} x(t)\\ f\big(\sigma(t)\big)\end{pmatrix}\\ &\quad + \gamma e^{\gamma t}\left[x^2(t) + \beta\int\limits_0^{\sigma(t)} f(s)\,ds\right].\end{aligned} \tag{32}$$

Since from the condition (2) it follows that

$$\int_0^{\sigma(t)} f(s)\,ds \ \geq \frac{k_1}{2k_2^2} f^2\big(\sigma(t)\big),$$

using this, and the assumptions $\beta > 0$, $\gamma < 0$, we transform (32) to the form:

$$\begin{aligned}&\frac{d}{dt}V\big(x(t),\sigma(t),t\big)\\ &= -e^{\gamma t}\Big(x(t),\ f\big(\sigma(t)\big)\Big)\begin{bmatrix} 2a-\gamma & -b-\dfrac{1}{2}\,\beta c \\ -b-\dfrac{1}{2}\,\beta c & \beta\left(\rho-\dfrac{\gamma k_1}{2k_2^2}\right)\end{bmatrix}\begin{pmatrix} x(t) \\ f\big(\sigma(t)\big)\end{pmatrix}.\end{aligned}$$

We need to determine the constants β, γ so that the matrix $S_2\,[\beta,\gamma]$ is positive definite. Applying the Silvester criterion we get conditions

$$\begin{gathered} 2a-\gamma > 0, \\ \beta(2a-\gamma)\left(\rho-\frac{\gamma k_1}{2k_2^2}\right)-\left(b+\frac{1}{2}\,\beta c\right)^2 > 0, \end{gathered} \tag{33}$$

whence we have

$$\begin{gathered} \gamma < 2a, \\ F\,(\beta,\gamma) \equiv \left(\frac{\beta k_1}{2k_2^2}\right)\gamma^2-\beta\left(\rho+\frac{a k_1}{k_2^2}\right)\gamma+\left[2a\beta\rho-\left(b+\frac{1}{2}\beta c\right)^2\right] > 0. \end{gathered}$$

Because the relationship

$$F(\beta,\gamma)|_{\gamma=2a} = -\left(b+\frac{1}{2}\,\beta c\right)^2 < 0$$

is valid for $\gamma = 2\,a$, the value γ must satisfy the condition

$$\gamma < \gamma_1,$$

where γ_1 is the smaller root of the equation $F\,(\beta,\gamma) = 0$, given by (29).

The full derivative of the Lyapunov function (24) is further modified to the form

$$\frac{d}{dt}V\big(x(t),\sigma(t),t\big) \leq -e^{\gamma t}\lambda_{\min}\big(S_2[\beta,\gamma]\big)\Big[x^2(t)+f^2\big(\sigma(t)\big)\Big],$$

or, using the condition (2), we get

$$\begin{aligned}&\frac{d}{dt}V\big(x(t),\sigma(t),t\big)\\ &\leq -e^{\gamma t}\lambda_{\min}\big(S_2[\beta,\gamma]\big)x^2(t)-e^{\gamma t}\lambda_{\min}\big(S_2[\beta,\gamma]\big)k_1^2\sigma^2(t).\end{aligned} \tag{34}$$

The convergence estimation of the solution of the system (1) can be obtained from the estimation of $V\big(x(t), \sigma(t)\big)$, which is analogous to the proof of Theorem 3.1. Our calculation procedure is also split into the following two steps:

1. First, the right part of the inequality (31) is rewritten to the form

$$-e^{\gamma t} x^2(t) \le -V\big(x(t), \sigma(t), t\big) + e^{\gamma t} \frac{1}{2} \beta k_2 \sigma^2(t)$$

and, with respect to this, putting it into (34) we obtain

$$\begin{aligned} \frac{d}{dt} V\big(x(t), \sigma(t)\big) \le \lambda_{\min}\big(S_2[\beta,\gamma]\big) &\left[-V\big(x(t), \sigma(t), t\big) + e^{\gamma t} \frac{1}{2} \beta k_2 \sigma^2(t) \right] \\ &- e^{\gamma t} \lambda_{\min}\big(S_2[\beta,\gamma]\big) k_1^2 \sigma^2(t) \\ = &- \lambda_{\min}\big(S_2[\beta,\gamma]\big) V\big(x(t), \sigma(t), t\big) \\ &- e^{\gamma t} \lambda_{\min}\big(S_2[\beta,\gamma]\big) \left[k_1^2 - \frac{1}{2} \beta k_2 \right] \sigma^2(t). \end{aligned}$$

Provided that

$$k_1^2 > \frac{1}{2} \beta k_2$$

we get

$$\frac{d}{dt} V\big(x(t), \sigma(t), t\big) \le -\lambda_{\min}\big(S_2[\beta,\gamma]\big) V\big(x(t), \sigma(t), t\big).$$

Integrating the last differential inequality from t_0 to t we state claim

$$V\big(x(t), \sigma(t), t\big) \le V\big(x(t_0), \sigma(t_0), t_0\big) \exp\Big\{ -\lambda_{\min}\big(S_2[\beta,\gamma]\big)(t - t_0) \Big\}. \tag{35}$$

2. Now, the right part of the inequality (31) is rewritten to the form

$$-e^{\gamma t} \sigma^2(t) \le -\frac{2}{\beta k_2} V\big(x(t), \sigma(t), t\big) + e^{\gamma t} \frac{2}{\beta k_2} \sigma^2(t),$$

and putting it into (34) we obtain

$$\begin{aligned} \frac{d}{dt} V\big(x(t), \sigma(t), t\big) \le &-e^{\gamma t} \lambda_{\min}\big(S_2[\beta,\gamma]\big) x^2(t) \\ &+ \lambda_{\min}\big(S_2[\beta,\gamma]\big) \frac{2 k_1^2}{\beta k_2} \left[-V\big(x(t), \sigma(t), t\big) + e^{\gamma t} x^2(t) \right] \\ = &-\frac{2 k_1^2}{\beta k_2} \lambda_{\min}\big(S_2[\beta,\gamma]\big) V\big(x(t), \sigma(t), t\big) \\ &- e^{\gamma t} \lambda_{\min}\big(S_2[\beta,\gamma]\big) \left[1 - \frac{2 k_1^2}{\beta k_2} \right] x^2(t). \end{aligned}$$

Under the assumption

$$1 - \frac{2 k_1^2}{\beta k_2} \ge 0$$

we get
$$\frac{d}{dt} V\big(x(t), \sigma(t),\, t\big) \le -\frac{2\, k_1^2}{\beta\, k_2}\, \lambda_{\min}\big(S_2[\beta,\gamma]\big) V\big(x(t), \sigma(t),\, t\big),$$
and integrating the last differential inequality from t_0 to t we have
$$\begin{aligned} & V\big(x(t), \sigma(t),\, t\big) \\ & \le V\big(x(t_0), \sigma(t_0),\, t_0\big) \exp\left\{-\frac{2\, k_1^2}{\beta\, k_2}\, \lambda_{\min}\big(S_2\,[\beta,\gamma]\,\big)(t - t_0)\right\}. \end{aligned} \tag{36}$$
Again using the bilateral inequality (31) we get
$$\begin{aligned} e^{\gamma t}\left[x^2(t) + \frac{1}{2}\,\beta k_1 \sigma^2(t)\right] & \le V\big(x(t), \sigma(t), t\big) \\ & \le V\big(x(t_0), \sigma(t_0), t_0\big) \exp\big\{-\xi_2(\beta,\gamma)(t - t_0)\big\} \\ & \le e^{\gamma t_0}\left[x^2(t_0) + \frac{1}{2}\,\beta k_2 \sigma^2(t_0)\right] \exp\big\{-\xi_2(\beta,\gamma)(t - t_0)\big\}, \end{aligned}$$
whence the proved statement (27), (28) follows.

Finally, note that the $\lambda_{\min}\big(S_2[\beta,\gamma]\big)$ in the form (30) we obtain as the smaller root of the characteristic equation
$$\det\big(S_2[\beta,\gamma] - \lambda I\big) = \begin{vmatrix} 2\,a - \gamma - \lambda & -b - \dfrac{1}{2}\,\beta\, c \\ -b - \dfrac{1}{2}\,\beta\, c & \beta\left(\rho - \dfrac{\gamma\, k_1}{2\, k_2^2}\right) - \lambda \end{vmatrix} = 0.$$
□

5. The estimation of hybrid control system with time switching

In the main part of the article we consider the hybrid dynamic system (3) composed of nonlinear control subsystems each of which is defined on the fixed interval $T_i = [t_{i-1}, t_i]$, $i = 1, \ldots, N$. We assume that f_i are continuous Lipschitz functions at the appropriate intervals and satisfy the sector condition (4). The estimation of the solution of separated subsystems derived in the previous parts is used to establish the estimation of the solution of the hybrid system (3) at the moment of time $t = t_N$.

We will use denoting
$$\theta_i(\beta_i, \gamma_i) = \begin{cases} \xi_{1,\,i}(\beta_i) & \text{if } \exists\, \beta_i \text{ such that } S_1[\beta_i] > 0, \\ \xi_{2,\,i}(\beta_i,\, \gamma_i) & \text{if } \nexists\, \beta_i \text{ such that } S_1[\beta_i] > 0 \end{cases} \tag{37}$$
in the formulation of the next theorem.

Theorem 5.1. *The estimation of the final state of the hybrid system* (3) *at the moment of time $t = t_N$ can be written in the form*

$$\begin{aligned} \big|x(t_N)\big| \le{}& \Big[R_{11}^N(\beta)\big|x(t_0)\big| + R_{12}^N(\beta)\big|\sigma(t_0)\big|\Big] \\ & \times \exp\left\{-\frac{1}{2}\sum_{i=1}^{N}\theta_i\left(\beta_i,\gamma_i\right)\left(t_i - t_{i-1}\right)\right\}, \end{aligned} \tag{38}$$

$$\begin{aligned} \big|\sigma(t_N)\big| \le{}& \Big[R_{21}^N(\beta)\big|x(t_0)\big| + R_{22}^N(\beta)\big|\sigma(t_0)\big|\Big] \\ & \times \exp\left\{-\frac{1}{2}\sum_{i=1}^{N}\theta_i(\beta_i,\gamma_i)\left(t_i - t_{i-1}\right)\right\}, \end{aligned} \tag{39}$$

where $R_{11}^N(\beta)$, $R_{12}^N(\beta)$, $R_{21}^N(\beta)$, $R_{22}^N(\beta)$ depend on the system parameters and on the number of iterations.

Proof. Let us start to consider the first time interval $t_0 \le t \le t_1$. At the moment of time $t = t_1$, in accordance to Theorem 3.1 we have the estimation

$$\big|x\left(t_1\right)\big| \le \big[\varphi_{11}\big|x(t_0)\big| + \varphi_{12}\big|\sigma(t_0)\big|\big]e^{-\frac{1}{2}\theta_1(\beta_1,\gamma_1)(t_1-t_0)}, \tag{40}$$

$$\big|\sigma\left(t_1\right)\big| \le \big[\varphi_{21}\big|x(t_0)\big| + \varphi_{22}\big|\sigma(t_0)\big|\big]e^{-\frac{1}{2}\theta_1(\beta_1,\gamma_1)(t_1-t_0)}, \tag{41}$$

where

$$\varphi_{11} = 1, \quad \varphi_{12} = \sqrt{\frac{1}{2}\,\beta k_2}, \quad \varphi_{21} = \sqrt{\frac{2}{\beta k_1}}, \quad \varphi_{22} = \sqrt{\frac{k_2}{k_1}}.$$

Now, let us consider the second time interval $t_1 \le t \le t_2$. Repeating the process at this interval we come to estimate

$$\begin{aligned} |x(t_2)| &\le \big[\varphi_{11}|x(t_1)| + \varphi_{12}|\sigma(t_1)|\big]e^{-\frac{1}{2}\,\theta_2(t_2-t_1)} \\ &= \big[R_{11}^1|x(t_1)| + R_{12}^1|\sigma(t_1)|\big]e^{-\frac{1}{2}\,\theta_2(t_2-t_1)}, \end{aligned} \tag{42}$$

$$\begin{aligned} |\sigma(t_2)| &\le \big[\varphi_{21}|x(t_1)| + \varphi_{22}|\sigma(t_1)|\big]e^{-\frac{1}{2}\,\theta_2(t_2-t_1)} \\ &= \big[R_{21}^1|x(t_1)| + R_{22}^1|\sigma(t_1)|\big]e^{-\frac{1}{2}\theta_2(t_2-t_1)} \end{aligned} \tag{43}$$

for the moment of time $t = t_2$.

Taking the inequalities (40), (41) into account, we get

$$\begin{aligned}
|x(t_2)| &\leq \Big[\varphi_{11}\big[\varphi_{11}|x(t_0)| + \varphi_{12}|\sigma(t_0)|\big] \\
&\quad + \varphi_{12}\big[\varphi_{21}|x(t_0)| + \varphi_{22}|\sigma(t_0)|\big]\Big] e^{-\frac{1}{2}\theta_2(t_2-t_1)-\frac{1}{2}\theta_1(t_1-t_0)} \\
&= \big[(\varphi_{11}\varphi_{11} + \varphi_{12}\varphi_{21})|x(t_0)| \\
&\quad + (\varphi_{11}\varphi_{12} + \varphi_{12}\varphi_{22})|\sigma(t_0)|\big] e^{-\frac{1}{2}\theta_2(t_2-t_1)-\frac{1}{2}\theta_1(t_1-t_0)} \\
&= \big[R_{11}^2|x(t_0)| + R_{12}^2|\sigma(t_0)|\big] e^{-\frac{1}{2}\theta_2(t_2-t_1)-\frac{1}{2}\theta_1(t_1-t_0)},
\end{aligned}$$

$$\begin{aligned}
|\sigma(t_2)| &\leq \Big[\varphi_{21}\big[\varphi_{11}|x(t_0)| + \varphi_{12}|\sigma(t_0)|\big] \\
&\quad + \varphi_{22}\big[\varphi_{21}|x(t_0)| + \varphi_{22}|\sigma(t_0)|\big]\Big] e^{-\frac{1}{2}\theta_2(t_2-t_1)-\frac{1}{2}\theta_1(t_1-t_0)} \\
&= \big[(\varphi_{21}\varphi_{11} + \varphi_{22}\varphi_{21})|x(t_0)| \\
&\quad + (\varphi_{21}\varphi_{12} + \varphi_{22}\varphi_{22})|\sigma(t_0)|\big] e^{-\frac{1}{2}\theta_2(t_2-t_1)-\frac{1}{2}\theta_1(t_1-t_0)} \\
&= \big[R_{21}^2|x(t_0)| + R_{22}^2|\sigma(t_0)|\big] e^{-\frac{1}{2}\theta_2(t_2-t_1)-\frac{1}{2}\theta_1(t_1-t_0)}.
\end{aligned}$$

Repeating the process farther, we obtain the formulas (38), (39). □

6. Example

In this Section we illustrate the obtained results in an example. Let us consider the hybrid system (3) at three intervals with the end points:

$$t_0 = 0, \quad t_1 = 12, \quad t_2 = 52, \quad t_3 = 55$$

and coefficients:

$$\begin{aligned}
&a_1 = \ \ 1, \quad b_1 = 0.1,\ c_1 = 0.2,\ k_{11} = 0.2,\ k_{21} = 0.3,\ \rho_1 = 1 \quad \text{for} \quad 0 \leq t \leq 12; \\
&a_2 = \ \ 0.5,\ b_2 = 0.2,\ c_2 = 0.2,\ k_{12} = 0.1,\ k_{22} = 0.3,\ \rho_2 = 1 \quad \text{for } 12 < t \leq 52; \\
&a_3 = -0.5,\ b_3 = 0.2,\ c_3 = 0.2,\ k_{13} = 0.1,\ k_{23} = 0.3,\ \rho_3 = 1 \quad \text{for } 52 < t \leq 55.
\end{aligned}$$

It is clear that the points $t_1 = 12$, $t_2 = 52$ are two switching moments.

We estimate the solution of the system (3) at the final moment

$$t_3 = 55$$

if the initial state of the system satisfies the conditions

$$|x(t_0)| = 1, \qquad |\sigma(t_0)| = 1.$$

We divide our calculations into three steps in relation to each subsystem.

1. First time interval

At the first time interval $0 \le t \le 12$, the hybrid system (3) has the form

$$\begin{aligned} \dot{x} &= -x + 0.1 f_1(\sigma), \\ \dot{\sigma} &= 0.2x - f_1(\sigma), \end{aligned} \tag{44}$$

where the function f satisfies the inequality

$$0.2\sigma^2 \le f_1(\sigma)\sigma \le 0.3\,\sigma^2,$$

and the condition (7)

$$a_1\rho_1 - b_1c_1 = 1 - 0.02 > 0$$

holds. The estimation at the moment $t_1 = 12$ is obtained from Theorem 3.1, i.e., formulas (8), (9). First we choose such optimal number β_{10} that the value of the function $\lambda_{\min}\big(S_1\,[\beta_{10}]\,\big)$ (see (22)) will be maximum. Because

$$\beta_{10} = \frac{2\,(2\,a_1\,\rho_1 - b_1\,c_1)}{c_1^2} = 50 \times 1.98 = 99$$

then the Lyapunov function has the form

$$V\big(x, \sigma\big) = x^2 + 99 \int_0^{\sigma} f(s)\,ds.$$

Applying (11) and (10), we have

$$\lambda_{\min}\big(S_1[\beta_{10}]\big) = 25\left\{(0.04 + 2 - 0.01) - \sqrt{(0.04 - 2 + 0.01)^2 + 0.16}\right\} = 0.97,$$

$$\xi(\beta_{10}) = \lambda_{\min}\big(S_1[\beta_{10}]\big)\,\min\left\{1, \frac{k_{11}^2\,c_1^2}{k_{21}\,(2\,a_1\,\rho_1 - b_1\,c_1)}\right\} = 0.0026.$$

Finally, we obtain the estimation of solution using the formulas (8), (9)

$$\begin{aligned} \big|x(12)\big| &\le \left[|x(t_0)| + \sqrt{\frac{1}{2}\,\beta_{10}k_{21}}|\sigma(t_0)|\right] e^{-\frac{1}{2}\,\xi_1(\beta_{10})(12-0)} \\ &= [1 + 3.85 \times 1]e^{-\frac{1}{2}\times 0.0026(12-0)} = 4.7773, \\ |\sigma(12)| &\le \left[\sqrt{\frac{2}{\beta_{10}k_{11}}}|x(t_0)| + \sqrt{\frac{k_{21}}{k_{11}}}|\sigma(t_0)|\right] e^{-\frac{1}{2}\,\xi_1(\beta_{10})(12-0)} \\ &= [0.3178 \times 1 + 1.2247]\,e^{-\frac{1}{2}\times 0.0026(12-0)} = 1.5183. \end{aligned}$$

2. Second time interval

At the second time interval $12 \le t \le 52$, the system (3) has the form

$$\dot{x} = -0.5x + 0.2f_2(\sigma),$$
$$\dot{\sigma} = 0.2x - f_2(\sigma),$$

where the conditions (4) and (7)

$$0.1\,\sigma^2 \le f_2\,(\sigma)\,\sigma \le 0.3\,\sigma^2,$$
$$a_2\,\rho_2 - b_2\,c_2 = 0.5 - 0.04 = 0.46 > 0$$

are fulfilled. For estimation at the initial point of the second interval we already have

$$|x(12)| \le 4.7773,$$
$$|\sigma(12)| \le 1.5183.$$

In order to calculate the estimation of the solution at the moment $t_2 = 52$, we repeat the calculations according to Theorem 3.1. Successively, we obtain

$$\beta_{20} = \frac{2(2\,a_2\,\rho_2 - b_2\,c_2)}{c_2^2} = 48,$$
$$V\big(x,\sigma\big) = x^2 + 48\int_0^\sigma f(s)\,ds,$$

$$\begin{aligned}&\lambda_{\min}\big(S_1[\beta_{20}]\big)\\&= \frac{1}{0.2}\Big\{(0.5\times 0.2^2 + 2\times 0.5\times 1^2 - 0.2\times 0.2\times 1)\\&\quad - \sqrt{(0.5\times 0.2^2 - 2\times 0.5\times 1^2 + 0.2\times 0.2\times 1)^2 + 4\times 0.5^2\times 0.2^2\times 1^2}\Big\}\\&= 0.47,\end{aligned}$$

$$\xi(\beta_{20}) = \lambda_{\min}\big(S_1[\beta_{20}]\big)\,\min\left\{1, \frac{k_1^2c^2}{k_2(2a_2\rho_2 - b_2c_2)}\right\} = 0.00066.$$

Using formulas (8), (9) we obtain the estimation of solution

$$\begin{aligned}\big|x(52)\big| &\le \left[\big|x(12)\big| + \sqrt{\frac{1}{2}\,\beta_{20}\,k_{22}}\;\big|\sigma(12)\big|\right]e^{-\frac{1}{2}\,\xi(\beta_{20})(52-12)}\\&= [4.7773 + 2.6833\times 1.5183]\,e^{-\frac{1}{2}\times 0.00066(52-12)} = 8.7356,\\\big|\sigma\,(52)\,\big| &\le \left[\sqrt{\frac{2}{\beta_{20}\,k_{12}}}\;\big|x\,(12)\,\big| + \sqrt{\frac{k_{22}}{k_{12}}}\;\big|\sigma\,(12)\,\big|\right]e^{-\frac{1}{2}\xi(\beta_{20})(52-12)}\\&= [0.6455\times 4.7773 + 1.7321\times 1.5183]\,e^{-\frac{1}{2}\times 0.0006(52-12)} = 5.6388.\end{aligned}$$

3. Third time interval

The system (3) has the form

$$\dot{x} = 0.5x + 0.2f_1(\sigma), \quad \dot{\sigma} = 0.2x - f_1(\sigma),$$

and the condition

$$0.1\,\sigma^2 \le f_1\,(\sigma)\,\sigma \le 0.3\,\sigma^2,$$

is fulfilled. But the condition (7)

$$a_3\,\rho_3 - b_3\,c_3 = -0.5 - 0.04 = -0.54 < 0$$

is not satisfied, therefore, in this case we use Theorem 4.1. We choose the optimal number $\beta_{30} = 1 > 0$, so

$$V\big(x,\sigma\big) = x^2 + \int_0^\sigma f(s)\,ds.$$

Using (29) we have $\gamma_1 = -0.4556$, and taking into account the last result, by (28) we get

$$\gamma < \min\{2a_3, \gamma_1\} = \min\big\{2(-0.5), -0.4556\big\} = -1.1.$$

In accordance with the relation (30) we can find the optimal value of the function

$$\lambda_{\min}\big(S_2[\beta_{30},\gamma]\big) = 1.1845 \text{ and in accordance with (28) } \xi_2\,(\beta_{30},\gamma) = 0.0071.$$

Consequently, the estimation of the solution is obtained from the formulas (26), (27)

$$\begin{aligned}|x(55)| &\le \left[|x(52)| + \sqrt{\frac{1}{2}\,\beta_{30}k_{23}}|\sigma(52)|\right] e^{-\frac{1}{2}\,\xi_1(\beta_{30},\gamma)(55-52)}\\ &= [8.7356 + 0.3873 \times 5.6388]\,e^{-\frac{1}{2}\times 0.0071(55-52)} = 10.8,\end{aligned}$$

$$\begin{aligned}|\sigma(55)| &\le \left[\sqrt{\frac{2}{\beta_{30}\,k_{13}}}|x(52)| + \sqrt{\frac{k_{23}}{k_{13}}}|\sigma(52)|\right] e^{-\frac{1}{2}\,\xi_1(\beta_{30},\gamma)(55-52)}\\ &= [4.4721 \times 8.7356 + 1.7321 \times 5.6388]e^{-\frac{1}{2}\times 0.0071(55-52)} = 48.31.\end{aligned}$$

REFERENCES

[1] AJZERMAN, M. A.—HANTMAKHER, F. R.: *Absolute Stability of Regulation Systems.* Izd. AN USSR, Moscow, 1963.

[2] BARBASHIN, E. A.: *Method of Lyapunov Functions.* Nauka, Moscow, 1970.

[3] BRANICKY, M. S.: *Multiple Lyapunov functions and other analysis tools for switched and hybrid systems,* IEEE Trans. Automat. Control **43** (1998), 32–45.

[4] BRANICKY, M. S.: *Stability of switched and hybrid systems*, in: Proc. 33rd Conf. Decision and Control, Lake Buena Vista, FL, 1994, pp. 3498–3503.

[5] GELIG, A. KH.—LEONOV, G. A.—YAKUBOVICH, V. A.: *Stability of Nonlinear System with Unique State of Equilibrium*. Nauka, Moscow, 1978.

[6] KHUSAINOV, D.—LANGERAK, R.—KUZMYCH, O.: *Estimations of solutions convergence of hybrid systems consisting of linear equations with delay*, Nonlinear Dyn. Syst. Theory **7** (2007), 169–186.

[7] LIAO, X.—YU, P.: *Absolute Stability of Nonlinear Control Systems* (2nd ed.), in: Math. Model. Theory Appl., Vol. 25, Springer, New York, 2008.

[8] LIAO, X.—WANG, L.—YU, P.: *Stability of Dynamical Systems*. Monograph Ser. Nonlinear Sci. Complex., Vol. 5, Elsevier, Amsterdam, 2007.

[9] LURIE, A. I.: *Some Nonlinear Tasks of Theory of Automatic Control*. Gostehizdat, Moscow, 1951.

[10] MANNA, Z.—PNUELI, A.: *Verifying hybrid systems*, in: Hybrid systems, Lecture Notes in Comput. Sci., Vol. 736, Springer-Verlag, Berlin, 1993, pp. 4–35.

[11] MARTYNYUK, A. A.: *Stability of Motion of Complex Systems*. Naukova dumka, Kiev, 1975. (In Russian)

[12] XUE, D.—CHEN, Y.—ATHERTON, D.: *Linear Feedback Control: Analysis and Design with MATLAB*, in: Adv. Des. Control, Vol. 14, SIAM, Philadelphia, 2007.

[13] KHUSAINOV, D. YA.—KUZMICH, O. I.: *The estimations of logic-dynamical (hybrid) systems stability with time switching*, Bull. Univ. Kiev, Ser. Phys.-Math. **1** (2005), 230–237.

[14] KUZMICH, O. I.: *Estimations of stability of dynamics of the hybrid systems with the finite number of switching*, Bull. Univ. Kiev, Ser. Phys.-Math. **2** (2005), 260–267.

Received January 7, 2011

Miroslava Růžičková
The University of Žilina
Department of Mathematics
Univerzitná 8215/1
SK–010-26 Žilina
SLOVAKIA
E-mail: miroslava.ruzickova@fpv.uniza.sk

Olena Kuzmych
Volyn National
Lesya Ukrainka University
Voly 13
UA–43-000 Lutsk
UKRAINE
E-mail: lenamaks79@mail.ru

Denys Ya. Khusainov
Kyiv National
Taras Shevchenko University
Vladimirskaya 64
UA–01-033 Kyiv
UKRAINE
E-mail: khusainov@unicyb.kiev.ua