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Cycloids on the Spherical Surface

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Abstract

The article describes the creation of the normal cycloidal curve by rotation of the point about the normal of the circle on the spherical surface and binormal cycloidal curve created by rotation of the point about the binormal of the previous cycloidal curve. Cycloidal cyclical surfaces are created by moving circles along the curves lying in the normal plane of the curves. Described cycloidal cyclical surfaces are projected on the spherical surface. Varying parameters of the curves are generated different ornaments on the spherical surface.

Key words: cycloidal curve, cycloidal cyclical surface, Frenet-Serret moving trihedron, spherical surface

1. Introduction

Let the three-dimensional Euclidean space E^3 is determined by the Cartesian coordinate system (0, x, y, z). Let the spherical surface Φ of radius *r* and centre in the origin 0 is determined by the parametrical equations

$$x = r \cos u \cos v$$

$$y = r \cos u \sin v, u \in \langle -\pi/2, \pi/2, \rangle, v \in \langle 0, 2\pi \rangle.$$

$$z = r \sin u$$
(1)

Let c is the circle on this spherical surface determined by the vector function

$$\mathbf{r}(v) = (r\cos v, r\sin v, 0, 1), v \in \langle 0, 2\pi \rangle.$$
⁽²⁾

Let (0', n, b, t) be the Frenet-Serret moving trihedron of the circle *c*, where *n* is the principle normal, *b* is binormal and *t* is tangent of the circle *c* at the point $0' \in c$ with the unit vectors in the coordinate system (0, x, y, z) [1]

$$\mathbf{t}(v) = (t_x(v), t_y(v), t_z(v)) = \frac{\mathbf{r}'(v)}{|\mathbf{r}'(v)|} = (-\sin v, \cos v, 0),$$
(3)

$$\mathbf{b}(v) = \left(b_x(v), b_y(v), b_z(v)\right) = \frac{\mathbf{r}'(v) \times \mathbf{r}''(v)}{|\mathbf{r}'(v) \times \mathbf{r}''(v)|} = (0, 0, 1), \tag{4}$$

$$\mathbf{n}(v) = (n_x(v), n_y(v), n_z(v)) = \mathbf{b}(v) \times \mathbf{t}(v) = (-\cos v, -\sin v, 0).$$
(5)

Frenet-Serret moving trihedron (0', n, b, t) is represented by the regular square matrix [2]

$$\mathbf{M}(v) = \begin{pmatrix} n_x(v) & n_y(v) & n_z(v) & 0 \\ b_x(v) & b_y(v) & b_z(v) & 0 \\ t_x(v) & t_y(v) & t_z(v) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\cos v & -\sin v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin v & \cos v & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
 (6)



Figure 1: Frenet-Serret moving trihedron (0', n, b, t) of the circle *c*

In Fig.1 is displayed spherical surface Φ , circle *c* with Frenet-Serret moving trihedron (0', n, b, t) and cyclical surface Φ' created by the circle c' = (0', r') moving along the circle *c* and lying in the normal plane determined by the principal normal *n* and the binormal *b* of the circle *c* at the point 0'.

Then the vector function of the cyclical surface Φ' is

$$\mathbf{P}'(u,v) = \mathbf{r}(v) + \mathbf{c}'(u) \cdot \mathbf{M}(v), \ u \in \langle 0, 2\pi \rangle, \ v \in \langle 0, 2\pi \rangle,$$
(7)

where $\mathbf{r}(v)$ is vector function of the circle *c* expressed in equations (2), $\mathbf{M}(v)$ is the transformation matrix of the coordinate system (0', n, b, t) into the coordinate system (0, x, y, z), (Eq.6) and $\mathbf{c}'(u) = (r' \cos u, r' \sin u, 0, 1)$, $u \in \langle 0, 2\pi \rangle$ is the vector function of the circle *c'*.

2. Creating of the normal cycloidal curve c_1 of the circle c

Point $P_1(0,0,d_1,1)$ in the coordinate system (0',n,b,t), $0' \in c$ rotates about the normal *n* of the circle *c* with angular velocity m_1v , the orientation is determined by the parameter sgn₁ and creates the cycloidal curve c_1 which is expressed by vector function

$$\mathbf{r}_{1}(v) = (x_{1}, y_{1}, z_{1}, 1) = (r \cos v - d_{1} \cos m_{1}v \sin v, r \sin v + d_{1} \cos m_{1}v \cos v, - \operatorname{sgn}_{1} d_{1} \sin m_{1}v, 1), (8)$$

for parameter $v \in \langle 0, 2\pi \rangle$.

Cycloidal cyclical surface Φ'_1 is created by the movement of the circle $c'_1 = (0'', r'_1)$ along the cycloidal curve c_1 and lying in the normal plane determined by the principal normal n' and the binormal b' of the curve c_1 at the point 0''.

Let (0'', n', b', t') be the Frenet-Serret moving trihedron of the cycloidal curve c_1 , where n' is the principle normal, b' is binormal and t' is tangent of the curve c_1 at the point $0'' \in c_1$ with the unit vectors

$$\mathbf{t}'(v) = (t'_x, t'_y, t'_z) = \frac{\mathbf{r}_1(v)'}{\left|\mathbf{r}_1(v)'\right|},\tag{9}$$

$$\mathbf{b}'(\mathbf{v}) = (b'_x, b'_y, b'_z) = \frac{\mathbf{r}_1(\mathbf{v})' \times \mathbf{r}_1(\mathbf{v})''}{\left| \mathbf{r}_1(\mathbf{v})' \times \mathbf{r}_1(\mathbf{v})'' \right|},$$
(10)

$$\mathbf{n}'(v) = (n'_x, n'_y, n'_z) = \mathbf{b}'(v) \times \mathbf{t}'(v).$$
(11)

Matrix

$$\mathbf{M}_{1}(v) = \begin{pmatrix} n'_{x}(v) & n'_{y}(v) & n'_{z}(v) & 0\\ b'_{x}(v) & b'_{y}(v) & b'_{z}(v) & 0\\ t'_{x}(v) & t'_{y}(v) & t'_{z}(v) & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}$$
(12)

represents the transformation of the coordinate system (0'', n', b', t') (the moving trihedron of the curve c_1) into the coordinate system (0', n, b, t) (the moving trihedron of the circle c).

Then the vector function of the cyclical cycloidal surface Φ'_1 is

$$\mathbf{P}_{1}'(u,v) = \mathbf{r}_{1}(v) + \mathbf{c}_{1}'(u) \cdot \mathbf{M}_{1}(m_{1}v), \ u \in \langle 0, 2\pi \rangle, \ v \in \langle 0, 2\pi \rangle,$$
(13)

where $\mathbf{r}_1(v)$ is vector function of the cycloidal curve c_1 expressed in equations (8), $\mathbf{M}_1(m_1v)$ is the transformation matrix (Eq.12) and $\mathbf{c}'_1(u) = (r'_1 \cos u, r'_1 \sin u, 0, 1), u \in \langle 0, 2\pi \rangle$ is the vector function of the circle c'_1 moving along the curve c_1 and lying in its normal plane. [4], [6]



Figure 2

Figure 3

In Fig.2 are displayed spherical surface Φ , cyclical surface Φ' and cycloidal cyclical surface Φ'_1 , determined by parameters r = 15, $m_1 = 6$, $d_1 = 5$, r' = 0.35, sgn₁ = -1, $r'_1 = 0.5$.

If we project every point $X_1(x_1, z_1, z_1, 1)$ of the cycloidal curve c_1 on the spherical surface Φ by using the central projection with centre of the spherical surface, then we obtain points $X'_1(x'_1, y'_1, z'_1, 1)$ which create the cycloidal curve s'_1 on the spherical surface Φ . Then transformation between coordinates is determined by the equations

$$x_{1}' = \frac{r x_{1}}{\sqrt{x_{1}^{2} + y_{1}^{2} + z_{1}^{2}}}, y_{1}' = \frac{r y_{1}}{\sqrt{x_{1}^{2} + y_{1}^{2} + z_{1}^{2}}}, z_{1}' = \frac{r z_{1}}{\sqrt{x_{1}^{2} + y_{1}^{2} + z_{1}^{2}}}.$$
 (14)

Then cycloidal cyclical surface Φ_1'' on the spherical surface Φ is created by the movement of the circle $c_1' = (0'', r_1')$ along the cycloidal curve c_1' and lying in the normal plane determined by the principal normal n' and the binormal b' of the curve c_1' at the point 0''.

In Fig.3 are displayed spherical surface Φ , cyclical surface Φ' and cycloidal cyclical surface Φ''_1 determined by the same parameters as in the Fig.2.

In Fig.4 are displayed spherical surface Φ , cyclical surface Φ' and cycloidal cyclical surface Φ''_1 , determined by the same parameters as in the Fig.2., except sgn₁ = +1.

In Fig.5 are displayed spherical surface Φ , cyclical surface Φ' and cycloidal cyclical surface Φ''_1 , with same parameters as in the Fig.4., except $m_1 = 8$.



3. Creating of the binormal cycloidal curve c_2 of the curve c_1

Point $P_2(d_2,0,0,1)$ in the coordinate system (0'',n',b',t'), $0'' \in c_1$ rotates about the binormal b' of the cycloidal curve c_1 with angular velocity m_2v and orientation determined by the parameter sgn₂ and creates the cycloidal curve c_2 with vector function

$$\mathbf{r}_{2}(v) = (x_{2}, y_{2}, z_{2}, \mathbf{1}) = (x_{1} + x_{01}n'_{x} + z_{01}t'_{x}, y_{1} + x_{01}n'_{y} + z_{01}t'_{y}, z_{1} + x_{01}n'_{z} + z_{01}t'_{z}, \mathbf{1}), \quad (15)$$

where parameter $v \in \langle 0, 2\pi \rangle$ and

$$x_{01} = d_2 \cos m_2 v, z_{01} = \operatorname{sgn}_2 d_2 \sin m_2 v.$$
 (16)

Cycloidal cyclical surface Φ'_2 is created by the movement of the circle $c'_2 = (O''', r'_2)$ along the cycloidal curve c_2 and lying in the normal plane determined by the principal normal n'' and the binormal b'' of the curve c_2 at the point O'''.

Let (0'', n'', b'', t'') be the Frenet-Serret moving trihedron of the cycloidal curve c_2 , where n'' is the principle normal, b'' is binormal and t'' is tangent of the curve c_2 at the point $0''' \in c_2$ with the unit vectors

$$\mathbf{t}''(\mathbf{v}) = \left(t_x'', t_y'', t_z''\right) = \frac{\mathbf{r}_2(\mathbf{v})'}{\left|\mathbf{r}_2(\mathbf{v})'\right|},\tag{17}$$

$$\mathbf{b}''(v) = \left(b_x'', b_y'', b_z''\right) = \frac{\mathbf{r}_2(v)' \times \mathbf{r}_2(v)''}{\left|\mathbf{r}_2(v)' \times \mathbf{r}_2(v)''\right|},\tag{18}$$

$$\mathbf{n}''(\mathbf{v}) = \left(n_x'', n_y'', n_z''\right) = \mathbf{b}''(\mathbf{v}) \times \mathbf{t}''(\mathbf{v}).$$
⁽¹⁹⁾

Then matrix

$$\mathbf{M}_{2}(v) = \begin{pmatrix} n_{x}''(v) & n_{y}''(v) & n_{z}''(v) & 0\\ b_{x}''(v) & b_{y}''(v) & b_{z}''(v) & 0\\ t_{x}''(v) & t_{y}''(v) & t_{z}''(v) & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}$$
(20)

represents the transformation of the coordinate system (0''', n'', b'', t'') (the moving trihedron of the curve c_2) into the coordinate system (0'', n', b', t') (the moving trihedron of the curve c_1). [8], [9]



Figure 6: Transformations of the coordinate systems $(O''', n'', b'', t') \rightarrow (O', n, b, t) \rightarrow (O, x, y, z)$

In Fig. 6 are displayed transformations of the coordinate system (0'', n'', b'', t'') (the moving trihedron of the cycloidal curve c_2) to the coordinate system (0'', n', b', t') (the moving trihedron of the cycloidal curve c_1), (0'', n', b', t') to the (0', n, b, t) (the moving trihedron of the circle c) and (0', n, b, t) to the coordinate system (0, x, y, z).

The vector function of the cyclical cycloidal surface Φ_2' is

$$\mathbf{P}_{2}'(u,v) = \mathbf{r}_{2}(v) + \mathbf{c}_{2}'(u) \cdot \mathbf{M}_{2}(m_{2}v), \ u \in \langle 0, 2\pi \rangle, \ v \in \langle 0, 2\pi \rangle,$$
(21)

where $\mathbf{r}_2(v)$ is vector function of the cycloidal curve c_2 expressed in equations (15), $\mathbf{M}_2(m_2v)$ is the transformation matrix (Eq.20) and $\mathbf{c}'_2(u) = (r'_2 \cos u, r'_2 \sin u, 0, 1), u \in \langle 0, 2\pi \rangle$ is the vector function of the circle c'_2 moving along the curve c_2 and lying in its normal plane.



In Fig.7 are displayed spherical surface Φ , cyclical surface Φ' , cycloidal cyclical surface Φ'_1 and cycloidal cyclical surface Φ'_2 determined by the parameters r = 15, $m_1 = 6$, $d_1 = 5$, r' = 0.35, $\text{sgn}_1 = -1$, $r'_1 = 0.5$, $m_2 = 6m_1$, $d_2 = 3$, $\text{sgn}_2 = +1$, $r'_2 = 0.5$.

If we project every point $X_2(x_2, z_2, z_2, 1)$ of the cycloidal curve c_2 on the spherical surface Φ by using the central projection with centre of the spherical surface, then we obtain points $X'_2(x'_2, y'_2, z'_2, 1)$ creating the cycloidal curve s'_2 on the spherical surface Φ . Then transformation between coordinates is determined by the equations

$$x'_{2} = \frac{r x_{2}}{\sqrt{x_{2}^{2} + y_{2}^{2} + z_{2}^{2}}}, y'_{2} = \frac{r y_{2}}{\sqrt{x_{2}^{2} + y_{2}^{2} + z_{2}^{2}}}, z'_{2} = \frac{r z_{2}}{\sqrt{x_{2}^{2} + y_{2}^{2} + z_{2}^{2}}}.$$
 (22)

Then cycloidal cyclical surface Φ_2'' on the spherical surface Φ is created by moving circle $c_2' = (0''', r_2')$ along the cycloidal curve c_2' and lying in the normal plane determined by the principal normal and the binormal of the curve c_2' at the point 0'''.

In Fig.8 are displayed spherical surface Φ , cyclical surface Φ' , cycloidal cyclical surface Φ''_1 and Φ''_2 determined by the same parameters as in the Fig.7.

4. Creating the ornaments on the spherical surface by using normal and binormal cycloidal cyclical surfaces

We can create various ornaments on the spherical surface by the changing of the parameters of the cycloidal curves c_1, c_2 as a determining cuves of the cyclical cylindrical surfaces Φ'_1 and Φ'_2 and their projected surfaces Φ''_1 and Φ''_2 on the spherical surface Φ .



In Fig.9 are displayed spherical surface Φ , cyclical surface Φ' , cycloidal cyclical surface Φ''_1 and Φ''_2 , determined by the same parameters as in the Fig.7 except sgn₁ = -1, sgn₂ = -1, in Fig.10 are changed parameters sgn₁ = +1, sgn₂ = -1 and in Fig.11 are changed parameters $m_1 = 8$, $m_2 = 8m_1$, sgn₁ = -1, sgn₂ = +1.



Figure 12



Figure 14

In Fig.12 are displayed spherical surface Φ , cyclical surfaces ${}^{1}\Phi'$, ${}^{2}\Phi'$ created by using the circles ${}^{1}c$, ${}^{2}c$ determined by vector functions

$${}^{1}\mathbf{r}(v) = (r\cos\alpha\cos\nu, r\cos\alpha\sin\nu, r\sin\alpha, 1), v \in \langle 0, 2\pi \rangle,$$

$${}^{2}\mathbf{r}(v) = (r\cos\alpha\cos v, r\cos\alpha\sin v, -r\sin\alpha, 1), v \in \langle 0, 2\pi \rangle,$$

cycloidal cyclical surfaces ${}^{1}\Phi_{1}''$, ${}^{2}\Phi_{1}''$ and ${}^{1}\Phi_{2}''$, ${}^{2}\Phi_{2}''$ determined by vector functions

$${}^{1}\mathbf{P}_{1}'(u,v) = {}^{1}\mathbf{r}_{1}(v) + \mathbf{c}_{1}'(u) \cdot {}^{1}\mathbf{M}_{1}(m_{1}v), \ {}^{2}\mathbf{P}_{1}'(u,v) = {}^{2}\mathbf{r}_{1}(v) + \mathbf{c}_{1}'(u) \cdot {}^{2}\mathbf{M}_{1}(m_{1}v),$$
$${}^{1}\mathbf{P}_{2}'(u,v) = {}^{1}\mathbf{r}_{2}(v) + \mathbf{c}_{2}'(u) \cdot {}^{1}\mathbf{M}_{2}(m_{2}v), \ {}^{2}\mathbf{P}_{2}'(u,v) = {}^{2}\mathbf{r}_{2}(v) + \mathbf{c}_{2}'(u) \cdot {}^{2}\mathbf{M}_{2}(m_{2}v)$$

Determined by the parameters r = 15, $m_1 = 6$, $d_1 = 5$, r' = 0.35, $\text{sgn}_1 = -1$, $r'_1 = 0.5$, $d_2 = 3$, $m_2 = 6m_1$, $\text{sgn}_2 = +1$, $r'_2 = 0.5$, $\alpha = \pi/5$.



Figure 15

Figure 16

Figure 17

In Fig.14 is changed parameter $m_1 = 8$, in Fig.15 is changed parameter $\text{sgn}_2 = -1$, in Fig.16 is changed parameter $d_1 = 7$, in Fig.17 are changed parameters $d_1 = 8$, $\alpha = \pi/4$, in Fig.16 is change parameter $d_2 = 6$.

Conclusion

By using of the mathematical apparatus we can create spatial ornaments as an example of the utilization of mathematics in design. The design is also needed in addition purposeful and aesthetic.

The described method of the modeling of cyclical cycloidal surfaces on the spherical surface shows a simply way to modeling of different interesting ornaments on the spherical surface by changing of parameters.

In this paper was used rotations about the coordinate axes of the moving trihedrons of the curves lying on the spherical surface.

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