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A DECISION LOGIC APPROACH TO MILL'S ELIMINATIVE INDUCTION

Abstract. The subject of the paper is a contemporary interpretation of J.S. Mill's elimination method using selected concepts of Zdzisław Pawlak's decision logic. The aim of the interpretation is to reformulate the original rules (canons) of Mill's induction so that they correspond more precisely to his concept of cause as a complex sufficient condition. In the first part of the paper, we turn to Mill's writings and justify the thesis that in his understanding the cause is an aggregation of circumstances, and not a single circumstance; next, we point out that Mill's original canons (for example the canon of agreement and the canon of difference) do not allow causes-aggregations to be singled out from empirical data. In the second part of this paper, we present such aspects of Z. Pawlak's decision logic that serve as the basis for the formalisation of the method of eliminative induction. We describe exhaustively the schema of induction that involves a gradual – divided into three stages – simplification of a set of implications corresponding to the observed dependencies [*system of potential causes, effect*]. The simplification is deductive because it maintains consistency within the set of implications. We show that such schema is ideal for isolating complex causes (aggregations of circumstances), ultimately described using complex conditional formulas of decision logic.

Keywords: Mill's methods, eliminative induction, decision logic, concept of cause, John Stuart Mill, Zdzisław Pawlak.

Introduction

John Stuart Mill is the author of well-known canons of eliminative induction, often described in methodological literature. He is also the author of a certain conditional concept of causality, in light of which these canons should be considered. Typically, however, they are presented independently from the concept of cause, which can somewhat falsify the idea of eliminative induction associated with them. This is because Mill himself does not explicitly point to any link between his concept of cause and the canons of induction.

As one reads *A System of Logic*, it seems that its author examines the issue of induction in a manner partially inconsistent with his concept of cause. Discussing the inductive method, comparing the preceding and successive phenomena, he does not treat predecessors as a complex whole consisting of circumstances causing the effect, but only as a single circumstance. But, according to his concept, the cause of a phenomenon should be treated as an aggregation of circumstances – a whole preceding the effect. With this approach, the induction method could be applied to search for various causes of the same effect; and speaking the language of logic: to search for complex alternatives, whose components would be conjunctions of individual circumstances. In this paper, we will discuss (selectively) the system of logic which ensures that such alternatives are found.

This paper is divided into two parts. The leading theme of the first part (Chapter 1) is the analysis of Mill's concept of cause, as well as the relations of said concept with the (now historical) canons of induction. The subject of the second part (Chapters 2 and 3) is the modern method of eliminative induction, which we believe to be consistent with Mill's concept of causality. This method is a part of Zdzisław Pawlak's decision logic, which has been developed as a formal basis for data processing in computer systems. Despite such a modern origin, this system has its theoretical roots in Mill's logic, and the induction schema proposed in this system has its distant prototype in Mill's canons.[1]

1. J.S. Mill's concept of cause and eliminative induction

1.1. The concept of physical cause

The presence of causal relationships in the world directly rationalises inductive science in Mill's writings, and to deny them would negate this doctrine. In his view, the only concept of cause that the theory of induction needs is a notion that can be obtained through empirical experience. The law of causality, which says that every phenomenon has its cause, is the main pillar of inductive science in Mill's writings. According to Mill, the inductive process, by its nature is an examination of cases of causal relationships (Mill, 1851, p. 381). A cause whose character is physical (and only those may be the subject of scientific examination) can be established on the basis of observation. This assertion is opposed to Hume's belief that causal relationships cannot be grasped through experience.

In this regard, the physical cause[2] is a physical phenomenon which affects another physical phenomenon causing a certain physical effect therein. Thus, the field of the causal relationship is a set of physical, material phenomena that can be learned through sensory experience (and sometimes described in terms of attributes). A causal relationship occurs when there is a constant succession between phenomena and events that precede them, and when it is necessary. The cause for a phenomenon is a predecessor or a set of predecessors, after which the phenomenon occurs invariably and inevitably (unconditionally) (Mill, 1851, p. 352). The phenomenon occurs unconditionally after the occurrence of a certain fact, if the said phenomenon would not have occurred but for that fact. For example, food is the cause of a man's death, if he would not have died had he not eaten that food.

1.2. Cause as a complex sufficient condition

According to Mill, invariable succession occurs between a successor and a set of predecessors. For practical or psychological reasons, we often single out one of the predecessors and call it the cause, while other ones are called the conditions. We recognise that condition to be the cause, whose participation in inducing the effect seems most evident, or the one on which we focus at a given time. For example, if a person eats food and consequently dies, we are, according to Mill, liable to say that the cause of death was eating the food. However, it does not necessarily mean that there is an invariable relationship between eating that food and death. After all, while one person may die after eating food, another will stay alive after eating the same food. Survival will depend, inter alia, on the weight of the person, weather conditions, state of health, etc. If the man who ate the poisonous food weighed 140 kilograms and only consumed a small amount of toxin, he might not die, even though the same amount would kill a person with a significantly lower body weight. There is therefore a causal relationship between the whole of the circumstances and the phenomenon that invariably follows them (Mill, 1851, p. 340) – in our example it would be all the circumstances surrounding eating poisonous mushrooms and the phenomenon of death that invariably follows.

The whole of the circumstances is a set of predecessors that can be treated either mereologically as a physical whole, or distributively, as a structure defined on physical objects (individual conditions).[3] Therefore, the real cause is the whole of these predecessors. No isolated condition, which is a single phenomenon, is more closely related to the effect than any of the other conditions. Each of them is just as necessary for the effect to occur.

The whole of the phenomena of nature is a mixture of unconditional consequences of the preceding set of causes of a permanent character. Mill argued that invariable succession occurs between each phenomenon occurring in nature, and the phenomenon that preceded it (Mill, 1851, p. 339). All phenomena without exception, aside from primary causes, are either direct or indirect results of those primary facts or their connections. Consequently, all phenomena in the world are somehow conditioned. For every phenomenon, there is another phenomenon or a set of phenomena that can be called a confluence of positive and negative conditions. By negative condition we mean the absence of causes that could counteract the occurrence of the effect. Every negative phenomenon is the same as the absence (non-existence) of a positive phenomenon that could interfere with the occurrence of the effect. If a phenomenon occurs invariably, there is a certain set of positive phenomena, provided that there are no other positive phenomena. A negative condition should be understood, therefore, as the absence of a positive phenomenon (Mill, 1851, pp. 345–346). Since there are many possibilities of phenomena that could prevent the effect from occurring, and to enumerate them all is practically impossible, therefore they can be given a single, collective name and described as the absence of counteracting causes. For example, a sentry momentarily leaving his post in the barracks in the case of an attack by enemy troops can be considered the cause of the soldiers' surprise (Mill, 1851, p. 343).

According to Mill, for every phenomenon there is some collection of phenomena that is a confluence of positive and negative conditions, whose occurrence is always followed by that phenomenon, whereby those conditions must occur in a convergent manner. The word "convergent" does not mean "simultaneous" here. The convergence of conditions can be spread over a certain period and consist in the phenomena occurring in a sequence. For instance, nowadays it is considered that driving under the influence of alcohol is often the cause of accidents. In Mill's terms, drinking and driving a car would be a convergent occurrence of the condition of a road accident (although these two conditions alone are not a sufficient condition).

Mill argued that the reason why it is not customary to list every possible condition is either that they are assumed by default, or because, due to the purpose of the examination, they can be overlooked without any consequences (Mill, 1851, p. 341). It is also possible not to discover what that convergence ultimately is. Disclosed herein is indeed the problem of the selection criteria of conditions that should be included as the aforementioned sufficient condition. Bearing in mind that a whole set of phenomena or circumstances has an impact on the occurrence of a certain event, the question arises on the criterion of separating the whole of the conditions from other phenomena. For if we believe that all phenomena remain in causal relationships with some other phenomena, then with a given phenomenon, occurring here and now, they can be considered the result of the whole of the preceding phenomena. Such a solution would bring nothing to the issue of causality and would be completely futile.

1.3. Distinguished class of phenomena

In research practice, we isolate certain phenomena groups from other phenomena, which in the light of our knowledge, we consider to be unrelated to the examined phenomena (treated as effects). According to Mill, "The whole of the present facts are the infallible result of all past facts, and more immediately of all the facts which existed at the moment previous" (Mill, 1851, p. 382). This approach is impossible to employ in the search for causes and requires one to remain only within a narrow confluence of phenomena. The class of distinguished phenomena always occurs among other phenomena that, theoretically speaking, could affect the explicated result. It seems impossible to ascertain the absence of indirect relationships between the distinguished class of phenomena and phenomena co-occurring with them that are not taken into account, and the impact of the latter on the effect. In practice, however, it is impossible to consider all the phenomena preceding the effect. Hence there is a need to narrow down the set of phenomena to a certain distinguished class.

Distinguishing the class of phenomena preceding the effect brings to mind the concept of a relatively isolated system, which was introduced by Ingarden to considerations of causality (Ingarden, 1981).[4] It is a necessary procedure, because otherwise it should be assumed that a phenomenon has for its cause some indefinite confluence of preceding phenomena, which, in extreme interpretation, assuming that the whole world of nature is linked by causal relationships, could be called the preceding state of the world. We assume here clearly that there are a number of conditions essential for the occurrence of a given phenomenon.

1.4. Necessary conditions and sufficient condition

If the cause is always some whole of circumstances, which are invariably and necessarily followed by a certain effect, then if it is known that the subsequent effect is affected by several circumstances, they should be considered as one active cause. Thus, several active phenomena causing some effect comprise one complex cause. Such a cause can be called a sufficient condition consisting of certain necessary conditions.

In Mill's writings, there are two types of confluence of phenomena comprising the cause. The first is a set of necessary conditions comprising the sufficient condition, whose occurrence is followed by the effect of the same kind. In this case, the total value of the effect, say C, cause A and cause B is the sum of these causes. For example, the movement of a billiard ball is the result of two forces affecting it – the force of gravity and the force of impact of the cue. Each of these forces has a certain size and direction, which affect its value. The speed and direction of the ball set in motion depends on them. Secondly, it is a set of necessary conditions that make up a sufficient condition, followed by an effect of a different kind than each of the individual phenomena separately. For example, a combination of two chemicals – hydrogen and oxygen – results in a qualitatively new substance that is nothing like either of the elements. Mill believed that he distinguished regularities that in the first type of relationship operate jointly without change, and regularities that cease to function as a result of a concurrence and are replaced by new ones.

This theory undermined the traditional philosophical belief that effects must be proportionate to their causes. It was in fact the axiom of causality, which fails, according to Mill, where the convergence of causes leads to a change in the attributes of the body, and the effect falls under the new laws, which are unlike those applied to the causes (Mill, 1851, pp. 380–381).

The concurrence of causes entails active agents interacting together in a specific set of passive conditions. It should be assumed that Mill did not attribute to them a more significant role than that of passive agents. The change of both passive and active conditions causes a different effect to occur. If we assume that there are two agents affecting a certain set of co-occurring passive conditions, they are followed by a specific effect. This effect would differ from the effect resulting from one active preceding agent that would not be connected with another, with the set of conditions identical in all other respects. The combined effect of the two active agents is different from the effect of each of them individually (Mill, 1851, p. 371). For example, two identical forces that affect an object cause that object to move. Similarly, each of them separately causes that object to move (although not necessarily so in each case). As a result of the force, the object moves in both cases; however, the motion of the object has a different value (i.e. it has a different speed or direction) when one or the other force affects it individually, and a different one when there are two forces acting at the same time.

1.5. Causal relationship as functional relation

This question raises the issue of the branching of causes and effects, that is, whether a cause of a certain kind can have a cause of only one type and whether a cause of a particular type can produce only one type of effect. Whether or not the causal relationship is some type of functional assignment has operational significance. Unambiguous assignment poses research problems for, respectively, detecting the causes of events and predicting the effects of phenomena.

According to Mill, various phenomena which have no relation to one another whatsoever may depend on one and the same factor. After some event or some fact, two different types of effects may occur in another and the same time. Mill writes that "the sun produces the celestial motions; it produces daylight, and it produces heat," and further "when the same phenomenon is followed [...] by effects of different and dissimilar orders, it is usual to say that each different sort of effect is produced by a different property of the cause" (Mill 1851, p. 355–356). The sun can actually be the cause of daylight on earth and the cause of the elliptical motion of the earth around the sun. Assuming, however, the concept of cause in the strict sense, as an aggregation, i.e. a concurrence of conditions necessary for the occurrence of some effect, we see that the sun would not be the cause of the motion of the earth if the latter did not have a certain mass; it would not be the cause of daylight on earth, if the earth did not have its own atmosphere with air whose particles reflect light waves. The cause of motion of the earth is therefore both the mass of the sun and the mass of the earth, and the cause of daylight on earth – sunlight and air particles on earth. One can also repeat after Mill that each of these effects is the result of a separate attribute of the sun – its mass and light, respectively. Mill uses the term "cause" rather ambiguously, sometimes treating the cause as the whole of circumstances preceding the effect, and at other times when he speaks about the cause as the active agent occurring in these circumstances (Krajewski, 1967, pp. 56–57). Thus, it seems that there are no reasons to assume that, according to Mill, the same cause, in the same respect, could cause two different effects.

In contrast, one and the same effect can have many different causes, in which logical notation would be conveyed as their alternative. Different causes incite motion, different causes underlie the emergence of some kind of experience, and finally, different causes may lead to death. In consequence, an effect of some sort may arise as a result of some particular cause, but it can also arise without it – as a result of some other cause (Mill, 1851, pp. 440–441). The consequence of this is the unreliability of canons of eliminative induction discussed below, because when we restrict the number of output phenomena in the study of the cause, we cannot be sure of the veracity of the conclusions. Noting the regularity of the sequence of phenomena, we cannot ultimately be certain that there is no other configuration of phenomena, which in fact is the cause of the effect we are investigating.

1.6. Canons of induction

Mill's method for detecting causal relationships has five variants, the so-called canons, widely described in the literature.[5] Inference based on Mill's canons is categorised as inference by eliminative induction.[6] Underlying the schemes cited by Mill is the rule of *tollendo ponens*: $A \vee B$, and $\neg A$, thus B. This rule can be interpreted as a rule of keeping non-contradiction, which involves the removal of contradictory statements until consistency is achieved.[7] This is more evident in the case of complex alternative. For example, $A_1 \vee A_2 \vee A_3 \vee A_4 \vee A_5$, and, $\neg A_1 \wedge \neg A_2 \wedge \neg A_3$, therefore $A_4 \vee A_5$.

The principle of one of the canons, the most significant one for the rest of the paper, the so-called canon (method) of agreement, is as follows: *If two or more instances of the phenomenon under investigation have only one circumstance in common, the circumstance which alone all the instances agree, is the cause or effect of the given phenomenon* (Mill, 1851, p. 396). We denote with uppercase letters $A_1, A_2, A_3, \ldots, A_n$ certain observational phenomena preceding other phenomena and, lowercase letters $b_1, b_2, b_3, \ldots, b_n$ – the observational phenomena that occur after them. Watching sets of phenomena, one can search for causes or effects. The method used for this purpose is a method of eliminating phenomena.

Mill's canons are generally known. In this paper, we address the canon of agreement and canon of difference as the most natural starting point for the elimination method in decision logic presented below. Let us assume that we find during observation O_1 that phenomena A_1, A_2, A_3, occur, followed by b_1, b_2, b_3. Then, we find in observation O_2 that phenomena A_1, A_2, A_4 occur, followed by b_2, b_3, b_4, and during observation O_3 – A_2, A_3, A_4 occur, followed by b_2, b_3, b_4. Writing that in columns, we have as follows:

$$\begin{array}{lll} (O_1) & A_1, A_2, A_3 & b_1, \boldsymbol{b}_2, b_3 \\ (O_2) & A_1, A_2, A_4 & b_1, \boldsymbol{b}_2, b_4 \\ (O_3) & A_2, A_3, A_4 & \boldsymbol{b}_2, b_3, b_4 \end{array}$$

The phenomena whose cause we are searching for occur together with other phenomena. However, we are interested in one phenomenon that we explain; it is written as b_2 above as the common element in all observations (thus the symbol b_2 is written in bold).

Canons are used to search for causal relationships. We assume that the phenomenon whose cause we seek is known (b_2), while the phenomenon that constitutes the cause is unknown. We assume hypotheses according to which a single phenomenon A_k is invariably followed by phenomenon b_2, where $k = 1, 2, 3, 4$. In accordance with Mill's methodology, we assume that if after a certain phenomenon another phenomenon always occurs, there is a necessary relationship between them. Invariably consecutive phenomena can be considered to be linked by causal relationship R. Finding the cause of a phenomenon means discovering some phenomenon that invariably precedes it, and eliminating phenomena that do not always precede it. A causal relationship occurs between such phenomena that always co-exist with one another, or between such phenomena that whenever one of them does not occur, the other does not occur either.

Mill's canons can be written in the form of tables, where the number 1 symbolises the occurrence of a phenomenon, while the number 0 – the absence of phenomenon (it has been written this way in light of the second part of this paper):

	A_1	A_2	A_3	A_4	b_2
O_1	1	1	1	0	1
O_2	1	1	0	1	1
O_3	0	1	1	1	1

Referring to the table above, we can say that elimination entails omitting such phenomena A_k, for which there is 0 in any row of the table, that is A_k does not occur, while b_2 occurs. The logical principle of this abbreviated procedure is explained in the footnote.[8]

By using the canon of difference, we establish the occurrence of a cause-and-effect relationship on the basis of the absence of a certain pair of consecutive events, which in other circumstances always occurred successively. Therefore, we add further observations, which, in some cases, b_2 does not occur. Let us assume that further observations took the following form:

(O_4) A_1, A_2, A_3 $\quad$ b_2

(O_5) A_1, A_2, A_4 $\quad$ b_2

(O_6) A_1, A_3, A_4 $\quad$ $(no\ b_2)$

Similarly as in the case of the canon of agreement, we can convey them in a tabular form:

	A_1	A_2	A_3	A_4	b_2
O_1	1	1	1	0	1
O_2	1	1	0	1	1
O_3	1	0	1	1	0

Note that in the table illustrating the canon of difference we allow a symbol of absence of phenomenon b_2 (being explained). This is an important step towards the modern method of elimination described in the next chapter, which involves simplifying decision rules with different successors (for example 0 and 1).

If within a certain isolated system all the possible phenomena preceding a certain examined phenomenon are given, then Mill's eliminative induction is complete and as such is deductive inference. However, only in a handful of cases of inductive inference can all the predecessors be distinguished. Firstly, in practice, we narrow down the number of inference premises, because we cannot indicate all the circumstances preceding the phenomenon whose cause we are attempting to establish. Secondly, according to Mill's concept of causality, one would have to take into account complex phenomena (Mill, 1851, pp. 372–373) – drinking sour milk can trigger one effect, eating pickled cucumbers another – and yet another can be provoked by drinking sour milk and eating pickled cucumbers at the same time (Ajdukiewicz, 1974, p. 167). For these material reasons, eliminative inference is generally non-deductive inference, although its formal schema is deductive in character.

For the purposes of the rest of the article, the following issue should be explained. In the case of Mill's original canons (for example the canon of agreement described above), it is assumed that all such significant combinations of the preceding phenomena have been observed that guarantee a deductive isolation of a single cause A_k (the ones on which we selectively focus attention). But what to do if not all such combinations are given? Such a situation cannot be ruled out, and Mill's original canons do not describe it. In many cases, therefore, using them cannot deductively isolate a single cause (for example A_2). However, if we allow – according to Mill's concept of causality – complex causes, we can isolate such a complex cause. The solution of this problem is provided by decision logic discussed further.

2. Selected elements of decision logic

Decision logic mentioned above (in short: DL) is considered pragmatically in this article, that is, as an effective tool for describing eliminative induction. A comprehensive description of this system can be found both in the original work of Zdzisław Pawlak (Pawlak, 1991), as well as in various secondary studies, for example (Bolc, Cytowski, Stacewicz, 1996). Our goal is not to duplicate the contents contained therein, but to isolate and clarify only those notions that allow us to understand the essence of the new – compared to traditional Millean approaches – the schema of eliminative induction. It is extremely important for the conclusion of the article that this new schema remains consistent with Mill's concept of causality (as described above).

2.1. Language and syntax

The language of decision logic (DL) is used to describe a distinguished set of objects (for example items, people, or events) by a finite number of *attributes* and a finite number of their values. For the purposes of this text, these attributes will be denoted by lowercase letters, that is a, b, c, d, etc., and their values with digits, that is 0, 1, 2, 3, etc. (which complies with the original notation system of Z. Pawlak). At the same time, we shall assume that each attribute has a strictly defined set of allowed values.

The listed symbols allow us to build atomic formulas of the (*attribute, value*) type, which in a specific case means that a certain object has an attribute of such and such value, for example, a given object is red. Using the designations adopted above, atomic formulas are, for example: $(b, 2)$, b_2 in short; and $(c, 0)$, c_0 in short.

Using atomic formulas we create complex formulas, and the rules for their construction and available symbols of logical connectives are the same as in the classical propositional calculus. Here are two simple examples: the formula $(b, 2) \wedge (c, 0)$, or $b_2 \wedge c_0$, is a conjunction of two simple formulas; while the formula $(b, 2) \vee (c, 0) \rightarrow (a, 0)$, or $b_2 \vee c_0 \rightarrow a_0$, is an implication with a complex predecessor and simple successor.

2.2. Extensions of formulas

In actual applications of DL logic, such as undoubtedly eliminative induction, interpreted systems of formulas are considered; that is, such systems in which it is known which described objects satisfy individual formulas (they satisfy them within a specific domain in a relatively isolated system). In other words: we know what values of attributes (or their logical combinations) are assigned to individual objects.[9]

A set of objects that satisfy a given formula z (simple or complex) is called its extension or meaning, and is denoted as $|z|$. For example, if it is found that the objects, x_2, x_3, x_6 (for example specific cars) demonstrate an attribute c with a value of 1 (for example the colour red), the extensions of formula c_1, or $|c_1|$, shall be the set $\{x_2, x_3, x_6\}$.

Extensions of complex formulas are determined in accordance with the conventional meaning of logical connectives, known from first-order predicate logic. For example, if the formulas $z_1 = (a, 0)$ and $z_2 = (c, 1)$ are satisfied, respectively, by objects from the sets $Z_1 = \{x_1, x_4\}$ and $Z_2 = \{x_1, x_2, x_6\}$, then the extension of the conjunction $z_1 \wedge z_2$ is the set $Z_1 \cap Z_2 = \{x_1\}$.

The exact rules for the determination of extension of complex formulas are presented in the table (Z_1 and Z_2 are, respectively, extensions of formulas z_1 and z_2).[10]

negation	$\|(z_1)\| = (Z_1)'$
conjunction	$\|z_1 \wedge z_2\| = Z_1 \cap Z_2$
alternative (disjunction)	$\|z_1 \vee z_2\| = Z_1 \cup Z_2$
implication	$\|z_1 \rightarrow z_2\| = (Z_1)' \cup Z_2$
equivalence	$\|z_1 \leftrightarrow z_2\| = ((Z_1)' \cup Z_2) \cap (Z_1 \cup (Z_2)')$

2.3. Tabular representation of formulas

A particularly transparent method of representing DL logic formulas, as well as their extensions, is the tabular method. Information in the table can be interpreted as the results of observations – observations that involve recording the attributes of individually studied objects. This idea will be presented by a simple example.

Suppose we are given table T.

objects/attributes	a	b	c	d
x_1	1	0	1	0
x_2	2	1	2	1
x_3	1	2	1	1

- Subsequent rows of the table can be interpreted as conjunctive descriptions of subsequent objects from x_1 to x_3; for example, the first row tells us that object x_1 demonstrates attribute a with value 1 (for example colour: red), attribute b with value 0 (for example shape: longitudinal), attribute c with value 1 (for example size: small), and attribute d with value 0 (for example hardness: high).
- As explained above, subsequent rows of a table should be written as conjunctions; for example the first row corresponds to the conjunction $a_1 \wedge b_0 \wedge c_1 \wedge d_0$.
- Subsequent conjunctions are satisfied by subsequent objects; for example, the first conjunction $a_1 \wedge b_0 \wedge c_1 \wedge d_0$ is satisfied by the first object x_1, and the second conjunction $a_2 \wedge b_1 \wedge c_2 \wedge d_1$ is satisfied by object x_2. This issue can be described more narrowly; namely that individual components of a given conjunction (that is simple formulas) are satisfied by an object specified in the row; for example simple formula d_1 is satisfied by objects x_2 and x_3 (for this reason set $\{x_2, x_3\}$ is an extension of d_1).

As the above explanation indicates, propositional descriptions of objects (using DL logic formulas) can be treated interchangeably with their tabular description; particularly important for us is the description of implications presented below.

2.4. Decision rules

In the analysis of the notion of eliminative induction, the most important types of formulas are special kinds of implications, called decision rules. Such rules are used to make decisions about the values of distinguished attributes, the so-called decision attributes, on the basis of the value of other attributes, so-called condition attributes.[11] Therefore, whether an implication belongs to the decision rules set or not depends on the prior division of attributes into condition and decision attributes. For example, if in a set of attributes $\{a, b, c, d\}$ we choose attribute d as the decision attribute (and the rest as condition attributes), then implication $a_1 \wedge b_0 \rightarrow d_0$ is the decision rule, and implication $a_1 \wedge b_0 \rightarrow c_1$ is not.

Among decision rules, we need to further distinguish conjunctive rules – that is, those in whose predecessors there are conjunctions of simple formulas; and disjunctive rules – that is those in whose predecessors there are alternatives (disjunctions) of simple formulas or their conjunctions. For example, rule $a_1 \wedge b_0 \rightarrow d_0$ is a conjunctive rule, and rule $a_0 \vee (a_1 \wedge b_0) \rightarrow d_1$ – a disjunctive one.

In the induction schema presented further, what plays the most important role is the issue of the consistency of conjunction rules. Two rules of this kind are called *inconsistent*, if they have the same predecessor but different successors. Otherwise, they are referred to as *consistent*. And so: rules $a_1 \wedge b_0 \to d_0$ and $a_1 \wedge b_0 \to d_1$ are inconsistent, while rules $a_1 \wedge b_0 \to d_0$ and $a_1 \wedge b_2 \to d_0$ are consistent.

The discussed division applies also to the sets of rules: a set of rules R is defined as *inconsistent* if it contains at least two inconsistent rules; otherwise it is called *consistent*.

Each set of conjunctive decision rules has a clear graphical interpretation as the so-called decision table. For example, a set of three rules $(a_1 \wedge b_0 \wedge c_1 \to d_0)$, $(a_2 \wedge b_1 \wedge c_2 \to d_1)$, and $(a_1 \wedge b_2 \wedge c_1 \to d_1)$ can be summarised in the table below, which, in addition to the rules themselves, contains additional information regarding objects (for example events or observations) for which they are satisfied.

objects/attributes	a	b	c	d
x_1	1	0	1	0
x_2	2	1	2	1
x_3	1	2	1	1

Rows in the table above correspond to subsequent rules, white columns – subsequent attributes of the predecessors of the rules, and the grey column – the attribute that occurs in the successor of all rules (there may be more decision attributes, and thus elements of the successors of the rules). In addition, in the first column, objects that satisfy subsequent rules are indicated.

From the perspective of eliminative induction, decision tables, as well as their respective rules, should be interpreted empirically: each row of the table describes a separate result of observation, during which it has been concluded that such and such values of condition attributes are accompanied by such and such values of decision attributes.

2.5. Cores and reducts of rules

In order to understand the schema of eliminative induction below, it is of utmost importance to know the concepts directly related to the simplification of rules, that is, the concepts of core and reduct. Their meaning will become clear in the course of discussing the example of eliminative induction (see below); however, we shall provide the necessary definitions now.

A set of rules R' is called the *reduct of a set of rules* R (due to the set of condition attributes), which is designated as $RED(R)$, if two conditions are satisfied: (1) there are no inconsistent rules in R'; (2) all attributes in predecessors of rules R' are indispensable (that is the omission of even one of them would result in an inconsistent set of rules, and thus a set comprising of at least two inconsistent rules). Therefore, set $RED(R)$ consists usually of simpler rules than rules from set R – simpler in the sense that they do not contain certain redundant (that is, dispensable) condition attributes.

A set of attributes found in all the predecessors of rules from $RED(R)$ can be called the *reduct of condition attributes* of the set of rules R and marked as $RDC(R)$.[12] A set of indispensable attributes – those attributes that need to occur in every reduct – is called the *core* of set R and is written as $CORE(R)$.

If we refer to the example from the previous point, the analysis of the table shows that: there are two reducts of condition attributes, $RDC_1 = \{a, b\}$ and $RDC_2 = \{b, c\}$; and hence, there are two reducts of the set of rules: $RED_1 = \{(a_1 \wedge b_0 \to d_0), (a_2 \wedge b_1 \to d_1), (a_1 \wedge b_2 \to d_1)\}$ and $RED_2 = \{(b_0 \wedge c_1 \to d_0), (b_1 \wedge c_2 \to d_1), (b_2 \wedge c_1 \to d_1)\}$. In contrast, the core of set R, which is a set of attributes found in all reducts, is $CORE(R) = \{b\}$ (because $\{b\} = \{a, b\} \cap \{b, c\}$).

The concept of reduct and core for individual rules is defined in a similar way as above. Rule r' is called the *reduct of rule* r in a set of rules R, which is marked as $red(r)$,[13] if two conditions are satisfied: (1) rule r' is consistent with other rules in the set R, and (2) all attributes in the predecessor of r' are indispensable (their removal would result in inconsistency with other rules in the set R).

A set of attributes present in the predecessor of rule r can be called the *reduct of condition attributes* of rule r and marked as $rdc(r)$. A set of indispensable attributes in the predecessor of rule r – the attributes that must occur in every reduct – is called the *core of rule* r and marked as $core(r)$.

The description of the final stage of eliminative induction requires one more notion of reduct to be introduced, namely that of a disjunctive rule. Let us remind that a disjunctive rule means any implication $p \to q$, whose predecessor p contains an *alternative* of simple formulas or conjunctions of simple formulas. It can be written as follows: $p_1 \vee p_2 \vee \ldots \vee p_n \to q$ (index i does not denote the value of an attribute in this case, but the subsequent number of the component of the alternative).

In a given set of decision rules R, disjunctive rule r might be too extensive; that is, not all components of alternative p must be indispensable to make a decision on the values of decision attributes. Apart from some of these components, a rule just as "good" can be obtained, albeit a simpler one. This observation leads directly to the concept of reduct of the disjunctive rule.

Rule r': $p' \rightarrow q$ is called the *reduct of the disjunctive rule* r: $p_1 \vee p_2 \vee \ldots \vee p_n \rightarrow q$, if three conditions are satisfied:

(1) p' contains only some of the components $p_1, p_2, \ldots, p_n$

(2) $|p'| = |p| = |q|$ (equality of extension)

(3) p' has the smallest possible number of components from the set $\{p_1, p_2, \ldots, p_n\}$

3. Eliminative induction schema in decision logic

3.1. General description and schema

The concepts introduced above, as well as their clear tabular interpretation, will now be used to describe eliminative induction – the description which has a de facto algorithmic form, which in turn makes it possible for the procedure it describes to be automated.[14]

Suppose that there is a given set of conjunctive decision rules R, which can be interpreted – as we know – using a decision table. Let us remind that such a set is most often created by way of observation: in subsequent cases it has been registered that the respective values of condition attributes (pre-selected, and thus strictly defined) were accompanied by appropriate combinations of decision attributes (also strictly defined).

With these assumptions, the task of eliminative induction (EI) is as follows: to minimise the number of rules (table rows) and their elements (table columns and/or their individual components) – in such a way, however, that will allow us to obtain a consistent set R' of the same decision power as set R.[15]

This task is accomplished in stages: (1) first, we determine the attributes that can be eliminated from all the rules (meaning that some condition columns in the decision table are removed); (2) then, we simplify individual conjunction rules in the pre-reduced set R, eliminating the redundant factors (from predecessors of rules); (3) finally, we group rules with the same successors, use them to create disjunction rules, and simplify their predecessors. As a result, set R is reduced to the maximum, where some rules are disjunctive and other conjunctive.

Represented using the concept of *reduct* introduced above, the description would be as follows: (1) first, we find reducts of the initial set R;

(2) then, having limited ourselves to only one reduct R', we find reducts of individual rules from R'; (3) next, we create disjunction rules with different successors and find their reducts. The results of such a procedure are the final reducts of disjunctive rules.

In a quasi-algorithmic form, the schema of eliminative induction is as follows.

SCHEMA OF ELIMINATIVE INDUCTION

1. **Eliminate dispensable attributes in set *R***
   - 1a. Determine dispensable attributes in R, the core $CORE(R)$ and reducts $RED_i(R)$.
   - 1b. Select one $RED_i(R)$ and proceed to step 2 for that reduct (or R')
2. **Eliminate conjunctive rules components**
   - 2a. For each rule r_i from set R' find core $core(r_i)$.
   - 2b. For each rule r_i from set R' find a set of its reducts $RED(r_i) = \{r_{i1}, r_{i2} \ldots\}$.
   - 2c. Create a set composed of all reducts of all rules (or R'').
3. **Eliminate disjunctive rules components**
   - 3a. For each unique successor q of rules from R'' create a disjunctive rule;
   - 3b. Find reducts of subsequent disjunctive rules.

3.2. Example of elimination

Suppose we are given the following decision table, which represents the set of 7 decision rules (further simplified).[16]

The initial decision table T has the following form:

	a	b	c	d	e
x_1	1	0	0	1	1
x_2	1	0	0	0	1
x_3	0	0	0	0	0
x_4	1	1	0	1	0
x_5	1	1	0	2	2
x_6	2	2	0	2	2
x_7	2	2	2	2	2

Attributes a, b, c, and d are condition attributes, while attribute e is a decision attribute. Each row in the table corresponds to some decision rule; for example, the first row corresponds to rule r_1: $a_1 \wedge b_0 \wedge c_0 \wedge d_1 \rightarrow e_1$.[17]

Below we present the subsequent steps of elimination (according to the schema in 3.1), whose aim is to reduce the number of rules and number of components in successors of rules (while maintaining the original decision power of the whole system).

Step 1 (simplification of the whole set of rules; eliminating attributes/columns)

Dispensable attributes: c.

Indispensable attributes: a, b, d; $CORE(R) = \{a, b, d\}$.

Reduct (there is only one): $RED(R) = \{a, b, d\}$.

The new table T' (after eliminating attribute c)

	a	b	d	e
x_1	1	0	1	1
x_2	1	0	0	1
x_3	0	0	0	0
x_4	1	1	1	0
x_5	1	1	2	2
x_6	2	2	2	2
x_7	2	2	2	2

Step 2 (eliminating components from predecessors of rules)

For individual rules r_i – corresponding to the rows in the table above – dispensable attributes, cores and reducts should be determined. The results are summarised in the table.

Rules	Dispensable attributes	Cores of rules	Reducts
r_1	a, d	$\{b\}$	$\{b, a\}, \{b, d\}$
r_2	b, d	$\{a\}$	$\{a, b\}, \{a, d\}$
r_3	b, d	$\{a\}$	$\{a\}$
r_4	a	$\{b, d\}$	$\{b, d\}$
r_5	a, b	$\{d\}$	$\{d\}$
r_6	a, b, d	$\emptyset$	$\{a\}, \{b\}, \{d\}$

Reducts of individual rules r_i, i.e. r_{ij} (r_{ij} is the j-th reduct of i-th rule), are as follows (x is the lack of attribute in a given rule):[18]

Reducts of rules	a	b	d	e
r_{11}	1	0	×	1
r_{12}	×	0	1	1
r_{21}	1	0	×	1
r_{22}	1	×	0	1
r_{31}	0	×	×	0
r_{41}	×	1	1	0
r_{51}	×	×	2	2
r_{61}	2	×	×	2
r_{62}	×	2	×	2
r_{63}	×	×	2	2

Step 3 (eliminating components of final rules)

In the examined example, the final rule is a rule with successor e_k and predecessor being an alternative of all predecessors of rules r_{ij} with successor e_k.

Simplifying the final rule, that is determining its reduct, entails the removal of unnecessary components of the alternative from its predecessor. In determining the reduct, it is necessary to define the meanings/extensions of formulas f, that is sets $|f|$.

Subsequent final disjunctive rules are as follows:

z_1: $p_1 \to q_1$, i.e. $(a_1 \wedge b_0) \vee (b_0 \wedge d_1) \vee (a_1 \wedge d_0) \to e_1$

z_2: $p_2 \to q_2$, i.e. $a_0 \vee (b_1 \wedge d_1) \to e_0$

z_3: $p_3 \to q_3$, i.e. $a_2 \vee b_2 \vee d_2 \to e_2$

Each of them should be simplified by determining a suitable reduct (see 2.5.). For example, we present subsequent steps of simplifying the rule z_1.

Rule z_1: $p_1 \to q_1$, so $(a_1 \wedge b_0) \vee (b_0 \wedge d_1) \vee (a_1 \wedge d_0) \to e_1$

Successor of the rule and its extension: $q_1 = e_1$, $|q_1| = \{x_1, x_2\}$

Predecessor of the rule and its extension: $p_1 = (a_1 \wedge b_0) \vee (b_0 \wedge d_1) \vee (a_1 \wedge d_0)$, $|p_1| = \{x_1, x_2\}$

Subsequent components of the predecessor and their extensions:

$p_{11} = (a_1 \wedge b_0)$, $|p_{11}| = \{x_1, x_2\}$
$p_{12} = (b_0 \wedge d_1)$, $|p_{12}| = \{x_2\}$
$p_{13} = (a_1 \wedge d_0)$, $|p_{13}| = \{x_2\}$

Dispensable components: $\{p_{12}, p_{13}\}$

Indispensable component: p_{11}

Reduct of the rule: $p_{11} \rightarrow q_1$, i.e. $a_1 \wedge b_0 \rightarrow e_1$

The next two final rules are simplified in much the same way. Finally, after complete elimination, we obtain the following set of three decision rules (rather than the original seven):

$a_1 \wedge b_0 \rightarrow e_1$
$a_0 \vee (b_1 \wedge d_1) \rightarrow e_0$
$d_2 \rightarrow e_2$

3.3. Causal interpretation of the elimination schema

In order to link the presented schema with the problem of detecting causal relationships, which in Mill's terms has been analysed in detail in Chapter 1, one should assume the following *rules of interpretation*:

a) Distinguished attributes and their values are equivalent to Mill's distinguished class of phenomena (it is an invariable set of attributes and their values).
b) Condition attributes are interpreted as potential components of causes being sought (causes that are complex wholes); while decision attributes – as effects.
c) Decision rules (decision table rows) are interpreted as results of observation, from which the actual causes should be extracted – that is, causes consistent with all observations under the distinguished class of phenomena.
d) The result of elimination is a complex cause that is sought (as opposed to simple causes in Mill's canons), described by a complex logical formula, which is usually (though not necessarily so) an alternative. A complex cause consists of different condition attributes with specific values.

The example discussed in the previous chapter concerned a situation in which initially we have taken into account four condition attributes and one decision attribute (which describe a distinguished class of phenomena), made seven observations (seven preliminary decision rules were recorded), and then, using eliminative induction divided into three steps, we isolated two complex causes and one simple cause of three consecutive events (effects).

It should be emphasised that for a given set of observations, there can be more than one *result of elimination* based on the DL logic – depending, for example, what reduct of a set of rules is selected in step 1 of the elimination. It seems that all possible results can be treated as components of the logical alternative. Ultimately, therefore, the broadest convergence of causes consistent with observations is a certain alternative of the results of elimination initiated by the choice of reduct.[19]

In conclusion, let us note that the use of decision logic presented above applies to a set of individual observations, which correspond to individual propositions (implications) and not general ones.[20] From the implications representing observations we eliminate (deductively) redundant components of the predecessors, assuming – according to the causal interpretation of the schema – that they are not causally related to successors. As a result of the elimination, we obtain an individual formula (most often complex), whose predecessor we deem to be the cause of the successor – one that is well defined in the light of observations made. Generalisation of the formula thus obtained (that is, the assertion that it occurs for every possible observation of a given type) is a certain additional activity that goes beyond the presented elimination schema. Such generalisation – being de facto a certain human interpretation of the result – makes the elimination method an unreliable method of inference.

Due to the elimination methodology described above, the language of decision logic need not include quantifiers, and consequently it need not allow general formulas. Still, a natural extension of the presented solution seems to be such a method of elimination that takes into account general formulas (as the object of elimination). In this context, it applies to certain working hypotheses which are generalisations of the results of elimination based on DL logic. In order to choose from among alternatives of such hypotheses one that remains consistent with subsequent (new) observations, one would have to use a different elimination schema based on formalism allowing the use of quantifiers.[21] This, however, immediately raises the question whether a better result (i.e. a better founded hypothesis) could not be obtained by means of decision logic? Would it not suffice to add the above-mentioned subsequent observations to the set of observations made thus far, and then apply the elimination method described herein? Such research questions are worth addressing in another paper.

Summary

In *A System of Logic*, Mill presents the empirical concept of causality, which is a prerequisite for the reasonableness of the inductive method. The rejection of the idea of the existence of causal relationships between phenomena would render the canons of induction pointless.

In Mill's writings, cause cannot be understood as a single phenomenon.[22] We assume that cause is a specific convergence of phenomena. From this point of view, evidence of inferences described by Mill's canons of eliminative induction can be treated as a set of observation propositions that involve convergences of different circumstances.

Since Mill wrote of a convergence of phenomena that make up a cause, and claimed that among all phenomena that occur at some point and phenomena that occur in the next moment, there is an invariable and unconditional order of succession, therefore cause should be sought in a certain class of phenomena, and the cause itself (when being determined) should also be considered as, in the language of Ingarden, a relatively isolated system. Otherwise, it would be insurmountably difficult to isolate a series of links that may or may not have an impact on the occurrence of an effect.

Mill's canons describe an induction method of detecting causes, but Mill's interpretation tends to indicate that they serve to detect individual phenomena, not complexes of phenomena. Since Mill's concept of cause applies to complexes of phenomena, the original canons of induction are incompatible with it. Because of this incompatibility, we believe that inference based on Mill's canons is a partial realisation of the idea of eliminative induction – partial in the sense that using these canons, we have shown only some ways of eliminating attributes not causally related to the examined effect, and we have also isolated only individual attributes as causes.

The schema of elimination developed under decision logic must be considered a much broader solution to the problem of eliminative induction – provided, however, that examined phenomena are described in the language of that logic, based on the language of propositional calculus. However, the suitability of this language (and, more broadly, the entire formalism of decision logic), is proven by the fact that it allows to isolate from a given set of observations complex causes (and not just simple ones), described by complex logical formulas. This in turn is consistent with Mill's original understanding of cause as a complex of phenomena.

NOTES

[1] We realise that the modern developments of decision logic, as well as the associated rough set theory are extremely extensive, and are applied, among others fields, in computer science. In this text, however – due to the objective we have set for ourselves, namely to show that the method of eliminative induction can be consistent with Mill's concept of causality – we only refer to the basic approach to decision logic, presented by Z. Pawlak in: *Rough Sets – Theoretical Aspects on Reasoning about Data*.

[2] Mill sets it apart from efficient cause, which is not necessarily a physical object. The field of efficient causal relationship would be the set of all possible objects – both physical and non-physical (material and non-material). Such a broad field of causal relationship prevents empirical, and therefore intersubjective examination of causation. Mill associates the notion of efficient cause with deductions relating to the ultimate cause of all things (Mill, 1851, pp. 505–506).

[3] This distinction between mereological and distributive has been adopted due to the further portion of the paper, which does not discuss wholes, but sets of events linked by relationships. Mill writes about an aggregate, a whole – and therefore he means a set in the mereological sense (although such a concept, distinguished from the notion of a distributive set, had most likely not been well known). However, such an aggregate can be treated distributively, provided that objects form a structure that is a set of elements linked by a relationship or relationships (for example a set of events connected by the symbol of conjunction). This is how we shall proceed in the subsequent part of the paper. The notion of eliminative induction in Mill's writings will be interpreted using the apparatus of decision logic, which provides a tool for describing relationships between separate events. Given Mill's understanding of the cause as a certain whole, it might be interesting to examine the possibilities of interpreting the concept of cause in the language of mereology, and then link it to decision logic. In this paper, we do not address this issue in any detail in the course of our inquiry, albeit recognising its potential for new research and analyses. Explanation of different meanings of the term "set" may be found in Borkowski (Borkowski, 1970, p. 146). A more detailed discussion of the motivations behind mereology, as well as explanation of formal differences in relation to the distributive sets theory may be found in Leśniewski (Leśniewski, 1992, chapters 2, 3, 4) or Whitehead (Whitehead, 1919).

[4] A system can be called relatively isolated if it is isolated only to a certain extent, temporarily, and in some respects, but not isolated in other respects. A system is isolated in some respects when it is not subject to the effects of the world for some time and in this particular respect, or it does not affect the external world in this respect, and in that time.

[5] Polish authors writing about them include, among others (Ajdukiewicz, 1974, pp. 165–181), (Kotarbiński, 1990, pp. 283–296). A strict formulation of Mill's canons and their formal analysis is found in (Łoś, 1948, pp. 269–297].

[6] Inductive inference is inference that leads from individual premises to a general conclusion. As a result, a general proposition is formulated that A is the cause of B. Finding a causal relationship between two individual phenomena entails the conclusion that there is an invariable succession between these phenomena. Therefore, the sentence stating that phenomenon A is the cause of phenomenon B is a general statement. (Ajdukiewicz, 1974, p. 163, footnote 29). Inferences according to the elimination method, however, are neither deductive, nor reductive inferences (Ajdukiewicz, 1985, vol. 2, p. 222). We cannot indicate all circumstances preceding the phenomenon whose cause we are seeking; also we omit complex phenomenon. Ajdukiewicz gives the example that drinking sour milk can have one effect, eating pickled cucumbers – another, and drinking sour milk and eating a pickled cucumber at the same time – yet another (Ajdukiewicz, 1974, p. 167).

[7] As we will see later, the same rule of preserving consistency is used in the modern method of elimination; however, it applies to preserving consistency of decision logic rules (see 3.2).

[8] In the strict sense, Mill's canons of eliminative induction are based on the rule *tollendo ponens*, which applies here as follows: Let A be a set of preceding events, and b – a set of events that follow them: $A = \{A_1, A_2, A_3, A_4\}$, $b = \{b_2\}$
Cartesian product of sets A and b: $A \times b = \{\langle A_1, b_2\rangle, \langle A_2, b_2\rangle, \langle A_3, b_2\rangle, \langle A_4, b_2\rangle\}$
Our aim is to isolate such a subset of our Cartesian product whose elements (ordered pairs) make up the unknown (sought) causal relationship R. Upon subsequent observations, we find the existence of certain successions, but not between all elements of ordered pairs. Our observations can be written as follows:
$O_1 = \{\langle A_1, b_2\rangle \cup \langle A_2, b_2\rangle \cup \langle A_3, b_2\rangle \cup \langle A_4, b_2\rangle - \langle A_4, b_2\rangle$
$O_2 = \{\langle A_1, b_2\rangle \cup \langle A_2, b_2\rangle \cup \langle A_3, b_2\rangle \cup \langle A_4, b_2\rangle - \langle A_3, b_2\rangle$
$O_3 = \{\langle A_1, b_2\rangle \cup \langle A_2, b_2\rangle \cup \langle A_3, b_2\rangle \cup \langle A_4, b_2\rangle - \langle A_1, b_2\rangle$
On this basis, we formulate a hypothesis (regarding the unknown causal relationship R):
H1: $(\langle A_1, b_2\rangle \in R \vee \langle A_2, b_2\rangle \in R \vee \langle A_3, b_2\rangle \in R \vee \langle A_4, b_2\rangle \in R) \wedge \langle A_4, b_2\rangle \notin R.$
H2: $(\langle A_1, b_2\rangle \in R \vee \langle A_2, b_2\rangle \in R \vee \langle A_3, b_2\rangle \in R \vee \langle A_4, b_2\rangle \in R) \wedge \langle A_3, b_2\rangle \notin R.$
H3: $(\langle A_1, b_2\rangle \in R \vee \langle A_2, b_2\rangle \in R \vee \langle A_3, b_2\rangle \in R \vee \langle A_4, b_2\rangle \in R) \wedge \langle A_1, b_2\rangle \notin R.$
Thus we obtain:
$(\langle A_1, b_2\rangle \in R \vee \langle A_2, b_2\rangle \in R \vee \langle A_3, b_2\rangle \in R \vee \langle A_4, b_2\rangle \in R) \wedge \langle A_4, b_2\rangle \notin R \wedge$
$\langle A_3, b_2\rangle \notin R \wedge \langle A_1, b_2\rangle \notin R.$
P After removing inconsistencies as a result of the application of the *tollendo ponens* rule, we are left with $\langle A_2, b_2\rangle \in R$.
Hence, the conclusion that A_2 is the cause of b_2.

[9] In the context of the schema of induction presented further, objects x_i can be associated with observations, and more specifically with the observed sets of values of attributes of these objects (this is consistent with the intention of Z. Pawlak). And so: object x_i is actually the i-th observation (concerning certain attributes of object x_i).

[10] In the table we use traditional operations on sets: the sum (∪), the product (∩) and the complement (‘).

[11] Here, we need to emphasis the following: decision logic is a formal system and therefore the criteria used for the distinction between conditional and decision attributes are not relevant (for example the criterion of cause and effect, or temporal succession). Some attributes are simply distinguished as conditional attributes, and others as decision attributes, leaving the interpretation of these choices to the user of the system.

[12] Note the distinction between the two reducts: on the one hand, we have a reduct which is a set of rules (denoted as RED), while on the other hand, we have a reduct which is a set of attributes (denoted as RDC).

[13] Lowercase letters in the name $red(r)$ have been used in order to distinguish between reducts of rules and reducts of sets of rules (which are marked as RED). The same principle applies to $core(r)$ defined hereinafter.

[14] Applications of decision logic, also called applications of the rough set theory, are very much present in modern science. See for example (Słowiński, 1992).

[15] It is such a set that with fewer rules and fewer attributes examined allows us to make the same decisions as the original set R.

[16] A similar example is found in the book by Z. Pawlak. See (Pawlak, 1991).

[17] It should be noted that table T contains more information than the rules themselves, because indicated therein are not only rules, but also the objects/observations x_i, to which these rules refer. Such references designate extensions of formulas (that is sets of elements x_i), which extensions are used in the third step of the presented method.

[18] The table above does not include rule r_7, because it is identical with r_6.

[19] To put it pragmatically, the issue of the multiplicity of elimination results can also be approached in a different way, namely by adopting a certain overriding criterion for selecting the ultimate formula (for example "the shorter the better") and choose such components of the alternative that meet it.

[20] This seems to be in agreement with Mill's canons, where the objects of elimination are individual (single) and properly selected observations but not dependencies or general laws. What is general in Mill's method is thus the concept of cause, which is ascribed to a certain concrete phenomenon. Elimination, therefore, does not refer to general dependencies.

[21] A simple elimination rule of this type could be as follows:
$\forall x(A(x) \to E(x) \vee \forall x(B(x) \to E(x))$, but $A(m) \wedge \neg E(x)$, SO $\forall x(B(x) \to E(x)$.

[22] Although there are positions claiming that from a formal point of view, there is no reason not to treat a set of phenomena as a single phenomenon (Szaniawski, 1959, p. 291), (Krajewski, 1967, p. 91).

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