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SOLVABILITY OF A LINEAR NON-LOCAL BOUNDARY VALUE PROBLEM FOR NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. New sufficient conditions are established for the solvability as well as unique solvability of a linear non-local boundary value problem for nonlinear functional differential equations.

1. Introduction and notation

On the interval $[a, b]$, we consider the functional differential equation

$$u'(t) = F(u)(t), \tag{1}$$

where $F \colon C([a, b]; \mathbb{R}) \to L([a, b]; \mathbb{R})$ is a continuous (in general) nonlinear operator. As usual, by a solution of this equation we understand an absolutely continuous function $u \colon [a, b] \to \mathbb{R}$ satisfying the equality (1) almost everywhere on the interval $[a, b]$. For the basic theory of functional differential equations we refer to monographs [1, 4, 6, 8] (see also references therein). In spite of many interesting results there is a broad field for further investigations, namely, in the question on the solvability of various boundary value problems for the equation (1).

In the present paper, along with the equation (1), we consider the non-local boundary condition

$$h(u) = c, \tag{2}$$

where $h \colon C([a, b]; \mathbb{R}) \to \mathbb{R}$ is a (non-zero) linear bounded functional and c is a real number. The results presented below can be regarded as a generalization of those established in [2] (see also [3, Chapter 14]).

The following notation is used in the sequel.

(1) $\mathbb{R}$ is the set of all real numbers. $\mathbb{R}_+ = [0, +\infty]$.

(2) $C\big([a,b];\mathbb{R}\big)$ is the Banach space of continuous functions $v\colon [a,b] \to \mathbb{R}$ with the norm $\|v\|_C = \max\big\{|v(t)| : t \in [a,b]\big\}$.

(3) $C\big([a,b];\mathbb{R}_+\big) = \big\{u \in C\big([a,b];\mathbb{R}\big) : u(t) \geq 0 \text{ for } t \in [a,b]\big\}$.

(4) $AC\big([a,b];D\big)$, where $D \subseteq \mathbb{R}$, is the set of absolutely continuous functions $v\colon [a,b] \to D$.

(5) $L\big([a,b];\mathbb{R}\big)$ is the Banach space of Lebesgue integrable functions $p\colon [a,b] \to \mathbb{R}$ with the norm $\|p\|_L = \int_a^b |p(s)|\,\mathrm{d}s$.

(6) $L\big([a,b];\mathbb{R}_+\big) = \big\{p \in L\big([a,b];\mathbb{R}\big) : p(t) \geq 0 \text{ for almost all } t \in [a,b]\big\}$.

(7) $\mathcal{L}_{ab}$ is the set of linear operators $\ell\colon C\big([a,b];\mathbb{R}\big) \to L\big([a,b];\mathbb{R}\big)$ for which there is a function $\eta \in L\big([a,b];\mathbb{R}_+\big)$ such that

$$|\ell(v)(t)| \leq \eta(t)\|v\|_C \quad \text{for almost all} \quad t \in [a,b] \quad \text{and all} \quad v \in C\big([a,b];\mathbb{R}\big).$$

(8) P_{ab} is the set of operators $\ell \in \mathcal{L}_{ab}$ transforming the set $C\big([a,b];\mathbb{R}_+\big)$ into the set $L\big([a,b];\mathbb{R}_+\big)$.

(9) F_{ab} is the set of linear bounded functionals $h\colon C\big([a,b];\mathbb{R}\big) \to \mathbb{R}$.

(10) PF_{ab} is the set of functionals $h \in F_{ab}$ transforming the set $C\big([a,b];\mathbb{R}_+\big)$ into the set $\mathbb{R}_+$.

(11) $K\big([a,b] \times \mathbb{R}_+; \mathbb{R}_+\big)$ is the set of functions $f\colon [a,b] \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the Carathéodory conditions, i.e., $f(\cdot, x)\colon [a,b] \to \mathbb{R}_+$ is a measurable function for all $x \in \mathbb{R}_+$, $f(t, \cdot)\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function for almost all $t \in [a,b]$, and for every $r > 0$ there exists $q_r \in L\big([a,b];\mathbb{R}_+\big)$ such that

$$|f(t,x)| \leq q_r(t) \quad \text{for almost all} \quad t \in [a,b] \quad \text{and all} \quad x \in [0,r].$$

2. Main results

First formulate the following definition of a solution of the problem (1), (2).

DEFINITION 2.1. By a solution to the equation (1) we understand an absolutely continuous function $u\colon [a,b] \to \mathbb{R}$ satisfying the equality (1) almost everywhere on the interval $[a,b]$. A solution to the equation (1) satisfying the boundary condition (2) is said to be a solution to the problem (1), (2).

We assume in theorems below that the functional h in the boundary condition (2) admits the representation $h = h_0 - h_1$, where $h_0, h_1 \in PF_{ab}$. There is

no lost of generality in assuming this, because an arbitrary linear bounded functional can be expressed in this form (see [7]). Moreover, the following assumption is used:

(H) $F \colon C\big([a,b];\mathbb{R}\big) \to L\big([a,b];\mathbb{R}\big)$ is a continuous operator such that the relation

$$\sup\Big\{|F(v)(\cdot)| : v \in C\big([a,b];\mathbb{R}\big), \ \|v\|_C \le r\Big\} \in L\big([a,b];\mathbb{R}_+\big)$$

is satisfied for every $r > 0$.

Before formulation of the main results we introduce the following notation. Having a point $c \in [a,b]$ and a functional $h_0 \in PF_{ab}$, for any $\lambda \ge 0$, we put

$$h_{0,c}^{\lambda}(v) \overset{\text{def}}{=} h_0(v) - \lambda v(c) \qquad \text{for} \quad v \in C\big([a,b];\mathbb{R}\big). \tag{3}$$

Obviously, $h_{0,c}^0 \in PF_{ab}$. Therefore, we set

$$\lambda_c^* = \sup\Big\{\lambda \ge 0 : h_{0,c}^{\lambda} \in PF_{ab}\Big\}. \tag{4}$$

It is clear that $0 \le \lambda_c^* \le h_0(1)$ and

$$h_{0,c}^{\lambda_c^*} \in PF_{ab}. \tag{5}$$

THEOREM 2.1. *Let the assumption* (H) *be satisfied, the functional h admit the representation $h = h_0 - h_1$ with $h_0, h_1 \in PF_{ab}$, the number λ_a^* be defined by the formula* (4), *and the condition*

$$h_1(1) < \lambda_a^* \tag{6}$$

hold. Let, moreover, there exist

$$\ell_0, \ell_1 \in P_{ab} \tag{7}$$

such that, for any $v \in C\big([a,b];\mathbb{R}\big)$, the inequality

$$\big[F(v)(t) - \ell_0(v)(t) + \ell_1(v)(t)\big]\operatorname{sgn} v(t) \le q(t, \|v\|_C) \quad \text{for almost all } t \in [a,b] \tag{8}$$

holds, where the function $q \in K\big([a,b] \times \mathbb{R}_+; \mathbb{R}_+\big)$ satisfies the condition

$$\lim_{x \to +\infty} \frac{1}{x} \int\limits_a^b q(s,x)\,\mathrm{d}s = 0. \tag{9}$$

If, in addition, either

$$\|\ell_0\| < 1 - \frac{1}{\lambda_a^*}\,h_1(1) - \left(\frac{1}{\lambda_a^*}\big(h_0(1) - \lambda_a^*\big)\right)^2,$$

$$\|\ell_1\| < 2\sqrt{1 - \frac{1}{\lambda_a^*}\,h_1(1) - \|\ell_0\|} - \frac{1}{\lambda_a^*}\big(h_0(1) - \lambda_a^*\big), \tag{10}$$

or

$$\|\ell_0\| \geq 1 - \frac{1}{\lambda_a^*} h_1(1) - \left(\frac{1}{\lambda_a^*}\big(h_0(1) - \lambda_a^*\big)\right)^2,$$

$$\lambda_a^* \|\ell_0\| + \big(h_0(1) - \lambda_a^*\big)\|\ell_1\| < \lambda_a^* - h_1(1) \tag{11}$$

is fulfilled, then the problem (1), (2) *has at least one solution.*

Remark 2.1. Let $\mathcal{A}$ denote the set of $(x, y) \in \mathbb{R}_+^2$ such that either

$$x < 1 - \frac{1}{\lambda_a^*} h_1(1) - \left(\frac{1}{\lambda_a^*}\big(h_0(1) - \lambda_a^*\big)\right)^2,$$

$$y < 2\sqrt{1 - \frac{1}{\lambda_a^*} h_1(1) - x} - \frac{1}{\lambda_a^*}\big(h_0(1) - \lambda_a^*\big),$$

or

$$x \geq 1 - \frac{1}{\lambda_a^*} h_1(1) - \left(\frac{1}{\lambda_a^*}\big(h_0(1) - \lambda_a^*\big)\right)^2,$$

$$\lambda_a^* x + \big(h_0(1) - \lambda_a^*\big)y < \lambda_a^* - h_1(1),$$

where the functionals $h_0, h_1 \in PF_{ab}$ are fixed and the number λ_a^* is defined by the formula (4) (see Fig. 1). Assume that the condition (6) holds, there exist
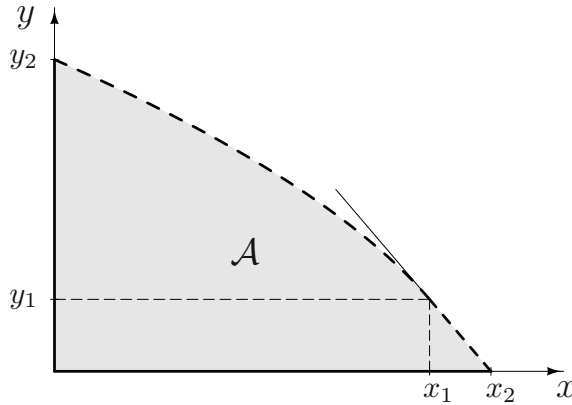

FIGURE 1. The set $\mathcal{A}$: $x_1 = 1 - \frac{1}{\lambda_a^*} h_1(1) - \left(\frac{1}{\lambda_a^*}\big(h_0(1) - \lambda_a^*\big)\right)^2$, $x_2 = 1 - \frac{1}{\lambda_a^*} h_1(1)$, $y_1 = \frac{1}{\lambda_a^*}\big(h_0(1) - \lambda_a^*\big)$, $y_2 = 2\sqrt{1 - \frac{1}{\lambda_a^*} h_1(1)} - \frac{1}{\lambda_a^*}\big(h_0(1) - \lambda_a^*\big)$.

$\ell_0, \ell_1 \in P_{ab}$ such that the inequality (8) is satisfied on the set $C\big([a, b]; \mathbb{R}\big)$, and

$$\big(\|\ell_0\|, \|\ell_1\|\big) \in \mathcal{A}.$$

Then the problem (1), (2) with $h = h_0 - h_1$ has at least one solution.

Remark 2.2. If the functional h is defined by the formula

$$h(v) \stackrel{\text{def}}{=} \alpha v(a) + \beta v(b) \qquad \text{for} \quad v \in C\big([a,b];\mathbb{R}\big)$$

with $\alpha, \beta > 0$, the assumptions (10) and (11) of the previous theorem take the forms

$$\|\ell_0\| < 1 - \left(\frac{\beta}{\alpha}\right)^2, \qquad \|\ell_1\| < -\frac{\beta}{\alpha} + 2\sqrt{1 - \|\ell_0\|}$$

and

$$\|\ell_0\| \geq 1 - \left(\frac{\beta}{\alpha}\right)^2, \qquad \alpha\|\ell_0\| + \beta\|\ell_1\| < \alpha,$$

respectively. Thus, in this case, Theorem 2.1 reduces to Theorems 14.1 and 14.6 established in [3].

THEOREM 2.2. *Let the assumption* (H) *be satisfied, the functional h admit the representation $h = h_0 - h_1$ with $h_0, h_1 \in PF_{ab}$, the number λ_b^* be defined by the formula* (4), *and the condition*

$$h_1(1) < \lambda_b^* \tag{12}$$

hold. Let, moreover, there exist $\ell_0, \ell_1 \in P_{ab}$ such that, for any $v \in C\big([a,b];\mathbb{R}\big)$, the inequality

$$\big[F(v)(t) - \ell_0(v)(t) + \ell_1(v)(t)\big] \operatorname{sgn} v(t) \geq -q(t, \|v\|_C) \qquad \text{for almost all } t \in [a,b],$$

where the function $q \in K\big([a,b] \times \mathbb{R}_+; \mathbb{R}_+\big)$ satisfies the condition (9), *holds. If, in addition, either*

$$\|\ell_1\| < 1 - \frac{1}{\lambda_b^*}h_1(1) - \left(\frac{1}{\lambda_b^*}\big(h_0(1) - \lambda_b^*\big)\right)^2,$$

$$\|\ell_0\| < 2\sqrt{1 - \frac{1}{\lambda_b^*}h_1(1) - \|\ell_1\|} - \frac{1}{\lambda_b^*}\big(h_0(1) - \lambda_b^*\big), \tag{13}$$

or

$$\|\ell_1\| \geq 1 - \frac{1}{\lambda_b^*}h_1(1) - \left(\frac{1}{\lambda_b^*}\big(h_0(1) - \lambda_b^*\big)\right)^2,$$

$$\big(h_0(1) - \lambda_b^*\big)\|\ell_0\| + \lambda_b^*\|\ell_1\| < \lambda_b^* - h_1(1) \tag{14}$$

is fulfilled, then the problem (1), (2) *has at least one solution.*

The next theorems deal with the unique solvability of the problem (1), (2).

THEOREM 2.3. *Let the assumption* (H) *be satisfied, the functional h admit the representation $h = h_0 - h_1$ with $h_0, h_1 \in PF_{ab}$, the number λ_a^* be defined by the*

formula (4), and the condition (6) hold. Let, moreover, there exist $\ell_0, \ell_1 \in P_{ab}$ such that the inequality

$$\big[F(v)(t) - F(w)(t) - \ell_0(v - w)(t) + \ell_1(v - w)(t)\big] \operatorname{sgn}\big(v(t) - w(t)\big) \leq 0$$
$$\text{for almost all } t \in [a, b] \quad (15)$$

holds on the space $C\big([a, b]; \mathbb{R}\big)$. If, in addition, either the conditions (10), or the conditions (11) are fulfilled, then the problem (1), (2) is uniquely solvable.

THEOREM 2.4. *Let the assumption* (H) *be satisfied, the functional* h *admit the representation* $h = h_0 - h_1$ *with* $h_0, h_1 \in PF_{ab}$, *the number* λ_b^* *be defined by the formula* (4), *and the condition* (12) *hold. Let, moreover, there exist* $\ell_0, \ell_1 \in P_{ab}$ *such that the inequality*

$$\big[F(v)(t) - F(w)(t) - \ell_0(v - w)(t) + \ell_1(v - w)(t)\big] \operatorname{sgn}\big(v(t) - w(t)\big) \geq 0$$
$$\text{for almost all } t \in [a, b]$$

holds on the space $C\big([a, b]; \mathbb{R}\big)$. *If, in addition, either the conditions* (13), *or the conditions* (14) *are fulfilled, then the problem* (1), (2) *is uniquely solvable.*

3. Proofs

The main results are proved using the lemma on a priory estimate due to K i g u r a d z e and P ů ž a. This lemma can be formulated as follows.

LEMMA 3.1 ([5, Cor. 2]). *Let there exist a positive number* ρ *and an operator* $\ell \in \mathcal{L}_{ab}$ *such that homogeneous problem*

$$u'(t) = \ell(u)(t), \qquad h(u) = 0 \tag{16}$$

has only the trivial solution and, for every $\delta \in\ ]0, 1[$, *an arbitrary function* $u \in AC\big([a, b]; \mathbb{R}\big)$ *satisfying the relations*

$$u'(t) = \ell(u)(t) + \delta\big[F(u)(t) - \ell(u)(t)\big] \quad \text{for almost all } t \in [a, b], \quad h(u) = \delta c \tag{17}$$

admits the estimate

$$\|u\|_C \leq \rho. \tag{18}$$

Then the problem (1), (2) *has at least one solution.*

Moreover, we need the following two lemmas.

LEMMA 3.2. *Let the operator* $\omega \colon C\big([a, b]; \mathbb{R}\big) \to C\big([a, b]; \mathbb{R}\big)$ *be defined by the formula*

$$\omega(z)(t) \stackrel{\text{def}}{=} z(a + b - t) \qquad \text{for} \quad t \in [a, b], \quad z \in C\big([a, b]; \mathbb{R}\big).$$

Put

$$\widehat{F}(z)(t) \overset{\text{def}}{=} -F\big(\omega(z)\big)(a+b-t) \qquad \textit{for almost all } t \in [a,b] \quad \textit{and all } z \in C\big([a,b];\mathbb{R}\big)$$

and

$$\widehat{h}(z) \overset{\text{def}}{=} h\big(\omega(z)\big) \qquad\qquad \textit{for} \quad z \in C\big([a,b];\mathbb{R}\big).$$

Then u is a solution of the problem (1), (2) *if and only if the function $v = \omega(u)$ is a solution of the problem*

$$v'(t) = \widehat{F}(v)(t), \qquad \widehat{h}(v) = c.$$

LEMMA 3.3. *Let the assumption* (H) *be satisfied, the functional h admit the representation $h = h_0 - h_1$ with $h_0, h_1 \in PF_{ab}$, the number λ_a^* be defined by the formula* (4), *and the condition* (6) *hold. Let moreover, the operators $\ell_0, \ell_1 \in P_{ab}$ be such that either conditions* (10), *or conditions* (11) *are fulfilled. Then there exists $r > 0$ such that, for arbitrary $c^* \in \mathbb{R}_+$ and $q^* \in L\big([a,b];\mathbb{R}_+\big)$, every function $u \in AC\big([a,b];\mathbb{R}\big)$ satisfying the inequalities*

$$h(u)\operatorname{sgn} u(a) \le c^*, \tag{19}$$

$$\big[u'(t) - \ell_0(u)(t) + \ell_1(u)(t)\big]\operatorname{sgn} u(t) \le q^*(t) \qquad \textit{for almost all } t \in [a,b] \tag{20}$$

admits the estimate

$$\|u\|_C \le r(c^* + \|q^*\|_L). \tag{21}$$

P r o o f. Let $c^* \in \mathbb{R}_+$, $q^* \in L([a,b];\mathbb{R}_+)$, and $u \in AC\big([a,b];\mathbb{R}\big)$ satisfy the conditions (19) and (20). We shall show that the estimate (21) holds, where the number r depends only on $\|\ell_0\|$, $\|\ell_1\|$, λ_a^*, $h_0(1)$, and $h_1(1)$. It is clear that

$$u'(t) = \ell_0(u)(t) - \ell_1(u)(t) + \widetilde{q}(t) \qquad \text{for almost all } t \in [a,b], \tag{22}$$

where

$$\widetilde{q}(t) = u'(t) - \ell_0(u)(t) + \ell_1(u)(t) \qquad \text{for almost all } t \in [a,b]. \tag{23}$$

From the condition (20) we get

$$\widetilde{q}(t)\operatorname{sgn} u(t) \le q^*(t) \qquad\qquad \text{for almost all } t \in [a,b]. \tag{24}$$

First suppose that the function u does not change its sign. Put

$$\bar{M} = \max\big\{|u(t)| : t \in [a,b]\big\} \tag{25}$$

and choose $t_{\bar{M}} \in [a,b]$ such that

$$|u(t_{\bar{M}})| = \bar{M}. \tag{26}$$

It is clear that $\bar{M} \ge 0$ and, moreover, according to (7), (24), and (25), from (22) we obtain

$$|u(t)|' \le \bar{M}\,\ell_0(1)(t) + q^*(t) \qquad\qquad \text{for almost all } t \in [a,b]. \tag{27}$$

In view of (3), it is clear that

$$h(u)\operatorname{sgn}u(a) = \lambda_a^*|u(a)| - h_1(u)\operatorname{sgn}u(a) + h_{0,a}^{\lambda_a^*}(u)\operatorname{sgn}u(a)$$

and thus the relations (5), (19), and (25) yield

$$|u(a)| \le M\,\frac{h_1(1)}{\lambda_a^*} + \frac{c^*}{\lambda_a^*}\,. \tag{28}$$

The integration of (27) from a to $t_{\bar M}$, on account of (7) and (28) results in

$$\bar M - \bar M\,\frac{h_1(1)}{\lambda_a^*} - \frac{c^*}{\lambda_a^*} \le \bar M\,\|\ell_0\| + \|q^*\|_L.$$

Note that the relations (10) and (11) yield that $\|\ell_0\| < 1 - \frac{h_1(1)}{\lambda_a^*}$. Therefore, from the last inequality we get

$$\|u\|_C \le r_0\,\max\left\{\frac{1}{\lambda_a^*}, 1\right\}\,(c^* + \|q^*\|_L),$$

where

$$r_0 = \left(1 - \frac{h_1(1)}{\lambda_a^*} - \|\ell_0\|\right)^{-1} > 0,$$

and thus the estimate (21) holds, where r is defined by the formula

$$r = r_0\,\max\left\{\frac{1}{\lambda_a^*}, 1\right\}.$$

Now suppose that the function u changes its sign. Put

$$M = \max\big\{u(t) : t \in [a,b]\big\}, \qquad m = -\min\big\{u(t) : t \in [a,b]\big\} \tag{29}$$

and choose $t_M, t_m \in [a,b]$ such that

$$u(t_M) = M, \qquad u(t_m) = -m. \tag{30}$$

Obviously, $M > 0$, $m > 0$, and either

$$t_m < t_M, \tag{31}$$

or

$$t_m > t_M. \tag{32}$$

First suppose that the relation (31) holds. It is clear that there exists $\alpha_2 \in\ ]t_m, t_M[$ such that

$$u(t) > 0 \quad \text{for} \quad \alpha_2 < t \le t_M, \qquad u(\alpha_2) = 0. \tag{33}$$

Let

$$\alpha_1 = \inf\big\{t \in [a, t_m] : u(s) < 0 \quad \text{for} \quad t \le s \le t_m\big\}.$$

Obviously,

$$u(t) < 0 \quad \text{for} \quad \alpha_1 < t \le t_m \qquad \text{and} \qquad u(\alpha_1) = 0 \quad \text{if} \quad \alpha_1 > a. \tag{34}$$

In view of (3), it is clear that

$$\lambda_a^* u(a) = h_1(u) - h_{0,a}^{\lambda_a^*}(u) + h(u)$$

and thus we obtain from (5), (19), (29), and (34) that

$$u(\alpha_1) \ge - m \frac{h_1(1)}{\lambda_a^*} - M \frac{1}{\lambda_a^*} \big(h_0(1) - \lambda_a^* \big) - \frac{c^*}{\lambda_a^*}. \tag{35}$$

The integration of (22) from α_1 to t_m, and from α_2 to t_M, in view of (7), (20), (29), (30), and (33)–(35), yields

$$m - m \frac{h_1(1)}{\lambda_a^*} - M \frac{1}{\lambda_a^*} \big(h_0(1) - \lambda_a^* \big) - \frac{c^*}{\lambda_a^*}$$
$$\le M \int_{\alpha_1}^{t_m} \ell_1(1)(s) \, \mathrm{d}s + m \int_{\alpha_1}^{t_m} \ell_0(1)(s) \, \mathrm{d}s + \int_{\alpha_1}^{t_m} q^*(s) \, \mathrm{d}s$$

and

$$M \le M \int_{\alpha_2}^{t_M} \ell_0(1)(s) \, \mathrm{d}s + m \int_{\alpha_2}^{t_M} \ell_1(1)(s) \, \mathrm{d}s + \int_{\alpha_2}^{t_M} q^*(s) \, \mathrm{d}s.$$

Hence, we have

$$m \left(1 - \frac{h_1(1)}{\lambda_a^*} - C \right) \le M \left(\frac{1}{\lambda_a^*} \big(h_0(1) - \lambda_a^* \big) + A \right) + \|q^*\|_L + \frac{c^*}{\lambda_a^*},$$
$$M \big(1 - D \big) \le mB + \|q^*\|_L, \tag{36}$$

where

$$A = \int_{\alpha_1}^{t_m} \ell_1(1)(s) \, \mathrm{d}s, \qquad B = \int_{\alpha_2}^{t_M} \ell_1(1)(s) \, \mathrm{d}s$$

and

$$C = \int_{\alpha_1}^{t_m} \ell_0(1)(s) \, \mathrm{d}s, \qquad D = \int_{\alpha_2}^{t_M} \ell_0(1)(s) \, \mathrm{d}s.$$

Due to the relations (10) and (11) we have $\|\ell_0\| < 1 - \frac{1}{\lambda_a^*} h_1(1)$ and thus it is clear that $C < 1 - \frac{h_1(1)}{\lambda_a^*}$ and $D < 1$. Consequently, (36) implies

$$
\begin{aligned}
&m\left(1 - \frac{h_1(1)}{\lambda_a^*} - C\right)(1 - D) \\
&\qquad \leq mB\left(\frac{1}{\lambda_a^*}\big(h_0(1) - \lambda_a^*\big) + A\right) + \left(\|q^*\|_L + \frac{c^*}{\lambda_a^*}\right)\left(\frac{h_0(1)}{\lambda_a^*} + A\right), \\
&M\left(1 - \frac{h_1(1)}{\lambda_a^*} - C\right)(1 - D) \qquad\qquad\qquad\qquad\qquad (37) \\
&\qquad \leq MB\left(\frac{h_0(1)}{\lambda_a^*} - 1 + A\right) + \left(\|q^*\|_L + \frac{c^*}{\lambda_a^*}\right)(B + 1).
\end{aligned}
$$

It is clear that

$$
\left(1 - \frac{h_1(1)}{\lambda_a^*} - C\right)(1 - D) \geq 1 - \frac{h_1(1)}{\lambda_a^*} - (C + D) \geq 1 - \frac{h_1(1)}{\lambda_a^*} - \|\ell_0\|. \quad (38)
$$

Let first the inequalities (10) hold. Obviously,

$$
B\left(\frac{h_0(1)}{\lambda_a^*} - 1 + A\right) \leq \frac{1}{4}\left(A + B + \frac{h_0(1)}{\lambda_a^*} - 1\right)^2 \leq \frac{1}{4}\left(\|\ell_1\| + \frac{h_0(1)}{\lambda_a^*} - 1\right)^2.
$$

By the last inequality, (38) and the second inequality in (10), from (37) we get

$$
\begin{aligned}
m &\leq r_1 \max\left\{\frac{1}{\lambda_a^*}, 1\right\}\left(\|q^*\|_L + c^*\right)\left(\frac{h_0(1)}{\lambda_a^*} + \|\ell_1\|\right), \\
M &\leq r_1 \max\left\{\frac{1}{\lambda_a^*}, 1\right\}\left(\|q^*\|_L + c^*\right)(1 + \|\ell_1\|),
\end{aligned}
$$

where

$$
r_1 = \left[1 - \frac{h_1(1)}{\lambda_a^*} - \|\ell_0\| - \frac{1}{4}\left(\|\ell_1\| + \frac{h_0(1)}{\lambda_a^*} - 1\right)^2\right]^{-1}.
$$

Consequently, the estimate (21) holds, where the number r is defined by the formula

$$
r = r_1 \max\left\{\frac{1}{\lambda_a^*}, 1\right\}\left(\frac{h_0(1)}{\lambda_a^*} + \|\ell_1\|\right),
$$

because we have $\lambda_a^* \leq h_0(1)$.

Now let the inequalities (11) hold. Using the relation $\lambda_a^* \leq h_0(1)$, from the inequalities (11) we get

$$
B \leq \|\ell_1\| < \frac{h_0(1)}{\lambda_a^*} - 1,
$$

and thus

$$B\left(\frac{h_0(1)}{\lambda_a^*} - 1 + A\right) \leq B\left(\frac{h_0(1)}{\lambda_a^*} - 1\right) + A\left(\frac{h_0(1)}{\lambda_a^*} - 1\right) \leq \|\ell_1\|\left(\frac{h_0(1)}{\lambda_a^*} - 1\right).$$

By the last inequality, (38) and the second inequality in (11), it follows from (37) that

$$m \leq r_2 \max\left\{\frac{1}{\lambda_a^*}, 1\right\} (\|q^*\|_L + c^*)\left(\frac{h_0(1)}{\lambda_a^*} + \|\ell_1\|\right),$$

$$M \leq r_2 \max\left\{\frac{1}{\lambda_a^*}, 1\right\} (\|q^*\|_L + c^*)(1 + \|\ell_1\|),$$

where

$$r_2 = \left[1 - \frac{h_1(1)}{\lambda_a^*} - \|\ell_0\| - \|\ell_1\|\left(\frac{h_0(1)}{\lambda_a^*} - 1\right)\right]^{-1}.$$

Consequently, the estimate (21) holds, where the number r is given by the formula

$$r = r_2 \max\left\{\frac{1}{\lambda_a^*}, 1\right\}\left(\frac{h_0(1)}{\lambda_a^*} + \|\ell_1\|\right).$$

If the relation (32) holds, the validity of estimate (21) can be proved analogously. □

Now we are in a position to prove Theorems 2.1–2.4.

Proof of Theorem 2.1. Let $\ell = \ell_0 - \ell_1$. It is clear that $\ell \in \mathcal{L}_{ab}$ and the assumptions of Lemma 3.3 are satisfied. Let r be the number appearing in this lemma. According to (9), there exists $\rho > 2r|c|$ such that

$$\frac{1}{x}\int_a^b q(s, x)\,\mathrm{d}s < \frac{1}{2r} \qquad \text{for} \quad x > \rho.$$

First note that the homogeneous problem (16) has only the trivial solution. Indeed, if u is a solution of the problem (16) then the function u satisfies the inequalities (19) and (20) with $c^* = 0$ and $q^* \equiv 0$ and thus Lemma 3.3 guarantees that $u \equiv 0$.

Now assume that a function $u \in AC([a, b]; \mathbb{R})$ satisfies the conditions (17) for some $\delta \in]0, 1[$. Then we obtain from (8) that inequalities (19) and (20) are fulfilled with $c^* = |c|$ and $q^* \equiv q(\cdot, \|u\|_C)$. Hence, using Lemma 3.3 and the definition of the number ρ, we get the estimate (18). Indeed, assuming that $\|u\|_C > \rho$, from the estimate (21) we get

$$1 \leq \frac{r|c|}{\|u\|_C} + \frac{r}{\|u\|_C}\int_a^b q(s, \|u\|_C)\,\mathrm{d}s < 1,$$

which is a contradiction.

Since ρ depends neither on u nor on δ, it follows from Lemma 3.1 that the problem (1), (2) has at least one solution. □

P r o o f   o f   T h e o r e m   2.2. According to Lemma 3.2, the validity of the theorem follows immediately from Theorem 2.1. □

P r o o f   o f   T h e o r e m   2.3. First note that the assumptions of Lemma 3.3 are satisfied.

It follows from the condition (15) that the inequality (8) is fulfilled on the set $C\big([a,b];\mathbb{R}\big)$, where $q \equiv |F(0)|$. Consequently, all the assumptions of Theorem 2.1 are satisfied and thus the problem (1), (2) has at least one solution. It remains to show that problem (1), (2) has at most one solution.

Let u_1, u_2 be arbitrary solutions of the problem (1), (2). Put

$$u(t) = u_1(t) - u_2(t) \qquad \text{for} \quad t \in [a, b].$$

Then $h(u) = 0$ and, by virtue of the inequality (15), we have

$$\big[u'(t) - \ell_0(u)(t) + \ell_1(u)(t)\big] \operatorname{sgn} u(t) \le 0 \qquad \text{for almost all } t \in [a, b].$$

Consequently, the inequalities (19) and (20) are satisfied with $c^* = 0$ and $q^* \equiv 0$. Therefore, Lemma 3.3 guarantees that $u \equiv 0$ and thus $u_1 \equiv u_2$. □

P r o o f   o f   T h e o r e m   2.4. According to Lemma 3.2, the validity of the theorem follows immediately from Theorem 2.3. □

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