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NEGATING AS TURNING UPSIDE DOWN

Abstract. In order to understand negation as such, at least since Aristotle's time, there have been many ways of conceptually modelling it. In particular, negation has been studied as inconsistency, contradictoriness, falsity, cancellation, an inversion of arrangements of truth values, etc. In this paper, making substantial use of category theory, we present three more conceptual and abstract models of negation. All of them capture negation as turning upside down the entire structure under consideration. The first proposal turns upside down the structure almost literally; it is the well known construction of opposite category. The second one treats negation as a contravariant functor and the third one captures negation as adjointness. Traditionally, negation was investigated in the context of language as negation of sentences or parts of sentences, e.g. names. On the contrary we propose to negate structures globally. As a consequence of our approach we provide a solution to the ontological problem of the existence of negative states of affairs.

Keywords: negation, opposition, adjointness, duality, category theory, negative states of affairs.

1. Introduction

In order to understand negation as such, at least since Aristotle's time, there have been many ways of conceptually modelling it. In particular, negation was studied as inconsistency, contradictoriness, falsity, cancellation, as an inversion of the arrangements of truth values, as failure, as a role-switch between two players, as an "empirical" negation, etc.[1] In this paper we will present three more conceptual models of negation. All of them capture negation as turning upside down the structure under consideration. The first one does it almost literally, the second one as a special kind of functor and the third one captures negation as adjointness.

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Negation is understood as a rather linguistic or logical phenomenon. In particular, we think of it as a language operator that leads somehow from one expression to the opposite expression. The most common expression used in research on negation is a sentence or some part of the sentence. From the logical point of view the connective $\neg$ is a negation if:

$$\phi \in \mathrm{Cn}(X) \quad \text{iff} \quad \mathrm{Cn}(X \cup \{\neg\phi\}) = \mathrm{Form}$$

where Form is a set of all formulas, $\phi \in \mathrm{Form}$, $X \subset \mathrm{Form}$ and Cn is a given logical consequence, see (Pogorzelski and Wojtylak, 2008, chapter 4). Less frequently, negation of a paragraph, or a chapter, or larger wholes, such as scientific theories or world views are examined. Aristotle used argument from the negation of ideas in the metaphysical battle between himself and Plato on the ontological status of ideas. Ideas can be regarded as metaphysical examples of objects, that can be negated *in toto*. In this paper, we present how one can understand the negation of larger wholes, which may be generally called structures.[2]

It may be safely assumed that content is carried by some larger whole. Without going into detail, we follow the Gestaltists: the whole is not only the sum of its parts and any part should be studied as a part of some greater whole. For example, the content of a sentence A is immersed in the entire scientific theory of which A is a part and should be examined in the context of the whole theory. Moreover, the content of a theory T fundamentally depends on the subject matter of T, which does not need to be a language object. Therefore, in order to understand what the negation of a sentence is, we need to understand the fragment of the area to which it refers. Another assumption that we accept for the purposes of this paper is that theories as such (as well as larger and consistent sets of sentences) relate to structures of which they are theories. These structures are non-linguistic objects that are described by their theories. Much then depends on these structures. To our knowledge the most general theory of structures is category theory, not Aristotle's, but the one created by Eilenberg and Mac Lane in the first half of the 20th century. Therefore, we consider how one can generally understand the process of negation in category theory, especially how one can consider the negation of a category as such.

Negation consists of at least what is negated and the result of the negation. These two objects are strongly related. The understanding and description of negation is actually a description of this dependency. The logical square is a traditional and well-known description of this dependence. It is also known that the existential quantifier could be defined by using a general quantifier and by means of the negation as a one-place logical connective. However, the question may be asked: is the link between the two quantifiers not a stronger structural link? What is the negation here? Is it only "one-place truth/falsit-toggling negation operator"? In the lens of category theory the link is much deeper. Quantifiers are functors which are adjoint. The adjointness is not a one-place operator, instead it is a subtle relationship between two functors.

There are at least three activities within category theory that could be considered as negation. Obviously nobody calls these operations «negation», because they have their own specific names and negation is commonly understood in mathematics as a one-place logical connective. Nevertheless, we want to show that there are conceptual similarities between these three activities and the negation *sensu largo*. The first of these is the reversal of the direction of arrows in a given category: that is to say the process of creating the opposite category. In other words it is placing the structure (here: a category) upside down. The second—related to the first—is the construction of a contravariant functor. The third process is finding the conceptual reversal of a given phenomenon (here: construction, operation, transformation). In this article we squeeze negative juice, if one can say so, from these three counterparts of negation.

The plan of our paper is as follows. The next section provides elementary concepts from category theory. Subsequently, the third section begins with the idea of negation as a "geometric turning" upside down. We show that this idea of negation can be presented more generally in category theory as a construction of the opposite category. An interesting consequence of our approach is the principle of duality—in its full generality as recognised in category theory—which permeates the whole of mathematics. In section 4, negation as a contravariant functor in the context of dual equivalence is examined. Then we investigate, in section 5, a third approach to negation, namely negation as adjointness. At the end we draw ontological consequences from the two presented models of negation. We propose *inter alia* a new solution to the problem of the existence of negative states of affairs. We also arrived at unexpected consequences, first noticed by William Lawvere, where it turns out that semantics and syntax are dual to each other.

2. Categories in short

In this section, we will rather informally introduce the concept of category.[3] Categories are found across all mathematics. An aggregate **Set** consisting of sets and functions defined on them is the first example. The second example is **Top**: the category of topological spaces and continuous functions. This aggregate consists of all topological spaces and all possible continuous functions between them. In the abstract case a category consists of two kinds of data: objects $a, b, c, \ldots$ and arrows $f, g, h, \ldots$. In **Set**, objects are sets and arrows are functions; in **Top**, objects are topological spaces and arrows are continuous functions. Each arrow is endowed with the object that is its source and the object that is its goal, namely: its domain and codomain. If $a = \mathrm{dom} f$ and $b = \mathrm{cod} f$, then we can write $f\colon a \to b$. Each arrow goes with its own domain and codomain: that is to say if we treat an inclusion as an arrow, these are two different arrows in **Set**: $\mathbb{N} \subseteq \mathbb{N}$ and $\mathbb{N} \subseteq \mathbb{Z}$, despite the fact that they are identical sets of ordered pairs. If $f\colon a \to b$ and $g\colon b \to c$ are given, then an arrow is also given: $g \circ f\colon a \to c$, called the composition of f and g: that is to say "first apply f, then g" (instead of the long version '$g \circ f$' we will prefer to use the abbreviation 'gf'). For each object a there is given an identity arrow: $1_a\colon a \to a$. That is all the data we need. The only thing that is still required are two simple laws: associativity of the composition and unity law. The former is stated as follows: $f(gh) = (fg)h$, the latter as follows: $f1_a = f = 1_b f$ (identity does not change anything). Everything that meets these conditions is called a category.

These two identities $f1_a = f = 1_b f$ can be expressed with the help of the commutative diagram:

$$\begin{tikzcd}
a \arrow[r, "1_a"] \arrow[dr, "f"'] & a \arrow[d, "f"] \arrow[dr, "f"] & \\
& b \arrow[r, "1_b"] & b
\end{tikzcd}$$

The fact that this diagram commutes means that identities $f1_a = f = 1_b f$ are valid. Generally speaking, we say that a diagram commutes if all paths (through composition) with the same beginning and end yield the same result. The commutating diagrams used in category theory are equivalent to algebraic equations.

There are a lot of categories, for ontological minimalists perhaps even too many. Each set, endowed with a certain structure and functions preserving this structure, is a category. As examples of these, "structured sets" may serve: groups and group homomorphisms; monoids and monoid homomorphisms; graphs and graph homomorphisms; differentiable manifolds and smooth mappings; posets (and preorders) and monotone functions; the natural numbers $\mathbb{N}$ and all recursive functions $\mathbb{N} \to \mathbb{N}$ and a huge number of others. There are categories in which arrows are not the functions. Consider the category **Rel**: where sets are objects and binary relations are arrows. There are also categories in which the collection of objects is not a set: consider the just mentioned category of groups and group homomorphisms. Let us give one more example. Let $\mathbf{P}$ be a preorder $\langle X, \leq \rangle$ equipped with a reflexive and transitive relation $p \leq q$. $\mathbf{P}$ is a category. Simply take as the objects the elements of P and the arrows $p \to q$ if and only if $p \leq q$. In the category $\mathbf{P}$ there is at most one arrow between any two objects.

The arrows between categories are called functors. Functors preserve the compositions of arrows, domains and codomains, and identities. One can think of them as homomorphisms of categories. An example of a functor is a forgetful functor. If **Grp** is a category of groups, then $F\colon \mathbf{Grp} \to \mathbf{Set}$ is a forgetful functor, namely F maps every group to the underlying set of its elements and every group homomorphism to the underlying set-theoretical function. Quantifiers $\forall$ and $\exists$ are also examples of functors. To see it, one has to select categories (domains and codomains) properly. If $\mathrm{Form}(\overline{x})$ is a set of all formulas (of the first order language) with, at most, free variables from the list $\overline{x} = x_1, x_2, \ldots, x_n$, then $\mathrm{Form}(\overline{x})$ is a category. It is a preorder set under the entailment relation $\phi(x) \vdash \psi(x)$. If y is a variable that is not in the list $\overline{x}$, then the universal quantifier is a functor $\forall y\colon \mathrm{Form}(\overline{x}, y) \to \mathrm{Form}(\overline{x})$.

Having functors, i.e. arrows between categories, it can be asked whether the categories with functors form a category. That is the case, but there are some limitations. If an aggregate of objects and an aggregate of arrows in a given category are sets, then this category is called a small category. Otherwise we say that the category is large. All small categories with functors as arrows form a (large) category of categories **Cat**. In fact category theory is a theory of transformation, in which the basic role is played by the concept of composition of transformations, not the concept of membership, as in set theory. It is a theory of dynamic processes, not a static substance. The dynamic aspect was summed up by Steve Awodey (2006, p. 8) in the following way:

> One important slogan of category theory is, *It's the arrows that really matter!*

3. Negating as turning upside down

The idea of negation as inversion was suggested by Frank Ramsey:

> [w]e might, for instance, express negation not by inserting a word "not", but by writing what we negate upside down. Such a symbolism is only inconvenient because we are not trained to perceive complicated symmetry about a horizontal axis, and if we adopted it, we should be rid of the redundant "not-not", for the result of negating the sentence "p" twice would be simply the sentence "p" itself (Ramsey, 1927, 161–162).

In this passage Ramsey is thinking only about the notation of negation and not about the meaning (semantics) of negation. In fact, this is only an idea of how to use mathematical symbols. The same idea is adopted in writing "$\top$" for truth and "$\bot$" for false. That was Behmann's typographical convention (see Varzi and Warglien (2003, p. 10)).[4] However, negation as such could be treated as reversing *things* upside down, not only the symbols. Suppose a man is talking to a woman: "I didn't leave you. It was the other way around". He claims that she was the person that left. Things were the other way round. Therefore, it can be argued that:

> (...) negation is a form of *reversal* or *inversion*: to deny a given proposition is to say that things are *the other way around* (Varzi and Warglien, 2003, p. 10).

Varzi and Warglien to model the negation use topologically rigid structures, which they call the *truth-polygons*. The negation of some truth-value is illustrated by a *rotation* or a *reflection*. In spite of the interesting results of their work, the negation understood as rotation or reflection is too narrow. They do not turn the structure around, but rather illustrate negation as a procedure on polygons. Our approach is to understand negation as a reversal of the structure *in toto*, the whole engaged context. Therefore, we are considering how to understand the negation of entire categories. Of course, there is no negation of a category in the sense that all non-groups and non-group-homomorpisms do not form a non-group category. But there are other solutions.

If we assume that the structures negated (i.e. turned upside down) are categories, then the natural counterpart of negation of a category is an opposite category. Let $\mathbf{C}$ be a category. Then the opposite category is the category $\mathbf{C}^{\mathrm{op}}$ which has the same objects as $\mathbf{C}$ but all of the arrows from $\mathbf{C}$ are formally turned around in $\mathbf{C}^{\mathrm{op}}$. The following figure explains this reversal. Let's assume that in $\mathbf{C}$ this is the case:

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[dr, "gf"'] & b \arrow[d, "g"] \\ & c \end{tikzcd}$$

Then in the upside down world of $\mathbf{C}^{\mathrm{op}}$ this would be the case:

$$\begin{tikzcd} \bar{a} & b \arrow[l, "\bar{f}"'] \\ & c \arrow[u, "\bar{g}"'] \arrow[ul, "\bar{f}\bar{g}"] \end{tikzcd}$$

In more formal terms, if $f\colon a \ \to b$ is an arrow in $\mathbf{C}$, then $\bar{f}\colon b \to a$ is an arrow in $\mathbf{C}^{\mathrm{op}}$.

Assume that $\mathbf{P}$ is a poset $\langle P, \leq \rangle$. Then $\mathbf{P}$ is a category in which objects are elements of P and there is an arrow $p \to q$ if and only if $p \leq q$, for $p, q \in P$. In this category there is at most one arrow between two objects. The operation turning upside down $\mathbf{P}$ consists of reversing the order $\leq$. Then the category $\mathbf{P}^{\mathrm{op}}$ is a category with the new order $\leq_{op}$ defined as follows: $q \leq_{op} p$ in $\mathbf{P}^{\mathrm{op}}$ if and only if $p \leq q$ for $p, q \in P$ in $\mathbf{P}$.

A *lattice* is a poset $\langle \mathbf{L}, \leq \rangle$ in which each of the two elements a, b have both the supremum (i.e. the least upper bound) $a \vee b$ and the infimum (i.e. the greatest lower bound) $a \wedge b$. Note that if $\mathbf{L}$ is viewed as a category then $a \vee b$ is simply the coproduct of the objects a and b, while $a \wedge b$ is the product of a and b. If the elements of $\mathbf{L}$ are supposed to represent some logical values (e.g. when they are the equivalence classes of certain formulas) then $a \vee b$ and $a \wedge b$ are viewed as the alternative and the conjunction. Passing to the opposite category $\mathbf{L}^{\mathrm{op}}$, which is again a lattice, we see that the conjunction and alternative are interchanged.

The abstract category-theoretic concept of a product and coproduct are defined as follows. Given objects A, B of a category $\mathbf{C}$, their *coproduct* is an object $A \oplus B$ together with $\mathbf{C}$-arrows $i_A\colon A \to A \oplus B$, $i_B\colon B \to A \oplus B$ satisfying the universality condition: For every $\mathbf{C}$-arrow $f_A\colon A \to X$, $f_B\colon B \to X$ there exists a unique $\mathbf{C}$-arrow $h\colon A \oplus B \to X$ such that $f_A = h \circ i_A$ and $f_B = h \circ i_B$. It turns out that coproducts (if they exist) are determined uniquely, up to isomorphisms. The definition of a *product* is dual to that of the coproduct. Formally, the product of A and B is the same as the coproduct of A, B in $\mathbf{C}^{\mathrm{op}}$.

3.1. Negation as duality

Negation traditionally (through a square of opposition) is understood as a contradiction. Modelling negation as an upside down operation allows us to capture the negation as duality. The phenomenon of duality in all its glory has been described in category theory and has been named as the duality principle.

In the elementary language of category theory[5] there are symbols of objects $a, b, c, \ldots$, arrows $f, g, h, \ldots$ and four operations: domain $\mathrm{dom}(f)$, codomain $\mathrm{cod}(f)$, identity 1_A and composition $\circ$. The formal definition of category consists of seven axioms:

A1 $\mathrm{dom}(1_A) = A$

A2 $\mathrm{cod}(1_A) = A$

A3 $f \circ 1_{\mathrm{dom}(f)} = f$

A4 $1_{\mathrm{cod}(f)} \circ f = f$

A5 $\mathrm{dom}(g \circ f) = \mathrm{dom}(f)$

A6 $\mathrm{cod}(g \circ f) = \mathrm{cod}(g)$

A7 $h \circ (g \circ f) = (h \circ g) \circ f$

It turns out that each sentence Σ expressed in the elementary language of category theory has its dual equivalent Σ^*. In order to receive a dual sentence Σ^* for Σ, it is sufficient to make the following replacements: $g \circ f$ for $f \circ g$, cod for dom and dom for cod.

Then we have a formal duality principle (Awodey, 2006, p. 48):

Formal duality principle

For any statement Σ in the language of category theory, if Σ follows from the axioms for categories, then so does Σ^*:

$$CT \vdash \Sigma \quad \text{implies} \quad CT \vdash \Sigma^*$$

The following principle results directly from the construction of the opposite category:

$$(\mathbf{C}^{\mathrm{op}})^{\mathrm{op}} = \mathbf{C}$$

The principle could be treated in our approach as a double negation principle. The construction of an opposite category also has other consequences, which we present in the next section.

4. Negation as dual equivalence

The simple construction of the opposite category has the following main features: We have two categories $\mathbf{C}$ and $\mathbf{C}^{\mathrm{op}}$ and two contravariant functors $R\colon \mathbf{C} \to \mathbf{C}^{\mathrm{op}}$, $L\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{C}$, both identically sending the objects while at the same time "reversing" the arrows. Formally, $R(f) = L(f) = f$ for every $\mathbf{C}$-arrow f; however L and R are not identities, because they transform a category to its opposite one. What is perhaps most important, the equations $LR = 1_{\mathbf{C}}$ and $RL = 1_{\mathbf{C}^{\mathrm{op}}}$ hold, meaning that one can come back to the original situation by using double negation.

It is perhaps time to explain what a contravariant functor is. Namely, a functor F from a category $\mathbf{C}$ to a category $\mathbf{D}$ is a mapping that sends the objects of $\mathbf{C}$ to the objects of $\mathbf{D}$ and sends $\mathbf{C}$-arrows to $\mathbf{D}$-arrows. A functor should of course send identities to identities and it should preserve the composition operation. Here, there are two possibilities. The first one is $F(f \circ g) = F(f) \circ F(g)$ for every compatible $\mathbf{C}$-arrow f, g. On the other hand, it is possible to declare that $F(f \circ g) = F(g) \circ F(f)$, thus assuming that F "reverses" the arrows. Such a functor is called *contravariant* while the usual functors (not reversing the arrows) are called *covariant*.

Summarizing, a contravariant functor $R\colon \mathbf{C} \to \mathbf{D}$ could be considered as a (more generalized) negation, as it does not forget too much information. Taking into account the rather acceptable "principle of return", namely, that the process of negating things can be reversed by applying a possibly different negation, we should assume that there is another contravariant functor $L\colon \mathbf{D} \to \mathbf{C}$ that serves as the inverse of R. The simple meaning of the "inverse" is of course declaring $LR = 1_{\mathbf{C}}$, $RL = 1_{\mathbf{D}}$. On the other hand, there are several concrete examples of functors L, R that look like inverses to each other, however they formally satisfy weaker equations, where $=$ is replaced by the isomorphism relation $\equiv$.

Perhaps one of the simplest examples comes from linear algebra: both $\mathbf{C}$ and $\mathbf{D}$ are finite-dimensional linear spaces and both R and L are the functors that produce dual spaces. It is well known that the second dual of a finite-dimensional vector space V has the same dimension as V and therefore it is isomorphic to V. On the other hand, it is not formally equal to V. Thus, double negation produces an object isomorphic to the original one, but not necessarily equal to it.

Another example of a pair (R, L) as above is $\mathbf{C}$ the category of finite sets, $\mathbf{D}$ the category of finite Boolean algebras, and $R(X)$ is the power-set of a set X, while $L(B)$ is the set of atoms of the Boolean algebra B. The finite version of Stone duality says that the functors R, L satisfy $LR \equiv 1_{\mathbf{C}}$, $RL \equiv 1_{\mathbf{D}}$. It turns out that the isomorphisms witnessing this fact form two compound structures, namely natural transformations between identities of $\mathbf{C}$, $\mathbf{D}$ and the composite functors RL, LR. Formally, a *natural transformation* from a functor $F: \mathbf{C} \to \mathbf{D}$ to a functor $G: \mathbf{C} \to \mathbf{D}$ is a mapping η transforming each $\mathbf{C}$-object C to a $\mathbf{D}$-arrow $\eta_C: F(C) \to G(C)$ satisfying $G(f) \circ \eta_C = \eta_{C'} \circ F(f)$ for every $\mathbf{C}$-arrow $f: C \to C'$. In our case, in order to speak about a *dual equivalence* we need to have natural transformations $\eta: 1_{\mathbf{C}} \to LR$, $\xi: 1_{\mathbf{D}} \to RL$ and moreover η_C and ξ_D should be isomorphisms, so that the principle of double negation is valid. The next section describes a slightly different situation, where the natural transformations have a special universal property, stated below in the definition of dual adjointness of two (not necessarily contravariant) functors.

5. Negating as categorical adjointness

Traditional logic has used specific forms of sentences; especially since Aristotle's times, sentences of the following kind have been studied: *Every S is P*, *No S is P*, *Some S is P* and *Some S is not P*. The logical relationships between these sentences are illustrated by a traditional and well-known square of opposition. The negation of the form-sentence *Every S is P* is a form-sentence *Some S is not P*. They are contradictory. Thanks to that, in order to find out the logical value of one of them, it is enough to know the logical value of the second one. For example, if we know that the sentence *Every politician is a wise man* is a false sentence, then we also know that the sentence *Some politicians are not wise men* is a true sentence—assuming that there are politicians and wise people. The logical value of one sentence is somehow included in the negation of the second sentence. Due to negation we can move freely between these sentences. The first one can be used to construct the second one. An adjoint situation in category theory is similarly constructed; having one functor, one can ask about the existence of its adjoint, which could be treated as an inverse of the initial functor.[6] We do not take logical values into account; in the case of functors, the considered structures (categories) and their properties are valid.

Let us consider, more specifically, the forgetful functor mentioned in **section 1**. $U\colon \mathbf{Grp} \to \mathbf{Set}$ forgets about the group structure, namely U maps every group G to the underlying set of its elements $U(G)$ and every group homomorphism $f\colon G \to H$ in $\mathbf{Grp}$ to the underlying function $U(f)\colon U(G) \to U(H)$ in $\mathbf{Set}$. A natural question arises: is it possible to return from $\mathbf{Grp}$ to $\mathbf{Set}$ by reversing U? The categories $\mathbf{Grp}$ and $\mathbf{Set}$ are not isomorphic, so there is no standard return. However, from any set X in $\mathbf{Set}$ it is possible to construct a free group. In this way we obtain a functor $F\colon \mathbf{Set} \to \mathbf{Grp}$, which assigns to X a free group $F(X)$ and to each function $f\colon X \to Y$ a group homomorphism $F(f)\colon F(X) \to F(Y)$. What is the relationship between U and F? They are not inverses in a standard way; namely, it is not true that UF is the identity functor on $\mathbf{Set}$ and FU is the identity functor on $\mathbf{Grp}$. Let us examine this in more detail. We start with a group G, then after applying the functor U we obtain the set $U(G)$, then after the application of the functor F we receive again the group $FU(G)$. Thus, what is the relationship between G and $FU(G)$? These groups are different, but they are strongly linked. There is a natural function $\epsilon\colon FU(G) \to G$ called co-unit of the adjunction with the following universal property: for any group homomorphism $g\colon F(X) \to G$, there is a unique function $h\colon X \to U(G)$ such that $\epsilon \circ F(h) = g \circ FU(G)$.

Let us apply first F on set X, then apply U, so that we will obtain the set $UF(X)$; they are not the same sets. However, again there is a strong link between X and $UF(X)$. There is a natural function $\eta\colon X \to UF(X)$, known as the unit of the adjunction, satisfying the following universal property: for any function $g\colon X \to U(G)$ there is is a unique group homomorphism $h\colon F(X) \to G$ such that $U(h) \circ \eta = g$. The construction $FU(X)$ could be treated as an insertion of generators, namely the best possible solution to inserting elements of X into a group. In a case such as this, we say that the forgetful functor U is right adjoint of "free construction functor" F which we indicate as $F \dashv U$. We are now able to proceed with the definition of adjunction.

An *adjunction* (Awodey 2016, p. 180–181) between categories $\mathbf{C}$ and $\mathbf{D}$ consists of functors $F\colon \mathbf{C} \to \mathbf{D}$ and $U\colon \mathbf{D} \to \mathbf{C}$ and a natural transformation $\eta\colon 1_{\mathbf{C}} \to UF$ with the universal property: for any $C \in \mathbf{C}$, $D \in \mathbf{D}$ and $f\colon C \to U(D)$ in $\mathbf{C}$ there is a unique $g\colon F(C) \to D$ such that $f = U(g)\eta_C$. As in the example above, U is called the right adjoint of F. The natural transformation η is called the unit of the adjunction.

Among many examples of adjoint situations, one should mention a few other cases. Let us start with a functor from the category of abelian groups to the category of abelian monoids. It forgets about the inverse operation.

Following on, let us mention the forgetful functor from the category of compact Hausdorff spaces to the category of topological spaces. It forgets the compactness and the Hausdorff property of a given topological space; its left adjoint is the Stone-Cech compactification. There are many more adjoint situations in mathematics. The reader can find many examples in (Mac Lane, 1998, p. 87), (Awodey, 2006, p. 187–196) and (Marquis, 2015).

It should be noted that negation understood as a one-place sentential connective can be represented as an adjunction. However, it cannot be claimed that the possibility to represent a sentential negation as an adjunction (see Awodey (2006, p. 193)) in category theory determines negation as such. Indeed, all sentential connectives could be represented as adjunctions. On the other hand, it can be argued, as we do, that the general phenomenon of negation has been captured in adjoint situations. Let us consider the forgetful functors and note that the essence of the process of forgetting is the omission of certain properties. Not all properties can be omitted arbitrarily, because not every omission leads to a significant adjunction. Aristotle in his *Categories* noted this while analyzing negation as a form of deprivation:

> We say that that is capable of some particular faculty, or possession has suffered privation, when the faculty or possession in question is in no way present in that in which, and at the time at which, it should naturally be present. We do not call that toothless which has not teeth, or that blind which has not sight, but rather that which has not teeth or sight at the time when by nature it should. For there are some creatures which from birth are without sight, or without teeth, but these are not called toothless or blind. (*Categories*, 12a28–33, translated by E. M. Edghill)

It is one of the most surprising facts that—due to Lawvere's finding—quantifiers as functors are adjoint (for details see (Awodey, 2006, p. 193–195)). If $*$ will be a trivial operation: $*\colon \mathrm{Form}(\overline{x}) \to \mathrm{Form}(\overline{x}, y)$ sending each formula $\phi(\overline{x}) \in \mathrm{Form}(\overline{x})$ to itself, then it turns out that:

$$\exists \dashv * \dashv \forall$$

It is a trivial fact that, in predicate logic, quantifiers are co-determinable by De Morgan's laws: $\forall x\phi(x) = \neg(\exists x \neg\phi(x))$ and $\exists x\phi(x) = \neg(\forall x \neg\phi(x))$. Categorical adjointness of the quantifiers is not such a direct connection between them as co-determinableness is. However, it is a structural and profound connection. For example, the existential introduction rule $\phi(\overline{x}, y) \vdash \exists y\phi(\overline{x}, y)$ is just a unit of the adjunction $\exists \dashv *$.

6. Conclusions

Consider any two left adjoints F and F' of a functor U. It is the case that they are naturally isomorphic, which means that adjoints are unique up to isomorphism. It means that a negation (of a functor or of some construction) modelled as an adjoint is unique up to isomorphism. Thus, the negation of a phenomenon is not determined uniquely, but all its instances are isomorphic. There is an additional connection between negated entities. In the example of Aristotle, a toothless person, both a person with teeth and a person without teeth, have something in common: the presence or absence of teeth. Similarly, negation as an adjoint situation keeps certain structural properties, i.e. a left adjoint preserves all the co-limits and a right adjoint preserves all the limits.

Let us assume that the affirmative sentence α refers to the state of affairs which occurs. For example, α can be like this: *This rose is red*. It refers to the state of affairs which is "being the red of this rose". What is the negation of this sentence? It could be such a statement: *This rose is not red*, which refers to the following state of affairs "being not the red of this rose". What is the entity that the negation of α refers to? This is a negative state of affairs. Of course, many philosophers would say that they do not exist (these problems are discussed in (Chrudzimski, 2012)). Because the only thing that exists is the state of affairs that happen. And that is not the case with "being not the red of this rose". Our solution to this problem is as follows. Suppose that category theory is the formal ontology of a state of affairs. To be more precise, let us assume that we represent states of affairs as functors. Then the negative state of affairs exists if there is an adjoint functor to the initial positive state of affairs. In mathematics it is known that the problem of the existence of an adjoint of a given functor is often a very challenging task. It should be analogous to ontology, where the existence of negative states of affairs should not be a general (and trivial) issue.

Negationality as such has always been studied in a syntactic and semantic context. A surprising result of the modeling of negation as an adjunction is a fact observed by Lawvere (1969). To put it broadly one can say that syntax and semantics are adjoints. To be more precise, if we consider the class of sentences and the class of its models (structures), there is a Galois connection—actually an adjoint situation—between them. The connection is generated by the relation "true of" (see Smith (2010)). Metaphorically speaking, semantics is a syntax turning upside down. There is a kind of opposition (or duality) between syntax and semantics. This consequence from a traditional point of view seems to be non-trivial: in what manner is it possible that sentences are facts, but turned upside down?

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NOTES

[1] An overview of these investigations together with a large bibliographic list is available in (Horn and Wansing, 2017).

[2] The structure could be treated as a model for an idea. Despite Aristotle's clever "negation" argument, Plato won the battle anyway. The understanding of negation elaborated in this paper can also be applied to reject Aristotle's argument. More details about how our considerations are related to this metaphysical disagreement can be found in (Skowron, 2015).

[3] All basic categorical concepts discussed in this paper are based on an easy-to-read textbook by Steve Awodey (2006). For readers working in mathematics, we recommend a classic title by Mac Lane (1998).

[4] Another inversion-oriented hero mentioned by Varzi and Warglien is Charles Sanders Peirce. Discussion of historical issues, however, would draw our attention too much from the main thread, hence we would refer the interested reader to the paper of Varzi and Warglien (2003).

[5] In the description of the phenomenon of duality we follow (Awodey, 2006, p. 48).

[6] The idea of treating adjoint situations as conceptual inverses is discussed in (Marquis, 2015). The analysis of the example of the forgetful functor given in the next paragraph also comes from (Marquis, 2015).

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