

APPLICATION OF HUREWICZ THEOREM TO CLASSIFICATION OF Π_1^1 -COMPLETE SETS

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ABSTRACT. We present a simple proof of a dichotomy theorem by Kechris-Louveau-Woodin [Kechris, A. S., Louveau, A.: *Descriptive set theory and the structure of sets of uniqueness*, London Math. Soc. Lecture Note Ser. **128** (1987), 104–138] telling that every Π_1^1 σ -ideal of compact subsets of a Polish space is either G_δ or else Π_1^1 -complete.

The descriptive hierarchy of sets raised a question of describing Π_1^1 (co-analytic) sets which are not Σ_1^1 (analytic), i.e., the true Π_1^1 sets. Following M. J. Souslin [4], we know that a set which is both analytic and coanalytic is Borel. Since there are Π_1^1 but not Borel sets, an important kind of true Π_1^1 sets is that of Π_1^1 -complete set. In particular, Π_1^1 -complete set on a hyperspace.

The main goal of this note is a simple proof of the Kechris-Louveau-Woodin Dichotomy Theorem [3]. The original proof by A. S. Kechris based on the game theory using Hurewicz Theorem [1] is elementary, however, rather complicated. We will use common set theoretical terminology and notations, say those of [2].

Let X be a Polish space. A subset P of X is called **Π_1^1 -hard** if for any Polish space Y and any Π_1^1 subset Q of Y , there is a Borel function $f: Y \rightarrow X$ such that $Q = f^{-1}[P]$. A Π_1^1 set which is Π_1^1 -hard is said to be **Π_1^1 -complete**. Replacing Π_1^1 with Σ_1^1 , we obtain the definition of Σ_1^1 -hard and Σ_1^1 -complete set.

Following Souslin's theorem again, we know that no Π_1^1 -complete set is Σ_1^1 . If X, Y are Polish spaces, $f: X \rightarrow Y$ is a Borel function, $P \subseteq X$ is Π_1^1 -complete and $Q \subseteq Y$ is Π_1^1 and such that $P = f^{-1}[Q]$, then Q is also Π_1^1 -complete. This is a standard procedure for showing the Π_1^1 -completeness of a set. Likewise, the Π_1^1 -hardness of a set.

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For each topological space X let $\mathcal{K}(X)$ denote the space of all compact subsets of X equipped with the **Vietoris topology** generated by the sets

$$\{K \in \mathcal{K}(X) : K \cap V \neq \emptyset\} \quad \text{and} \quad \{K \in \mathcal{K}(X) : K \subseteq V\},$$

V being a non-empty open set.

The Vietoris topology is induced by the **Hausdorff metric** on $\mathcal{K}(X)$ which is defined as

$$\delta(K, L) = \begin{cases} \max\left\{\max_{x \in K} \text{dist}(x, L), \max_{y \in L} \text{dist}(y, K)\right\} & \text{if } K, L \neq \emptyset, \\ 0 & \text{if } K = L = \emptyset, \\ 1 & \text{if exactly one of } K, L \text{ is } \emptyset. \end{cases}$$

Any subtree of ${}^{<\omega}\omega$ can be identified with its characteristic function as a member of ${}^{<\omega}\omega 2$. If ${}^{<\omega}\omega 2$ is given the product topology with $2 = \{0, 1\}$ discrete, then it is homeomorphic to the Cantor space ${}^\omega 2$. The set of all subtrees of ${}^{<\omega}\omega$ is a closed subset of ${}^\omega 2$. Moreover, the set of those subtrees of ${}^{<\omega}\omega$ having all branches finite, is a Π_1^1 -complete set.

A. S. Kechris [3] has proved the next result. We now present a slightly modified version of his proof.

LEMMA 1. *If Q is a countable dense subset of ${}^\omega 2$ then $\mathcal{K}(Q)$ is Π_1^1 -complete.*

Proof. Let $Q \subseteq {}^\omega 2$ be a countable dense set. Q is an F_σ set, so $\mathcal{K}(Q)$ is Π_1^1 .

Let $P \subseteq {}^\omega 2$ be a Π_1^1 -complete set as mentioned above. Since ${}^\omega 2 \setminus P$ is Σ_1^1 , there exists a continuous mapping $\lambda: {}^\omega \omega \rightarrow {}^\omega 2$ such that $\lambda({}^\omega \omega) = {}^\omega 2 \setminus P$. Let $\pi: \omega \times \omega \rightarrow \omega$ be any bijection and $t: {}^\omega \omega \rightarrow {}^\omega 2$ be the mapping defined by

$$t(v)(\pi(n, m)) = \begin{cases} 0 & \text{if } v(n) = m, \\ 1 & \text{other,} \end{cases}$$

for $v \in {}^\omega \omega$. Then the set $t({}^\omega \omega)$ is equal to

$$\bigcap_n \bigcup_m \left\{ \delta \in {}^\omega 2 : \delta(\pi(n, m)) = 0 \right\} \\ \cap \bigcap_n \bigcap_m \bigcap_{k \neq m} \left\{ \delta \in {}^\omega 2 : \delta(\pi(n, m)) = 0 \rightarrow \delta(\pi(n, k)) = 1 \right\}$$

and therefore, is a G_δ subset of ${}^\omega 2$. Let $G \subseteq {}^\omega 2 \times {}^\omega 2$ be such that

$$(x, y) \in G \Leftrightarrow \lambda(t^{-1}(y)) = x.$$

Then G is a G_δ set. If we denote $F = ({}^\omega 2 \times {}^\omega 2) \setminus G$, then

$$x \in P \Leftrightarrow (\forall y \in {}^\omega 2) (x, y) \in F.$$

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Since ${}^\omega 2 \times {}^\omega 2$ is homeomorphic to ${}^\omega 2$, there exists a continuous $h: {}^\omega 2 \times {}^\omega 2 \rightarrow {}^\omega 2$ such that $h^{-1}(Q) = F$. The mapping $\mu: {}^\omega 2 \rightarrow \mathcal{K}({}^\omega 2)$ defined by

$$\mu(x) = h[\{x\}, {}^\omega 2]$$

is continuous and

$$x \in P \Leftrightarrow \mu(x) \in \mathcal{K}(Q).$$

Thus, $P = \mu^{-1}(\mathcal{K}(Q))$ and therefore $\mathcal{K}(Q)$ is Π_1^1 -complete. $\square$

Note that if Q is a set of all eventually periodic sequences in ${}^\omega 2$, then we can reduce $\mathcal{K}(Q)$ to $\mathcal{K}(\mathbb{Q} \cap [0, 1])$. So, we obtain

COROLLARY 2 ([1, W. Hurewicz]). $\mathcal{K}(\mathbb{Q} \cap [0, 1])$ is Π_1^1 -complete set.

A complementary result is that the set $\{K \in \mathcal{K}([0, 1]) : K \cap \mathbb{I} \neq \emptyset\}$ is Σ_1^1 -complete. Moreover, if X is a Polish, and $G \subseteq X$ is a G_δ set but not an F_σ set, then $\{K \in \mathcal{K}(X) : K \cap G \neq \emptyset\}$ is Σ_1^1 -complete. Thus, if $F \subseteq X$ is an F_σ set but not a G_δ set, then $\mathcal{K}(F)$ is Π_1^1 -complete.

A subset $\mathcal{I} \subseteq \mathcal{K}(X)$ is called a σ -ideal of compact sets of X if the following conditions are fulfilled

- (1) if $K \in \mathcal{K}(X), K \subseteq L \in \mathcal{I}$ then $K \in \mathcal{I}$,
- (2) if $K_n \in \mathcal{I}, n \in \omega$ and $\bigcup_n K_n \in \mathcal{K}(X)$ then $\bigcup_n K_n \in \mathcal{I}$.

Let $\mathcal{S}$ be a non-empty subset of $\mathcal{K}(X)$. We denote by $\mathcal{S}_\sigma$ the set

$$\mathcal{S}_\sigma = \left\{ M \in \mathcal{K}(X) : M = \bigcup_n M_n, M_n \in \mathcal{S}, n \in \omega \right\}.$$

Let us remark that $\mathcal{S}_\sigma$ is closed under countable unions which are in $\mathcal{K}(X)$, i.e., it satisfies condition (2) of the previous definition, but need not be σ -ideal of compact sets of X .

THEOREM 3. Let X be a Polish space, and $\mathcal{S}$ be a non-empty subset of $\mathcal{K}(X)$ such that $\mathcal{S}$ contains a countable perfect subset $\mathcal{L}$ in $\mathcal{S}$.¹ If $\mathcal{L}$ is closed in $\mathcal{S}_\sigma$ and

$$(\forall \mathcal{M} \in \mathcal{K}(\overline{\mathcal{L}})) \left(\bigcup \mathcal{M} \in \mathcal{S}_\sigma \wedge M \in \mathcal{M} \rightarrow M \in \mathcal{S}_\sigma \right)$$

then $\mathcal{S}_\sigma$ is Π_1^1 -hard.

Proof. $\overline{\mathcal{L}}$ is a closed subset of $\mathcal{K}(X)$, so we consider $\overline{\mathcal{L}}$ with a subspace topology of $\mathcal{K}(X)$. Let $h: \mathcal{K}(\overline{\mathcal{L}}) \rightarrow \mathcal{K}(X)$ be the mapping defined by

$$h(\mathcal{M}) = \bigcup \mathcal{M} = \bigcup \{M : M \in \mathcal{M}\}.$$

Then h is continuous, see, e.g., [3]. By the assumptions, we can verify that

$$\mathcal{M} \subseteq \overline{\mathcal{L}} \cap \mathcal{S} = \mathcal{L} \Leftrightarrow \bigcup \mathcal{M} \in \mathcal{S}_\sigma$$

¹A perfect set is a non-empty closed set without isolated points.

for any $\mathcal{M} \in \mathcal{K}(\overline{\mathcal{L}})$. Thus,

$$h^{-1}(\mathcal{S}_\sigma) = \mathcal{K}(\mathcal{L}) = \{\mathcal{M} \in \mathcal{K}(\overline{\mathcal{L}}) : \mathcal{M} \subseteq \mathcal{L}\}.$$

Since $\mathcal{L}$ is an F_σ , but not a G_δ set, then $\mathcal{K}(\mathcal{L})$ is Π_1^1 -complete. Thus $\mathcal{S}_\sigma$ is Π_1^1 -hard. $\square$

COROLLARY 4. *Let X be a Polish space, and $\mathcal{S}$ be a non-empty subset of $\mathcal{K}(X)$ such that $\mathcal{S}$ contains a countable perfect subset $\mathcal{L}$ in $\mathcal{S}$. If there exists a σ -ideal $\mathcal{I} \supseteq \mathcal{S}_\sigma$ of compact sets of X such that $\mathcal{L}$ is closed in $\mathcal{I}$, then $\mathcal{S}_\sigma$ is Π_1^1 -hard.*

Proof. Let $\mathcal{I} \supseteq \mathcal{S}_\sigma$ be a σ -ideal of compact sets of X such that $\mathcal{I} \cap \overline{\mathcal{L}} = \mathcal{L}$. Then it is easy to see that $\mathcal{L}$ is closed in $\mathcal{S}_\sigma$. Let $\mathcal{M}$ be any non-empty set from $\mathcal{K}(\overline{\mathcal{L}})$ such that $\bigcup \mathcal{M} \in \mathcal{S}_\sigma$. If $M \in \mathcal{M}$, then $M \subseteq \bigcup \mathcal{M} \in \mathcal{S}_\sigma \subseteq \mathcal{I}$. Thus, $M \in \mathcal{I} \cap \overline{\mathcal{L}} = \mathcal{L} \subseteq \mathcal{S} \subseteq \mathcal{S}_\sigma$. By the previous theorem, $\mathcal{S}_\sigma$ is Π_1^1 -hard. $\square$

If we take $X = [0, 1]$ and $\mathcal{S} = \mathcal{K}(\mathbb{Q} \cap [0, 1])$, then the set of all singletons on $\mathcal{K}(\mathbb{Q} \cap [0, 1])$ is a countable perfect subset of $\mathcal{S}$. Therefore, by Corollary 4, the set $\mathcal{S} = \mathcal{S}_\sigma = \mathcal{K}(\mathbb{Q} \cap [0, 1])$ is Π_1^1 -hard, and hence Π_1^1 -complete.

A. S. Kechris [3] presents a proof of Corollary 5 based on game theory, in which he uses a theorem of Hurewicz [1], telling that if A is a coanalytic non- G_δ set in a Polish space X , then there exists a homeomorphism $f: {}^\omega 2 \rightarrow X$ and a countable dense set $Q \subseteq {}^\omega 2$ such that $f[Q] \subseteq A$ and $f[{}^\omega 2 \setminus Q] \subseteq X \setminus A$. An elementary proof of a generalization of Hurewicz's theorem, using the notion of Hurewicz scheme, is presented in [5]. As its application, we suggest a simple proof of the next assertion.

COROLLARY 5. *Let $\mathcal{I}$ be a Π_1^1 σ -ideal of compact sets of a Polish space X and $\mathcal{S}$ be a non-empty subset of $\mathcal{I}$. If $\mathcal{S}, \mathcal{I}$ cannot be separated by a G_δ set, then $\mathcal{S}_\sigma$ is Π_1^1 -hard.*

Proof. Assume that there is no G_δ set separating $\mathcal{S}, \mathcal{I}$, i.e., $\mathcal{K}(X) \setminus \mathcal{I}, \mathcal{K}(X) \setminus \mathcal{S}$ cannot be separated by an F_σ set. According to the main theorem of [5], there exists a countable perfect subset $\mathcal{L} \subseteq \mathcal{S}$ such that $\overline{\mathcal{L}} \setminus \mathcal{L} \subseteq \mathcal{K}(X) \setminus \mathcal{I}$. Then, by Corollary 4, $\mathcal{S}_\sigma$ is Π_1^1 -hard. $\square$

In particular, if we take $\mathcal{S} = \mathcal{I}$, we get the Kechris-Louveau-Woodin's Dichotomy Theorem [3].

THEOREM 6 ([3, Kechris, Louveau, Woodin]). *Every Π_1^1 σ -ideal of compact subsets of a Polish space X is either G_δ or else Π_1^1 -complete.*

Thus, we know that any Π_1^1 -complete set is true Π_1^1 , and from the previous theorem, that every true Π_1^1 σ -ideal of compact subsets is Π_1^1 -complete.

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