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WEAK FORMS OF OPEN MAPPINGS AND α-TOPOLOGY

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ABSTRACT. The following notations associated with a mapping $f\colon (X,\tau)\to(Y,\sigma)$ have been introduced in [D. S. Janković: *A note on mappings of extremally disconnected spaces*, Acta Math. Hungar. **46** (1985), 83–92]:

$$f_*\colon (X,\tau^\alpha)\to(Y,\sigma^\alpha),\quad f_\alpha\colon (X,\tau^\alpha)\to(Y,\sigma),\quad f^\alpha\colon (X,\tau)\to(Y,\sigma^\alpha),$$

where $f_*(x)=f_\alpha(x)=f^\alpha(x)=f(x)$ for each $x\in X$. Interrelationships between well-known openness-type properties (a.o.S., w.o., α-openness, a.o.W.) for $f, f_*, f^\alpha, f_\alpha$ are studied.

1. Preliminaries

Throughout the present paper (X,τ) and (Y,σ) stand for topological spaces on which no separation axioms are assumed. Let S be a subset of (X,τ). The *closure* (the *interior*) of S in (X,τ) will be denoted by $\mathrm{cl}\,(S)$ or $\mathrm{cl}_\tau(S)$ ($\mathrm{int}\,(S)$ or $\mathrm{int}_\tau(S)$). The set S is said to be *regular open*, *α-open* [12], *semi-open* [8], *semi-closed* [2], *preopen* [9] in (X,τ) if $S=\mathrm{int}\big(\mathrm{cl}\,(S)\big)$, $S\subset\mathrm{int}\big(\mathrm{cl}\,(\mathrm{int}\,(S))\big)$, $S\subset\mathrm{cl}\big(\mathrm{int}\,(S)\big)$, $\mathrm{int}\big(\mathrm{cl}\,(S)\big)\subset S$, $S\subset\mathrm{int}\big(\mathrm{cl}\,(S)\big)$, respectively. The family of all closed (regular open, α-open, semi-open, semi-closed, preopen) subsets of (X,τ) is denoted by

$$\mathrm{c}\,(X,\tau)\ (\mathrm{RO}\,(X,\tau),\ \tau^\alpha,\ \mathrm{SO}\,(X,\tau),\ \mathrm{SC}\,(X,\tau),\ \mathrm{PO}\,(X,\tau)).$$

It is known [12] that τ^α is always a topology on X. Also, $\tau^\alpha=\mathrm{SO}\,(X,\tau)\cap\mathrm{PO}\,(X,\tau)$ [17, Lemma 3.1] and $\tau\subset\tau^\alpha$ for every space (X,τ).

A mapping $f\colon (X,\tau)\to(Y,\sigma)$ is said to be *semi-continuous* or briefly *s.c.* [8] (*irresolute* [3], *a.c.H.* [6]) if

$$f^{-1}(V)\in\mathrm{SO}\,(X,\tau)$$

($f^{-1}(V)\in\mathrm{SO}\,(X,\tau)$, $f^{-1}(V)\in\mathrm{PO}\,(X,\tau)$) for each $V\in\sigma$ (resp. $V\in\mathrm{SO}\,(Y,\sigma)$, $V\in\sigma$).

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In [20, Theorem 4] D. A. Rose observed that a.c.H. and precontinuity [9] are equivalent. In [11] A. Neubrunnová showed that a.c.H. and s.c. are independent of each other.

A mapping $f\colon (X,\tau) \to (Y,\sigma)$ is said to be *almost open in the sense of Singal & Singal* or briefly *a.o.S.* [21] (α-open [10]) if $f(U) \in \sigma$ ($f(U) \in \sigma^{\alpha}$) for every $U \in \mathrm{RO}\,(X,\tau)$ ($U \in \tau$). A mapping

$$f\colon (X,\tau) \to (Y,\sigma)$$

is said to be *weakly open* (briefly *w.o.*) [19] if $f(U) \subset \operatorname{int}\big(f(\operatorname{cl}(U))\big)$ for every $U \in \tau$. A mapping

$$f\colon (X,\tau) \to (Y,\sigma)$$

is said to be *almost open in the sense of Wilansky* [23] (briefly *a.o.W.*) if

$$f^{-1}\big(\operatorname{cl}(V)\big) \subset \operatorname{cl}\big(f^{-1}(V)\big) \quad \text{for each} \quad V \in \sigma.$$

Recall that Wilansky originally considered a.o.W. mappings for injections only. In 1982, A. S. Mashhour *et al.* [9] introduced preopen mappings as the mappings with images of open sets being preopen. The same definition was independently used by D. A. Rose a little bit later [19]. In [20, Theorem 11] Rose proved that a.o.W. and preopenness coincide.

Relations between the above described notions are depicted in a diagram below (compare [4]) where an arrow stands for the implication, an arrow crossed out means the implication that does not hold, and a double arrow crossed out twice means that none of the two implications holds.

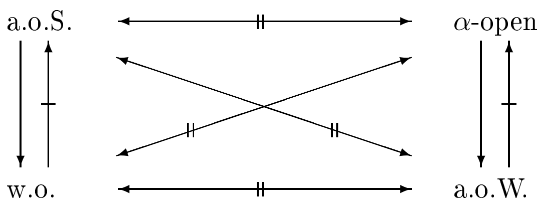

Diagram.

2. Weakly open mappings

The following known facts will be used.

Lemma 1. *Let a space (X,τ) be arbitrary. Then*

(**a**) [5, Lemma 1(i)] $\operatorname{cl}_{\tau}(S) = \operatorname{cl}_{\tau^{\alpha}}(S)$ *for any* $S \in \mathrm{SO}\,(X,\tau)$;

(**b**) [5, Lemma 1(ii)] $\mathrm{int}_\tau(S) = \mathrm{int}_{\tau^\alpha}(S)$ *for any* $S \in \mathrm{SC}\,(X,\tau)$*;*

(**c**) $\mathrm{RO}\,(X,\tau^\alpha) = \mathrm{RO}\,(X,\tau)$ (*follows from* [7, Corollary 2.4(a)])*;*

(**d**) $\mathrm{SC}\,(X,\tau^\alpha) = \mathrm{SC}\,(X,\tau)$ (*follows from* [7, Proposition 2.1])*;*

(**e**) [7, Corollary 2.5(a)] $\mathrm{PO}\,(X,\tau^\alpha) = \mathrm{PO}\,(X,\tau)$*;*

Theorem 1. *For any mapping* $f\colon (X,\tau) \to (Y,\sigma)$

(1) f *is a.o.S.* $\Leftrightarrow$ f_α *is a.o.S.,*

(2) f_* *is a.o.S.* $\Leftrightarrow$ f^α *is a.o.S.,*

(3) f *is a.o.S.* $\Rightarrow$ f^α *is a.o.S.*

P r o o f. (1) and (2) follow from Lemma 1(**c**), while (3) is obvious by the inclusion $\sigma \subset \sigma^\alpha$. □

The reverse implication to that of (3) above may be false, as it is seen by the following.

Example 1. Let

$$X = \{a,b,c,d,e\},\ Y = \{a,b,c\},\ \tau = \bigl\{\emptyset, X, \{a,b\}, \{c,d\}, \{a,b,c,d\}\bigr\},$$

and

$$\sigma = \bigl\{\emptyset, Y, \{a\}\bigr\}.$$

Define $f\colon (X,\tau) \to (Y,\sigma)$ as follows: $f(a) = f(c) = a$, $f(b) = f(d) = b$, $f(e) = c$. We have

$$\mathrm{RO}\,(X,\tau) = \bigl\{\emptyset, X, \{a,b\}, \{c,d\}\bigr\} \quad \text{and} \quad \sigma^\alpha = \bigl\{\emptyset, Y, \{a\}, \{a,b\}, \{a,c\}\bigr\}.$$

Thus, f^α is a.o.S., but f is not.

Recall that a mapping $f\colon (X,\tau) \to (Y,\sigma)$ is called *pre-semi-closed* [1, p. 8] if $f(F) \in \mathrm{SC}\,(Y,\sigma)$ for each set $F \in \mathrm{SC}\,(X,\tau)$.

Lemma 2 ([16, Theorem 2.1])**.** *A mapping* $f\colon (X,\tau) \to (Y,\sigma)$ *is a.o.S. if and only if* $f\bigl(\mathrm{int}\,(F)\bigr) \subset \mathrm{int}\,\bigl(f(F)\bigr)$ *for each set* $F \in \mathrm{SC}\,(X,\tau)$.

Theorem 2. *Let an* $f\colon (X,\tau) \to (Y,\sigma)$ *be pre-semi-closed. Then*

$$f^\alpha \quad \text{is a.o.S.} \quad \Longrightarrow \quad f \quad \text{is a.o.S.}$$

P r o o f. Let f^α be a.o.S. and pick an arbitrary $F \in \mathrm{SC}\,(X,\tau)$. By Lemma 2, we have $f\bigl(\mathrm{int}_\tau(F)\bigr) = f^\alpha\bigl(\mathrm{int}_\tau(F)\bigr) \subset \mathrm{int}_{\sigma^\alpha}\bigl(f^\alpha(F)\bigr)$. As f is pre-semi-closed, hence applying Lemma 1(**b**), we obtain $f\bigl(\mathrm{int}_\tau(F)\bigr) \subset \mathrm{int}_\sigma\bigl(f(F)\bigr)$. □

Corollary 1. *Let a mapping* $f\colon (X,\tau) \to (Y,\sigma)$ *be pre-semi-closed. Then*

$$f^\alpha \quad \text{is a.o.S.} \quad \Longleftrightarrow \quad f \quad \text{is a.o.S.}$$

P r o o f. Theorem 1 (3) and Theorem 2. □

Remark 1. The assumption that f is pre-semi-closed in Theorem 2 is necessary (Example 1). Consider, for instance, the image $f\big(\{c,d\}\big)$.

Theorem 3. *For any* $f\colon (X,\tau)\to(Y,\sigma)$

(1) f *is w.o.* $\Leftrightarrow$ f_α *is w.o.*,

(2) f_* *is w.o.* $\Leftrightarrow$ f^α *is w.o.*,

(3) f *is w.o.* $\Rightarrow$ f^α *is w.o.*

Proof.

(1): (⇒): Let f be w.o. and let $U\in\tau^\alpha$. Then we have

$$\begin{aligned} f_\alpha(U)=f(U) &\subset f\Big(\mathrm{int}_\tau\big(\mathrm{cl}_\tau(\mathrm{int}_\tau(U))\big)\Big)\\ &\subset \mathrm{int}_\sigma\bigg(f\bigg(\mathrm{cl}_\tau\bigg(\mathrm{int}_\tau\Big(\mathrm{cl}_\tau\big(\mathrm{int}_\tau(U)\big)\Big)\bigg)\bigg)\bigg)\\ &\subset \mathrm{int}_\sigma\Big(f\big(\mathrm{cl}_\tau(U)\big)\Big). \end{aligned}$$

Since every α-open set is semi open, by Lemma 1(**a**) we get

$$f_\alpha(U)\subset \mathrm{int}_\sigma\Big(f\big(\mathrm{cl}_{\tau^\alpha}(U)\big)\Big)=\mathrm{int}_\sigma\Big(f_\alpha\big(\mathrm{cl}_{\tau^\alpha}(U)\big)\Big).$$

(⇐): Let f_α be w.o. and let $U\in\tau$. Then, by Lemma 1(**a**), we get

$$f(U)=f_\alpha(U)\subset \mathrm{int}_\sigma\Big(f_\alpha\big(\mathrm{cl}_{\tau^\alpha}(U)\big)\Big)=\mathrm{int}_\sigma\Big(f\big(\mathrm{cl}_\tau(U)\big)\Big).$$

(2): It follows from calculations that consider the use of Lemma 1(**a**).

(⇒): Let $U\in\tau$ be arbitrary. Since $U\in\tau^\alpha$ and f_* is w.o., we get:

$$f^\alpha(U)=f_*(U)\subset \mathrm{int}_\sigma\Big(f^\alpha\big(\mathrm{cl}_{\tau^\alpha}(U)\big)\Big)=\mathrm{int}_\sigma\Big(f^\alpha\big(\mathrm{cl}_\tau(U)\big)\Big).$$

(⇐): Let $U\in\tau^\alpha$, i.e., $U\subset \mathrm{int}_\tau\big(\mathrm{cl}_\tau(\mathrm{int}_\tau(U))\big)$. By hypothesis we have

$$\begin{aligned} f_*(U) &\subset f^\alpha\Big(\mathrm{int}_\tau\Big(\mathrm{cl}_\tau\big(\mathrm{int}_\tau(U)\big)\Big)\Big)\\ &\subset \mathrm{int}_{\sigma^\alpha}\Big(f^\alpha\Big(\mathrm{cl}_\tau\big(\mathrm{int}_\tau(U)\big)\Big)\Big)\\ &\subset \mathrm{int}_{\sigma^\alpha}\Big(f_*\big(\mathrm{cl}_\tau(U)\big)\Big)\\ &= \mathrm{int}_{\sigma^\alpha}\Big(f_*\big(\mathrm{cl}_{\tau^\alpha}(U)\big)\Big). \end{aligned}$$

(3): Let f be w.o. and U be from τ. In a view of the inclusion $\sigma\subset\sigma^\alpha$, the following is clear:

$$f^\alpha(U)=f(U)\subset \mathrm{int}_\sigma\Big(f\big(\mathrm{cl}_\tau(U)\big)\Big)\subset \mathrm{int}_{\sigma^\alpha}\Big(f^\alpha\big(\mathrm{cl}_\tau(U)\big)\Big).$$

□

The converse of Theorem 3(3) is not true, in general.

EXAMPLE 2. Let $X = \{a, b, c, d, e\}$, $Y = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a, b\}, \{c, d\}, \{a, b, c, d\}\}$, and $\sigma = \{\emptyset, Y, \{a\}\}$. Let $f\colon (X, \tau) \to (Y, \sigma)$ be formally defined as in Example 1. One easily checks that f^α is w.o. At the same time f is not w.o., since $f\bigl(\{a, b\}\bigr) \not\subset \mathrm{int}_\sigma\bigl(f(\mathrm{cl}_\tau(\{a, b\}))\bigr) = \{a\}$.

A mapping $f\colon (X, \tau) \to (Y, \sigma)$ is said to be *semi-closed* [14] if $f(F) \in \mathrm{SC}\,(Y, \sigma)$ for every set $F \in \mathrm{c}\,(X, \tau)$. It is well-known that pre-semi-closedness of f implies f is semi-closed, but not conversely.

THEOREM 4. *Let $f\colon (X, \tau) \to (Y, \sigma)$ be semi-closed. Then*

$$f^\alpha \quad \text{is w.o.} \quad \Longrightarrow \quad f \quad \text{is w.o.}$$

Proof. Let f^α be w.o. and $U \in \tau$. We get

$$f(U) = f^\alpha(U) \subset \mathrm{int}_{\sigma^\alpha}\bigl(f^\alpha(\mathrm{cl}_\tau(U))\bigr) = \mathrm{int}_{\sigma^\alpha}\bigl(f(\mathrm{cl}_\tau(U))\bigr),$$

where $f\bigl(\mathrm{cl}_\tau(U)\bigr) \in \mathrm{SC}\,(Y, \sigma)$. Thus, by Lemma 1(**b**), f is w.o. □

COROLLARY 2. *Assume that $f\colon (X, \tau) \to (Y, \sigma)$ is semi-closed. Then*

$$f^\alpha \quad \text{is w.o.} \quad \Longleftrightarrow \quad f \quad \text{is w.o.}$$

Proof. This equivalence follows from Theorems 3(3) and 4. □

Remark 2. One can easily see that the mapping f^α from Example 2 is w.o., but f is not semi-closed (consider, for instance, the image $f\bigl(\{c, d, e\}\bigr)$).

Clearly (see the proof of [15, Lemma 1.4]), for any $f\colon (X, \tau) \to (Y, \sigma)$ we have

$$f_* \quad \text{is a.o.S.} \quad \Longrightarrow \quad f_* \quad \text{is w.o.}$$

The converse may fail, as it is seen by [15, Example 1.5]. For spaces (X, σ), (Y, τ) from that example, $\sigma = \sigma^\alpha$ and $\tau = \tau^\alpha$.

In [4] the author has shown the following.

LEMMA 3. *Let a mapping $f\colon (X, \tau) \to (Y, \sigma)$ be a.c.H. and pre-semi-closed. Then*

$$f \quad \text{is a.o.S.} \quad \Longleftrightarrow \quad f \quad \text{is w.o.}$$

COROLLARY 3. *Let $f\colon (X, \tau) \to (Y, \sigma)$ be pre-semi-closed and let f_* be a.c.H. Then*

$$f_* \quad \text{is w.o.} \quad \Longleftrightarrow \quad f_* \quad \text{is a.o.S.}$$

Proof. By Lemma 1(**d**), pre-semi-closedness of f and f_* are equivalent. Thus, Lemma 3 yields the result. □

LEMMA 4. *For any $f\colon (X, \tau) \to (Y, \sigma)$,*

$$f_* \quad \text{is a.c.H.} \quad \Longrightarrow \quad f \quad \text{is a.c.H.}$$

P r o o f. Clear by the inclusion $\sigma \subset \sigma^\alpha$ and Lemma 1(**e**). □

The reverse implication may be false.

EXAMPLE 3. Let $X = \{a, b, c\}$, $\tau = \big\{\emptyset, X, \{a\}, \{c\}, \{a, c\}\big\}$, and $\sigma = \big\{\emptyset, X, \{a\}\big\}$. The identity mapping $\mathrm{id}\colon (X, \tau) \to (X, \sigma)$ is continuous and so a.c.H., but id_* is not a.c.H. since $\mathrm{id}_*^{-1}\big(\{a, b\}\big) \notin \mathrm{PO}\,(X, \tau) = \mathrm{PO}\,(X, \tau^\alpha)$.

COROLLARY 4. *Let $f\colon (X, \tau) \to (Y, \sigma)$ be pre-semi-closed and f_* be a.c.H. Then*

(**a**) *f_* is a.o.S. $\Leftrightarrow$ f is w.o.;*

(**b**) *f is a.o.S. $\Leftrightarrow$ f_* is w.o.*

P r o o f. Every pre-semi-closed mapping is semi-closed and therefore we can utilize Corollary 2. (**a**) Use Corollaries 3 and 2, and Theorem 3(2), (**b**) follows by Lemmas 4 and 3, Corollary 2, Theorem 3(2). □

3. a.o.W. mappings

LEMMA 5 ([12, Proposition 10])**.** *In every space (X, τ), $\tau^\alpha = \big(\tau^\alpha\big)^\alpha$.*

THEOREM 5. *For any $f\colon (X, \tau) \to (Y, \sigma)$*

(1) *f^α is α-open $\Leftrightarrow$ f is α-open;*

(2) *f_* is α-open $\Leftrightarrow$ f_α is α-open;*

(3) *f_α is α-open $\Rightarrow$ f is α-open.*

P r o o f. (1) follows from Lemma 5, (2) is obvious, while (3) is clear by the inclusion $\tau \subset \tau^\alpha$. □

The implication converse to (3) may be not true.

EXAMPLE 4. Let $X = \{a, b, c\}$, $\tau = \big\{\emptyset, X, \{a\}\big\}$, and $\sigma = \big\{\emptyset, X, \{a\}, \{b, c\}\big\}$. The identity $\mathrm{id}\colon (X, \tau) \to (X, \sigma)$ is open (and so α-open), but id_α is not α-open since $\mathrm{id}_\alpha\big(\{a, b\}\big) \notin \sigma^\alpha = \sigma$.

A mapping $f\colon (X, \tau) \to (X, \sigma)$ is called *pre-irresolute* [18] if the preimage $f^{-1}(V) \in \mathrm{PO}\,(X, \tau)$ for each $V \in \mathrm{PO}\,(Y, \sigma)$. In [4] we have proved the following.

LEMMA 6. *If a mapping is a.o.S. and pre-irresolute, then it is α-open.*

LEMMA 7. *If an $f\colon (X, \tau) \to (Y, \sigma)$ is a.o.S. and pre-irresolute, then f_* (or equiv. f_α) is α-open.*

Proof. By Lemma 1(**e**), pre-irresolutness of f and pre-irresolutness of f_* are equivalent. From Theorem 1 we infer that f_* is a.o.S. Thus, the result follows directly from Lemma 6. □

Each pre-irresolute mapping is a.c.H. but [18, Example 3] guarantees the existence of an a.c.H. mapping which is not pre-irresolute.

Lemma 8 ([4]). *If a mapping $f\colon (X,\tau) \to (Y,\sigma)$ is a.o.S. and a.c.H., then f is α-open if and only if it is pre-irresolute.*

Theorem 6. *Let $f\colon (X,\tau) \to (Y,\sigma)$ be a.c.H. and a.o.S. Then*

$$f \quad \text{is } \alpha\text{-open} \quad \Longrightarrow \quad f_\alpha \quad \text{is } \alpha\text{-open}.$$

Proof. Lemmas 8 and 7. □

Corollary 5. *Let $f\colon (X,\tau) \to (Y,\sigma)$ be a.c.H. and a.o.S. Then*

$$f \quad \text{is } \alpha\text{-open} \quad \Longleftrightarrow \quad f_\alpha \quad \text{is } \alpha\text{-open}.$$

Proof. From Theorems 5 and 6. □

Theorem 7. *For every $f\colon (X,\tau) \to (Y,\sigma)$ we have*

(1) *f^α is a.o.W. $\Leftrightarrow$ f is a.o.W.;*

(2) *f_* is a.o.W. $\Leftrightarrow$ f_α is a.o.W.;*

(3) *f_α is a.o.W. $\Rightarrow$ f is a.o.W.*

Proof. (1) and (2) follow by Lemma 1(**e**) while (3) is obvious by $\tau \subset \tau^\alpha$. □

The implication in Theorem 7(3) is not reversible.

Example 5. Let $X = \{a,b,c\}$, $\tau = \big\{\emptyset, X, \{a\}\big\}$, $\sigma = \big\{\emptyset, X, \{a\}, \{c\}, \{a,c\}\big\}$. The identity $\mathrm{id}\colon (X,\tau) \to (X,\sigma)$ is a.o.W. (open, in fact), but id_α is not a.o.W. since $\{a,b\} \notin \mathrm{PO}\,(X,\sigma)$.

Lemma 9 ([22, Lemma 3.9]). *A mapping $f\colon (X,\tau) \to (Y,\sigma)$ is a.o.W. if and only if*

$$f^{-1}\big(\mathrm{cl}\,(V)\big) \subset \mathrm{cl}\,\big(f^{-1}(V)\big) \qquad \text{for every} \quad V \in \mathrm{SO}\,(Y,\sigma).$$

Theorem 8. *Let a mapping $f\colon (X,\tau) \to (Y,\sigma)$ be s.c. Then*

$$f \quad \text{is a.o.W.} \quad \Longrightarrow \quad f_\alpha \quad \text{is a.o.W.}$$

Proof. Let $V \in \mathrm{SO}\,(Y,\sigma)$ be arbitrary. By [13, Lemma 2], $\mathrm{cl}\,(V) = \mathrm{cl}\,\big(\mathrm{int}\,(V)\big)$. Thus, by our assumption and by Lemma 1(**a**), we get

$$\begin{aligned} f_\alpha^{-1}\big(\mathrm{cl}_\sigma(V)\big) &= f^{-1}\Big(\mathrm{cl}_\sigma\big(\mathrm{int}_\sigma(V)\big)\Big) \subset \mathrm{cl}_\tau\Big(f^{-1}\big(\mathrm{int}_\sigma(V)\big)\Big) \\ &= \mathrm{cl}_{\tau^\alpha}\Big(f^{-1}\big(\mathrm{int}(V)\big)\Big) \subset \mathrm{cl}_{\tau^\alpha}\big(f_\alpha^{-1}(V)\big). \end{aligned}$$

By Lemma 9, f_α is a.o.W. □

Corollary 6. *Let $f\colon (X,\tau) \to (Y,\sigma)$ be s.c. Then*

$$f \quad \textit{is a.o.W.} \quad \Longleftrightarrow \quad f_\alpha \quad \textit{is a.o.W.}$$

Remark 3. It is not difficult to see that an $f\colon (X,\tau) \to (Y,\sigma)$ is a.o.W. if and only if $f^{-1}\big(\mathrm{cl}\,(V)\big) \subset \mathrm{cl}\,\big(f^{-1}(V)\big)$ for every $V \in \sigma^\alpha$ (use inclusions $\sigma \subset \sigma^\alpha \subset \mathrm{SO}\,(Y,\sigma)$).

Obviously, any mapping f fulfils the implication:

$$f_* \quad \text{is } \alpha\text{-open} \quad \Longrightarrow \quad f_* \quad \text{is a.o.W.}$$

However, the converse can fail.

Example 6. Let $X = \{a,b,c\}$, $\tau = \big\{\emptyset, X, \{a\}, \{b\}, \{a,b\}\big\}$, $\sigma = \big\{\emptyset, X, \{a\}, \{b,c\}\big\}$. It can be seen that $\tau = \tau^\alpha$ and $\sigma = \sigma^\alpha$. Hence, $\mathrm{id}\colon (X,\tau) \to (X,\sigma)$ is a.o.W. but it is not α-open as $\mathrm{id}\big(\{b\}\big) \notin \sigma$.

Lemma 10 ([4])**.** *Let a mapping f be pre-irresolute and pre-semi-closed. Then*

$$f \quad \textit{is a.o.W.} \quad \Longleftrightarrow \quad f \quad \textit{is } \alpha\textit{-open.}$$

Corollary 7. *Let an f be pre-irresolute and pre-semi-closed. Then*

$$f_* \quad \textit{is a.o.W.} \quad \Longleftrightarrow \quad f_* \quad \textit{is } \alpha\textit{-open.}$$

Proof. By Lemma 1(**d**),(**e**) we get that pre-irresoluteness of f and f_* coincide, and that pre-semi-closedness of f and f_* coincide also. Thus our result follows directly from Lemma 10. □

Corollary 8. *Let a mapping $f\colon (X,\tau) \to (Y,\sigma)$ be s.c., pre-irresolute, and pre-semi-closed. Then*

(**a**) *f_* is α-open $\Leftrightarrow f$ is a.o.W.;*

(**b**) *f is α-open $\Leftrightarrow f_*$ is a.o.W.*

Proof. (**a**) From Corollaries 7 and 6 and Theorem 7(2). (**b**) From Lemma 10, Corollary 6, and Theorem 7(2). □

The next examples prove that s.c. and pre-irresoluteness are independent of each other.

Example 7.

(**a**) Let $X = \{a,b,c\}$, $\tau = \big\{\emptyset, X, \{a\}, \{b\}, \{a,b\}\big\}$, $\sigma = \big\{\emptyset, X, \{a\}\big\}$. The identity $\mathrm{id}\colon (X,\tau) \to (X,\sigma)$ is continuous, but it is not pre-irresolute since $\{a,c\} \notin \mathrm{PO}\,(X,\tau)$.

(**b**) Let $X = \{a,b\}$, $\tau = \big\{\emptyset, X\big\}$, $\sigma = \big\{\emptyset, X, \{a\}\big\}$. The identity $\mathrm{id}\colon (X,\tau) \to (X,\sigma)$ is pre-irresolute and not s.c.

A bijection is said to be a *semi-homeomorphism* [3] if it is irresolute and pre-semi-open [3]. For a bijection, pre-semi-openness and pre-semi-closedness are equivalent notions.

The two concluding corollaries are immediate consequences of some previous results.

COROLLARY 9. *Let f be a semi-homeomorphism. Then*

(**a**) *f_* is a.o.S. $\Leftrightarrow$ f is a.o.S.;*

(**b**) *f_* is w.o. $\Leftrightarrow$ f is w.o.;*

(**c**) *f_* is a.o.W. $\Leftrightarrow$ f is a.o.W.*

Proof. (**a**) Corollary 1 and Theorem 1(2). (**b**) Corollary 2 and Theorem 3(2). (**c**) Corollary 6 and Theorem 7(2). □

COROLLARY 10. *For a homeomorphism f we have*

$$f \quad \text{is } \alpha\text{-open} \quad \Longleftrightarrow \quad f_* \quad \text{is } \alpha\text{-open}.$$

Proof. It follows from Corollary 5 and Theorem 5 (2). □

We recall that every homeomorphism is a semi-homeomorphism, but not conversely [3, Example 1.2].

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