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Witold Marciszewski
University of Bialystok

ON ACCELERATIONS IN SCIENCE DRIVEN BY DARING IDEAS: GOOD MESSAGES FROM FALLIBILISTIC RATIONALISM

> Daring ideas may be beaten, but they may start a winning game.
>
> Johann Wolfgang von Goethe

Abstract. The first good message is to the effect that people possess reason as a source of intellectual insights, not available to the senses, as e.g. axioms of arithmetic. The awareness of this fact is called *rationalism*. Another good message is that reason can daringly quest for and gain new plausible insights. Those, if suitably checked and confirmed, can entail a revision of former results, also in mathematics, and – due to the greater efficiency of new ideas – accelerate science's progress. The awareness that no insight is secured against revision, is called *fallibilism*.

This modern *fallibilistic rationalism* (Peirce, Popper, Gödel, etc. oppose the *fundamentalism* of the classical version (Plato, Descartes etc.), i.e. the belief in the attainability of inviolable truths of reason which would forever constitute the foundations of knowledge. Fallibilistic rationalism is based on the idea that any problem-solving consists in processing information. Its results vary with respect to informativeness and its reverse – certainty. It is up to science to look for highly informative solutions, in spite of their uncertainty, and then to make them more certain through testing against suitable evidence. To account for such cognitive processes, one resorts to the conceptual apparatus of logic, informatics, and cognitive science.

Keywords: a priori, complexity, computability, fallibilism, fallibilistic rationalism, fallibility, fundamentalism, Gödel's speedup strategy, guessing with clues, information processing, informativeness vs certainty, problem-solving, risk.

§1. The main notions of the cognitivist approach employed in the present discussion

In this essay, I adopt the cognitivist approach to reality. In the focus of its ideas there is the notion of *problem-solving with utm-computation*, where

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the prefix hints at computation defined in terms of the Universal Turing Machine. This notion is closely linked with the following ones: *computability, information processing, guessing with clues, complexity, informativeness vs security.*

The opposition in the last item refers to this paper's title. This is to mean that the more innovative an idea, the more it is informative, and the less secure, that is, less certain; a low security is what we call a high risk. Science accelerates when scientists succeed in creating bold, i.e. risky but highly informative conjectures. Their authors should be brave enough to risk failure, and in the next step clever enough to reduce the risk through observational confirmation.

Let us briefly discuss the concepts listed above in their mutual relations. The idea of utm-computation, taken as the focal point, is understood in the cognitivist framework as the strictest concept of computation. Thus it provides us with the most fitting departure point for successive extensions of scope. For instance, analog problem solving is computational as well, but not utm-computational. Thus the feature of being utm-computational constitutes a sufficient condition for any of the broader concepts of computation.

The broadest of them can be used in order to partially define the concept of *information processing.* This is to mean that every computation is a procedure of information processing, but not vice versa. What can belong yet, besides various kinds of computation, to the realm of information processing?[1]

Leaving apart much mental phenomena, like chains of pictures maundering in dreams, etc., let us focus on what is cognitively most significant: to wit, the act of guessing solutions for a problem one is trying to solve, not through random guesses, but intelligent guessing with clues.

This is like the departure point of a skilled detective's chain of reasoning. In the course of investigation, his guesses should be checked against new findings, then suitably modified, in order to ultimately find a correct, well confirmed, solution. Another exemplification can be taken from flashes of intuition as experienced by scientists at the dawning of a discovery. As an example there may serve Gödel's idea of analogy between the liar paradox and the behaviour of the concept of provability. The transition from such a first preapprehension to a final result is implicit (i.e., unconscious) information processing, much different from the utm-computational processes performed on explicitly recorded symbols.

I use the phrase "guessing with clues" as a commonplace counterpart of the concept of *intuition.* The colloquial character of this phrase is more likely to evoke experiences familiar to anybody, not only to those trained in philosophical nuances.

Such guessing is a clear example of information processing which is neither utm-computation nor any other computational proceeding. Thus, both the class of guessings with clues and the class of computations are proper subsets of the vast set of information-processing procedures. The issue of guessing, rather ignored in standard accounts of cognitive science, is crucial in the present discussion on account of its connections with innovativeness and with fallibility (the notions occurring in the title of this essay). This should become obvious soon; here let it just hint at a kinship of guessing with Turing's (1939) oracle.[2]

Informativeness is a gradable property of judgments and theories. The greater the amount of information carried by a text or another vehicle, the higher it is said to have informativeness. It is the least, amounting to zero, in logical tautologies, say $p \iff p$, whose probability degree is the highest, that is, which amounts to 1. And it is the greatest in contradictions, say $\neg(p \iff p)$, whose probability amounts to zero; the feature of the highest informativeness of contradiction gets reflected in the law $(p \wedge \neg p) \rightarrow q$.

The comparison of these extremes reveals their being the reverse of each other. That is to say, high informativeness (as a profit) compensates low probability, i.e. low safety, i.e. a risk (as a cost); whereas high probability (as a profit) compensates low informativeness (as a cost). This is why this relation is conveniently referred to as the *informativeness-safety tradeoff*.

The way toward a highly *informative theory* starts from a highly *innovative idea*. Either feature carries a risk. High informativeness of a theory means an abundance of diverse consequences that ought to be tested empirically; the greater such a wealth of inferences, the more it is likely that some of them turn out fallacious, and this implies the denial of conjectures entailing the error in question. Newton's theory of gravity, for instance, after its having been newly born – that is, before being exposed to observational tests, might have been contradicted by astronomical data, by observations concerning the speedup of falling bodies – observation of sea tides, etc.

After each such exam is successfully passed, the theory in question becomes more and more corroborated. Then corroborations reduce the risk – that is, increase certainty – whereas the high informativeness, measured with the wealth of significant consequences, remains the same.

As for the innovativeness of an idea, the risk it carries consists in its being quite new, that is, not exposed so far to any controlling observations.

After having been so exposed, if it gets corroborated by agreement with observations, then it turns out less risky than it had been before passing tests. At such a new stage it grows more widely accepted, hence more familiar, and thereby less innovative. Here appears a difference between informativeness and innovativeness. While the former, i.e., quantity of information, remains unchanged in the course of a theory's evolution, the latter is relative to the stages of evolution.

The concept of innovativeness helps us to state a basic problem of cognitive science (as well as of AI): can standard computer programs (i.e., those patterned on UTM instructions) can be innovative? In other words: they generate new axioms?

This problem is related to that of compressing the complexity of information as discussed by Gregory Chaitin in his algorithmic information theory. In it, the information content of a bit string is defined as the number of bits in the smallest computer program that will generate that string. A bit string S which can be generated by a program P shorter than S is said to be *compressible*. This amounts to saying that P is less complex than S. This notion of complexity can be applied both to mathematical and to empirical theories; a law of physics from which follows an enormous set of predictions is less complex than the statement being a conjunction of all those predictions. The Copernican theory was more compressed, i.e., less complex than Ptolemy's, and Brahe's and Kepler's theory still more compressed.

The quantity of information output provided by a program cannot be greater than the information quantity of the given input. As Chaitin put it, "if one has ten pounds of axioms and a twenty pound theorem, then that theorem cannot be derived from those axioms".[3] This entails the impossibility of creating a computer program (an algorithm) that would be innovative; that is, discover a new axiom not built into that program. The reason is obvious: computerized information processing consists of operations defined by rules of logic, and these cannot provide a greater amount of information in conclusions than that contained in premises.[4]

The list of notions which form this section's title, with the concept of problem-solving as its departure point, roughly determines the cognitive approach to reality, provided we conceive cognition as being mainly a problem-solving enterprise.

On the other hand, such an approach might be called *informational worldview*. For it is the notion of information that is focal in understanding the world, owing to the definitional links (as listed above) with each other constituent of the cognitive approach.[5]

However we name this approach, it is characterized by the features of being rationalistic, optimistic, and fallibilistic: Rationalistic and optimistic for its trust in the power of human reason as a source of reliable cognition, and fallibilistic for its awareness that, usually, the way towards reliability leads through risking the failure of our guesses. Anyway, such failures should not be seen as an oppressive evil, but be welcome as a means to eliminate errors and closer approximate truth.

§2. Exponentially accelerated progress of science: Can it last for ever?

When considering the progress of knowledge, two domains should be taken into account, each ruled by different kinds of laws of evolution: one concerned with physical factors of development, the other with intellectual factors. This latter are connected with the limitations of algorithms, as mentioned above in §1; to wit, their inability to produce axioms. Hence it is the privilege of human intelligence alone to move forward the frontiers of mathematics with innovative ideas, usually in the form of axioms. Not rarely, axioms happen to be evident to our intellect, but even if not, they have a chance to be justified through their ability to compress information. According to Gödel, such a progress, due to human inventiveness, may stretch out endlessly, as long as mankind will exist.

Physical factors are much more limited in their developmental potential. One sort consists in bounds imposed by such circumstances of experimenting as the range of telescopes, the speed of light, or the impossibility of obtaining the extremely high energies necessary for some observations. Another sort of physical boundary stems from the nature of vehicles of scientific communication; this is the issue to be discussed in the present section.

Let us start from surveying the situation in the time of Newton. His work marks a new era in the history of human cognition. Owing to Newton, the leisurely progress of science which had started in antiquity and continued through two millenia ended; then the evolution of science gained with increasing speed.

Newton and his contemporaries were not fully aware of such a turn. However, from our present perspective this was a border post to mark not only a new vision of the universe but also a new pattern of scientific dynamics. From that moment on, the graph of evolution was to reflect the exponential growth in various aspects of academic activities. Studies of this phenomenon and of its sources have formed an extensive domain since the pioneering publications by Derek J. de Solla Price.

> There is among them a much instructive and thought-provoking book *Science since Babylon* (Yale University Press, 1961). In chapter 5 we find a graph to represent the exponential growth of the number of scientific periodicals since the establishing of *Philosophical Transactions of Royal Society of London* in 1665.

We read in the graph that the number of periodicals, starting ca. 1700 with ten, has grow by ten times every 50 years with the exception of the 18th century when the growth was slower. The numbers are listed in the following columns (first the dates, next the numbers of titles).

1700 — 10^1
1800 — 10^2
1850 — 10^3
1900 — 10^4
1950 — 10^5
2000 — 10^6

The dates are rounded off, but this does not deform the picture.

Similar graphs presenting the exponential growth, though with different rates of increase, reflect the number of researchers, of publications, etc. Among them, the mentioned growth of periodical titles is specially representative as it hints at relations between the factors considered: the greater the number of periodicals, the greater the output of publications and the greater the amount of researchers as their authors.[6]

As for the increasing number of authors, the considering of this processs reveals a dramatic limitation of scientific growth to be expected in the not so distant future. Note that in the most developed countries the population growth rate is constant. This means that, contrary to Malthus' exponential growth model, the quantity of population does not grow. On the other hand, in the same countries the accelerating progress of research is exponential, and this determines the enormous increase in the number of authors. When comparing these figures, we find that in the relatively near future (say, two centuries) the population of scientific authors would equal the whole earth's population.

Such an obvious absurdity leads to a question about the future of science. Should scientific growth stop when it reachs a ceiling? Would this be an upper limit of growth?

To address this question, one should notice a certain sidedness of Price's approach. His notion of scientific development is closed within the category of numerical indexes concerning what can be called *physical vehicles of information*, such as printed publications, library and laboratory spaces, numbers of researchers, etc. The amount of them cannot grow to infinity, and ultimately they have to encounter a ceiling. In such a perspective, science appears helpless in the face of the unending complexity of nature's makeup.

However, there does exist a line of scientific growth which is not bound by any physical limitations. It consists in the increase of the *amount of information* conceived as an abstract entity. Such an entity does not reduce to any physical vehicles, though it needs them for communication purposes.

We obtain a striking example of the difference between the two types of increase in the realm of information, when taking into account the appearance of Newton's *Philosophiae Naturalis Principia Mathematica* in 1687. This happened in the same time interval in which the first scientific periodical, the *Philosophical Transactions* (1665) was established.

Either date opens a new era in the history of science, but in a very different way. Imagine that the theory of gravity was published in one of the first issues of *Philosophical Transactions*. It would appear as one among several articles in that issue. As a printed publication among some other ones, it would add a single item to the picture of quantitative growth, but this would be no significant event; a single article does not make any telling difference in number.

On the other hand, with respect to the informativeness of content, the publication of the theory of gravity means such an unimaginably great increase of information that no numerical estimation would render such an enormous scale of growth. The law of gravity is regarded as the greatest of scientific discoveries of all times, and the paradigm according to which other laws of physics have to be fashioned. It has proved to be in the highest degree explanatory and predictive. Therefore it has given the natural sciences a giant speedup whose scale surpasses the performances of exponential increase.

This fact sheds light on the issue of the rapidly increasing progress of science as raised by the title of the present paper. As for the quantitatively accelerated increase of the vehicles of information in its present form, it cannot be continued forever. As for the accelerated increase of the amount of information due to scientific discoveries, we should appreciate the power of the human mind. A source of such power is alluded to in this essay's title in terms of daring innovative ideas. This is to be inspected closer in what follows.

§3. How the acceleration of science depends on the emergence of new bold ideas

Any new bold idea, which pretends to be highly informative, carries a risk of being erroneous. This property – called *fallibility* – did not enjoy esteem in epistemology from Plato and Aristotle till the turn of the 19th and 20th century. It seemed obvious that fallibility must be counted among the drawbacks of an assertion or a theory. For a long time people did not perceive that it might be a merit. In fact, it is a merit insofar as it stems not from one's stupidity, but from creative curiosity, a desire for new experiences and understandings.

That the realization of this fact came so late may seem nowadays a bit strange. The very meaning of the phrase "likely to be erroneous" should make people think that fallibility is inseparable from inventiveness. Invariably, fallibility must accompany our looking for new understanding to surpass the knowledge existing so far. Hence it is likely to be erroneous, and such uncertainty remains unless the newly obtained information gets checked against some evidence. The philosopher who pioneered the understanding of the positive aspect of *fallibility* and coined a term to express his view, was Charles Sanders Peirce (1839-1914). The name he proposed for his point reads *fallibilism.*

This point is to the effect that one may be wrong in his beliefs, understanding of the world, etc., and yet still be justified in holding such incorrect beliefs as hypotheses to be checked. Without such a procedure, no scientific law would have been established, since each starts its career from being a conjecture which pretends to become a confirmed assertion; some of them succeed, some fail, but without such a casting of candidates, including those doomed to fail, there would be no progress of science.

At the time, Peirce's claim, though shared in the circle of American pragmatists, did not influence the academic mainstream. It became known, discussed, and fairly common among quite a number of philosophers only after Karl Popper's campaign, which started with his *Logik der Forschung*, 1935, against the Vienna Circle's *inductionism.*[7]

Not only philosophers but also practicing scientists through many centuries were not ready to endorse fallibilism. The need for daring conjectures which are so far-reaching that they involve a risk of error, was not more widely acknowledged until the emergence of the scientific revolutions at the turn of the 19th and the 20th century. This was the time in which the first message about quanta given by Max Planck appeared in 1900, and about relativity, by Einstein, in 1905.[8]

In the field of mathematics, the revolutionary turn was taken a bit earlier by Georg Cantor with his theory of infinite sets. This provided mathematics with a new unexpected basis; it complies with the concept of revolution in a science as a turn in its foundations. The next revolution in mathematics was to come with Kurt Gödel's discovery of the undecidability of arithmetic, in 1931.

Nobody in antiquity, in the Middle Ages, and even at the first stages of modernity, imagined such a course of affairs. The schoolmen as well as Renaissance scholars, were convinced that the whole of knowledge had been completed in antiquity as a body of fully reliable assertions. It was lost – they maintained – in the stormy times after the fall of Rome, and it was theirs to reconstruct that precious output as recorded in old manuscripts. After that work was done, it was believed, mankind should be in full possesion of knowledge to be only commented upon and diffused, without any need to add something essentially new.

In the 17th century the authority of the ancients faded, but there remained the same static picture of science. With the difference that not the teachings of Aristotle, but Newton's achievements became deemed as the last and ultimate word of science. This view dominated still in the 19th century. For example, Max Planck's professor advised him to engage rather in another science than physics; in physics, he argued, nothing essentially new could be added. The famous physicist Hermann von Helmholtz (1821–1894), and other great masters of physics by the end of that century, were of the same opinion.

This did not leave any place for failure in scientific cognition. Though individuals may have erroneous views, it was their problem, but the knowledge so far attained and accepted by the whole academic community was regarded immune against doubt. Like in the time of Plato and Aristotle, and in the following twenty centuries, the world of learning identified science with a set of fully reliable results, established forever.

§4. The accelerated progress of science faced with the universe's complexity

In our century's perspective, the frontiers of science appear to be moving forward and forward. Should this last as long as human civilization would exist? Or, does there have to come a point in which science's potential would be discharged?

Such a question has not arisen on the ground of the static conception of knowledge, for instance, that was cherished in the Middle Ages. In the 17th century there appeared two great seers, Pascal and Leibniz, who perceived and admired the infinite complexity of the universe, but their conceptual devices were too thin to translate these visions into a tractable research project. Only the advanced theory of infinity, due to such mathematicians as Dedekind and Cantor, and its applications to computational complexity (Gödel's and Turing's diagonal proofs), as well as discoveries of the complexity of the subatomic world (initiated by Maria Skłodowska-Curie), provided us with conceptual tools to address infinity and complexity.

The problem of limitations in exploring the universe is concerned with space, time, and the structure of matter. At each point there appear specific limitations. Inquiry into the furthest regions of the universe is limited by the finite, though enormous, speed of light. And tracing the history of the universe in the minutest fractions of a second after its having been born encounters difficulties concerned with singularity. We have no cognitive access to singularity since time and space did not exist then; hence no radiation can come from that stage of evolution to the present one.[9]

A controversial issue is that of the complexity of matter. The belief inherited after the ancient atomists that atoms are ultimate and indivisible elements of matter has been abandoned with the discoveries of ever smaller particles of those elements which previously were regarded as indivisible. May such a process of finding ever smaller constituents go on to infinity? Let us consider first the experimental point of view as expressed by an excellent popularizer of science, and next the opinion of a much renowned physicist of to-day.

> Every new accelerator, with its increase in energy and speed, extends science's field of view to tinier particles and briefer time scales, and every extension seems to bring new information. – J. Gleick, *Chaos: Making a New Science*, 1991, p. 115.

> Particles that were thought to be "elementary" twenty years ago are, in fact, made up of smaller particles. May these, as we go to still higher energies, in turn be found to be made from still smaller particles? This is certainly possible, but we have some theoretical reasons for believing that we do have, or are very near to, a knowledge of the ultimate building blocks of nature. – S. W. Hawking, *A Brief History of Time*, 1992, p. 66.

Gleick sounds optimistic about obtaining ever new information with every increase of the power of accelerators. However, we know that the opportunities for having higher and higher energies are limited on technical grounds. Should this halt the progress of our inquiries into the structure of matter? It depends on the fundamental question of whether the divisibility of matter into ever smaller elements is infinite or is finite.

Which alternative is the case? This has been a hot problem since antiquity, and it has got even hotter in our times. There are two camps among physicists and philosophers. The infinitist view is represented, e.g., by the philosopher Nicholas Rescher who, following the physicist David Bohm, postulates the following principle of unending complexity of nature's makeup. *At least as a working hypothesis science assumes the infinity of nature; and this assumption fits the facts much better than any other point of view that we know.*[10]

If there held the principle of unending complexity, then its exploration by experimental physics must stop at a stage at which no higher energies would be technically available. Then, after making a finite number of steps in exploring the structure of the universe, there would outstretch infinitely many strata of ever tinier elements unattainable for human cognition. This would be an impassable limit of scientific progress in that respect.

However, there are firm supporters of finitism who defend the point that the universe is a discrete system and suitably understood as an all-encompassing digital computer.[11]

The first to suggest this approach was Konrad Zuse (1967). Presently Edward Fredkin, a very prolific adherent of this point, provides us with extensive studies into the subject. John Wheeler, Stephen Wolfram, Set Lyod, Hector Zenil and quite a number of other physicists and computer scientists endorse such a cosmological vision that the universe is an enormous digital computer: maybe a quantum computer able to program its own evolution.

Then, if to suppose that our scientific civilization will enjoy the time sufficient to decode such cosmic software, we ought to attain a theory of everything, and this would mean a triumphant end of the history of human inquiries into the universe. As Stephen Hawking put it in the last paragraph of *A Brief History of Time*.

> Then we shall all, philosophers, scientists, and just ordinary people, be able to take part in the discussion of the question of why it is that we and the universe exist. If we find the answer to that, it would be the ultimate triumph of human reason – for then we would know the mind of God.

§5. Fallibilism associated with rationalism: Peirce and other advocates of this attitude

The concept of risk is crucial not only in the theory of games, or economics, but also in epistemology, methodology, philosophy of science, cognitive science. Fallibility, as a definitional feature of scientific theories, means facing a *risk of error*, while reliability means being *secure* against such a risk. Why incur risk instead of enjoying security?

The rationale lies in what is obtained for a given price. The return for a higher risk consists in an increase of *informativeness*. Let this term be an abbreviation for the phrase "a considerable amount of information". In such a context, "information" is not to mean "true message" but any message which tells something new, whether true or false. If I estimate the message as important, and do not see reasons to reject it as false, then the next step should consist in checking my risky guess against some indubitable facts.

When we make a test, then each result, whether recognizing information as true or as false, brings a profit. Thus the final return consists in the joint growth of informativeness and certainty (i.e. safety). At this stage, we are allowed to employ the term "information" in another sense, equivalent to that of the phrase "true message".

However, in what follows I mean "information" in that broader sense which does not exclude being false. This is the standard meaning introduced with Claude Shannon's theory of information. His famous formula states exactly the inverse relation between the amount of such broadly conceived information and the message's degree of probability.

The informativeness/risk-of-failure tradeoff is what provides a conceptual basis to develop the doctrine of fallibilism suggested by Charles Peirce – as discussed above in §3. Independently, similar views were later predicated by other eminent thinkers: Russell, Carnap, von Neurath, Quine, von Neumann, Popper, etc.

Fallibilism claims that any of one's current beliefs, including scientific theorems, might be mistaken for the lack of justification within the knowledge hitherto. The greater such a leap beyond what is known, the greater the risk of mistake, and at the same time the greater the novelty and informativeness of a given assertion, theory, etc. Taking such a risk is an essential prerequisite to the progress of knowledge. This moves forward the frontiers of knowledge through guessing and conjecturing about regions of reality unrecognized heretofore.

This point can be highlighted with the example of mechanized theorem proving. A machine makes random combinatorial guesses as to what possible chains of inferences form a proof providing a conclusion looked for, and eliminates those which do not turn conclusive. On the other hand, a machine's human adviser does not proceed randomly. He is able to give the machine heuristic hints which make the process much shorter, hence more economical. Such hints are due to sophisticated ideas which the machine cannot achieve by itself. This complies with Peirce's view that guesses, when guided by some ideas entertained by a human mind, are more often correct than there might be by random guessing.

Such intelligent guessing – steered both by knowledge formerly acquired and by creative inventive imagination – is what philosophers call intellectual *intuition*. Such non-random guesses derive from implicit information with the help of implicit rules of information processing. They act likewise from behind the scene, without one's being aware of their activity. Or, to use another analogy, like hidden programs which in the idiom of Unix are called "demons".

The process of non-random guessing with clues is nicely exemplified by the quest for new axioms. When a mathematician realizes, as did Andre Wiles who proved Fermat's Theorem, that no set of existing axioms, as known to him, is relevant to the purpose, he looks for theories likely to provide desired premises. After a difficult search Wiles found them in some of the most sophisticated fields of modern mathematics. This enabled him to transfer a problem of arithmetic to the more powerful framework of geometry and analysis (elliptic curves, modular forms).[12]

> A fascinating story about clues to guide Wiles' guess that these theories would supply the quested premises is told by Simon Singh in the book *The Epic Quest to Solve the World's Greatest Mathematical Problem* (Anchor Books, 1997).

The fallibilism which acknowledges the role of knowledge and perceptions due to our reason, and not available with senses – as discussed below in §6 – deserves to be called *rationalism*. This is a new kind of rationalism, much different from the classical kind, archetypally represented in antiquity by Plato and Aristotle, and centuries later by Descartes, Leibniz, and the rest of the 17th century rationalists. That classical version is *fundamentalistic*. That is to say, it holds that there exist ultimate and unshakable intellectual perceptions of reason to become foundations for the entire rest of human knowledge. Cartesian *Cogito*, the Leibnizian principle of sufficient reason, as well as the Euclidean principle – axiom 8 in *Elements*, that any whole is always greater that its proper part – can serve as classical illustrations of such fundamentalism.

> The case of that Euclidean axiom can be convincingly used as an argument against fundamentalism in mathematics, and thereby an argument for fallibilism. Its fallibility gets nicely displayed in the case of infinite sets of numbers where it ceases to hold.

The term “intellectual”, when predicated on reason’s understandings, is to mean that they do not reduce to sense perceptions but are *prior* to them, which is expressed by the Latin philosophical phrase *a priori*. E.g., in order to see with eyes that some configuration of physical objects forms a pair, a triple, etc., one should first possess a notion of number.

Fallibilistic rationalism shares with its classical ancestor the belief in the existence of intellectual perceptions, and their priority with respect to sensory perceptions. However, it does not share the fundamentalist belief in their ultimate infallibility. This is the issue examined in more detail in the next section.

§6. On aprioristic propositions whose fallibility does not make them arbitrary and untenable

A convincing argument for such reformed rationalism is found in a study by Barry Smith, an excellent historian of science, especially of logic, also a historian of philosophy. His contribution bears the title “In Defense of Extreme (Fallibilistic) Apriorism”.[13] The term “apriorism” in the title implies the aforementioned priority of reason, hence it consitutes a basic element of rationalism.

The argument, which culminates in items 3 and 4 below, starts from the following question:

1. Do the empirical theories with the help of which we seek to approximate a good or true picture of reality rest on any non-empirical presuppositions?

   Smith answers “YES”, while “no” is answered by extreme empiricists. If SO, then appears the next question.

2. Are the propositions which express these pre-empirical assumptions in every case analytic (tautological, lacking in content)?

   Smith answers “NO”, while “yes” is answered by logical empiricists. If NO, then appears the next question.

3. Do we have an <u>infallible knowledge</u> of all the synthetic pre-empirical propositions which are presupposed by the various sciences in the different phases of their development?

   Smith answers "NO", while "yes" is answered by extreme Cartesians. If NO, then appears the next question.

4. Could these pre-empirical assumptions, which are presupposed by the empirical sciences, be <u>arbitrary</u>?

   Smith answers "NO", while "yes" is answered by Feyerband. If NO, then appears the next question.

5. The propositions in question must therefore be characterized by a certain plausibility. Is this plausibility always a contextual affair?

   Smith answers "NO", while "yes" is answered by hermeneutic relativists. If NO, then appears the next question.

6. There is therefore something like an intrinsic tenability (plausibility). Are the intrinsically tenable pre-empirical synthetic propositions which play an indispensable role in the sciences given only individually, so that we have only a few isolated examples thereof between which no systematic relations would obtain?

   Smith answers "NO", and then appears the next question.

7. Is it really true that the intrinsically plausible or intelligible preempirical synthetic propositions here at issue are read into or imposed upon the world by us?

   Smith answers "NO", while "yes" is answered by Kantians. If NO, then arises the last question.

8. Might the intrinsically plausible pre-empirical synthetic propositions all be false?

   Smith answers "NO", and this is equivalent to the following conclusion.

9. *Certain pre-empirical synthetic intrinsically plausible propositions require ontological correlates which are their truth-makers. Hence, <u>there are intelligible structures in the world</u>, which we could also call "a priori structures".*

To sum up: the intelligible structures in the world are the truth-makers of judgements which are both a priori and fallible, but less fallible than random guesses.

The above course of reasoning is to the effect that by attaining ever more abstract and sophisticated theories our subjective picture of reality will approximate more and more closely the objective reality. This is exactly what Kurt Gödel thought, at a later stage of his investigations, about mathematical cognition. Following his own results concerning undecidability in mathematics, Gödel believed in the power of mathematical intuition which can guess solutions being beyond algorithmic decidability, but this involves the risk of false guesses.

Gödel cherished some thought-provoking ideas about how to reduce such risk and so increase the certainty of mathematical intuitions. His point can be called "fallibilistic rationalism". This is another name for fallibilistic apriorism, as defended in Smith's paper. The replacement is justified by the fact that from Plato to Descartes, Leibniz, etc. "apriorism" denotes the epistemological part of rationalistic doctrine, which is what Gödel meant and endorsed. This point is discussed in the next section.

§7. How to face complexity and reduce uncertainty? Gödel's speedup strategy

On June 19, 1934, at a seminar run by Karl Menger in Vienna, Kurt Gödel held the paper entitled "Über die Länge von Beweisen" – on the length of proofs, published in 1936. It was one of the most memorable days in the history of logic and of laying the logical foundations for computer science and cognitive science. Before we enter the heart of the matter, some introductory remarks are in order.

The size of a formalized proof is a measure of its complexity and this, in turn, provides a measure of how hard the problem is to be solved with a proof. For instance, Andrew Wiles' famous demonstration of Fermat's Theorem, an extremely hard task, took about 200 pages of manuscript, its presentation to an audience several days, and its working out cost Wiles many years of intense effort.

The more complex a proof or computation, the bigger the costs (of time, memory size, etc.) of its production, the more dangerous the risk of error, and hence the greater uncertainty as to the reliability of the result. In the first version of Wiles' proof, its reviewers found a mistake, and it took close to a year to find a way to circumvent it.

There are two ways of facing proof complexity, greatly differring as to the domain of discourse, one belonging to computer technology, the other one close to mathematics and its philosophy. It is the latter to be discussed in this context. An excellent account of the issue can be found in the paper by Jeremy Avigad and John Harrison "Formally Verified Mathematics".[14]

When discussing sources of uncertainty of mathematical results, the authors give numerous examples of "inferential gaps, misstatements, missing hypotheses, unstated background assumptions, imprecise definitions, misapplied results, and the like."

> E.g., in the text of an eminent author a plus sign got omitted, becoming in effect a multiplication sign. The resulting false formula got accepted as a premise for the ensuing erroneous argument.

Even a process of thorough reviewing does not ensure correctness, since there appear such intricate proofs that very long and effortful review work must be done. E.g., once, a proof was scrutinized over four years by three independent groups of reviewers; such mobilization is no unusual case. An editor reported about the hopeless situation in which "the referees have not been able to certify the correctness of the proof, and will not be able to certify it in the future, because they have run out of energy to devote to the problem." In such a strain, reviewers' inaccuracies are not unlikely, and in fact they turn out to be frequent. This is one of the reasons why there arises in mathematics a vast region of fallibility and uncertainty.

The main problem of Avigad's and Harrison's paper is concerned with those remedies which are offered with automated procedures, when both provers and checkers take advantage of very sophisticated software. But even then, as the authors conclude, and even when using the most powerful computers, there remain cases which are practically intractable for computation, too hard to bring to solution in a reasonable time. Hence there exists a region of intractability not to be conquered even by advanced technology.

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After having reached this conclusion, it is necessary to turn to the strategy devised by Kurt Gödel. The remedy is sought neither in software nor in hardware but in increasing the deductive power of a theory. This strategy is based on the fact demonstrated by Gödel in 1931 (the famous incompleteness theorem) that there must exist unprovable sentences in the first-order arithmetic of natural numbers, but if we suitably enlarge the set of axioms, we obtain a system enjoying a greater deductive power in which some sentences that at the former stage could not be proved become provable.

In the paper of 1936, “Über die Länge von Beweisen”, Gödel assumed a system of arithmetic with axioms containing variables of higher orders. Such reinforcing makes it possible to prove sentences hitherto unprovable. And even more: this new system allows much shorter proofs for many of the previously obtainable sentences. Within this new system it is possible to construct a new undecidable sentence. However, due to introducing a system of still higher order, we obtain new opportunities to prove sentences unprovable in the previous system. And those proofs which were previously available but so perplexingly long that in practice untractable, become enormously shortened by giving a new idea. Such a process can be repeated ad infinitum. To sum up this groundbreaking idea, let me repeat it by translating into English the original statement by Gödel (numbering added by WM).

> Passing to the logic of the next higher order has the effect, not only of (1) making provable certain propositions that were hitherto unprovable, but also of (2) making it possible to shorten, by an extraordinary amount, infinitely many of the proofs already available.

Thus we attain what one may call the *Gödelian speedup* in problem-solving. The term “speedup” renders the acceleration of the processes of proving due to reducing the number of steps. The concept of speedup reveals the relevance of item 2 – in the above framed text – to the issues discussed by Avigad’s and Harrison’s contribution on formally verified mathematics.

Owing to the Gödelian speedup, many setbacks issuing from complexity, having been insurmountable because of the unattainably long time of proof verification, get successfully removed with the dramatic shortening of a proof. That is, owing to the dramatic reduction of complexity.

There were important steps to continue the idea of speedup. The role of a milestone in this chain of contributions belongs to George Boolos’ seminal paper “A Curious Inference” (*Journal of Philosophical Logic* 16, 1987). Boolos considered an arithmetical theorem whose intuitive proof, plausible for the mathematical community, can be put down in a dozen or so lines. Boolos has formalized this proof in the second-order logic, and obtained ca. two pages of print, which may account for no more than ten thousands of symbols. At the same time, Boolos computed that the derivation of this theorem in the first-order logic would require the number of symbols represented by an exponential stack of as many 2's as 64536. It is larger than any integer that might appear in science.[15]

> Instead of going more into details, I refer to the paper of mine "The Gödelian Speedup and Other Strategies to Address Decidability and Tractability Issues" in vol. 22, 2006, of *Studies in Logic, Grammar and Rhetoric*, where the significance of the issue and contributions to it by various authors, including Boolos and experts in automated proving, are extensively discussed.[16]

Thus, going along the way shown by Gödel, we benefit from accelerated progress both in mathematics and in the many sciences for which mathematics is a locomotive. The progress consists in a very significant reduction of the complexity of proofs in a great class of cases which grows ever greater as we employ ever more powerful devices of deduction.

As a rule, to obtain a benefit one should be ready to bear some costs, and the tradeoff is profitable only if costs get surpassed by gains. Is there a price to be paid for so much significant reduction of the complexity of proofs or calculations?

Let us consider the Gödelian case of using higher-order logics. Applying them, we reduce the complexity of proofs, but at the cost of a more controversial ontological commitment, to wit the belief in the existence of sets, then sets of sets, etc. Should we regard such a tradeoff as profitable, or rather dubious with respect to outmatching cost by profit?

This depends on one's philosophical position. For a mathematical Platonist, the cost is near to zero, since the existence of abstract entities is for him like a basic evidence at the bottom of his inquiries. However, a nominalist sees the expense as too great, and resigns from diminishing the complexity size at such a cost. Other philosophers prefer to abstain from taking a stand.

As another example, we can take the axiom of choice. Some important theorems cannot be proved if one does not assume this axiom. However, is it legitimate to use it in order to prove statements involving infinite structures? Mathematical constructivism answers in the negative; other philosophies of mathematics are more permissive.

Even more debatable is the continuum hypothesis. Neither it nor its negation can be derived from the axioms of standard set theory. So, what reasons can we have to accept or to reject it? To accept it as a premise may carry the cost of risking a fallacious proof. This is why some mathematicians do not endorse this conjecture.

These examples demonstrate, contrary to the approach of Descartes, Leibniz, and other fundamentalists, that there are degrees of certainty and clarity in mathematics. The spectrum extends from small integers, most concrete and closest to our sensory experience, up to very abstract axioms of set theory.

A penetrative insight into that difference of degree is characteristic of Gödel's advanced reflection. We learn it from his talks with Hao Wang reported in Wang's book *A Logical Journey: From Gödel to Philosophy* (The MIT Press, 1997). Gödel emphasizes the fallibility of our knowledge, including mathematics, and the primary importance of number theory (i.e., the arithmetic of integers) rather than set theory.

This view of Gödel's is of special significance for discussing fallibility in the case of mathematics. It is to the effect that the degrees of clarity and certainty of different parts of mathematics tend to decrease as we move from concepts close to sense perceptions towards ever higher levels of abstraction; that is, from simple numerical computations to constructive and classical number theory, then to classical analysis, then full set theory. For example, the continuum is not seen as clearly as the physical world and the integers (op. cit., p. 212). Owing to the greater clarity and certainty of natural numbers theory, we can become more convinced about the certainty of set theory, provided that Gödel is right about the following conjecture:

> [A] "If set theory is inconsistent, then elementary number theory is already inconsistent." (p. 216, sec. 7.1.8)

This becomes elucidated with the following comment.

> [B] "Strictly speaking, we only have clear propositions about physically given sets and then only about simple examples of them." (p. 217, sec. 7.1.10).

From such statements there emerges a method for reducing uncertainty in mathematics, which is somewhat like checking hypotheses in natural science. It amounts to having some empirical evidence E, and some assumption A (maybe a hypothetical one) such that A entails logically E. Let A be the set of Peano axioms, and let E be visual perceptions of sets as consisting of physical objects, say a pair of fingers (P), three fingers (T), four (F), etc. One sees with eyes that T is greater than P, etc. This perception of being greater is entailed by the abstract theory of numbers, and so corroborates this theory, i.e., makes it more understandable and more certain. Next, when we succeed in deducing number theory from set theory, then the empirically based certainty of the former augments the certainty of the latter.

This does not mean that the certainty of set theory – owing to such indirect derivation – may match the certainty of the departure point: the perceiving of physically given sets. There arise new uncertainties with each successive step of abstraction. In other words, we are dealing with a decrease of reliability, hence we are dealing with growing fallibility. The increase of the fallibility of a theory, that is, the decrease of its certainty, involves the growth of informativeness, i.e., cognitive content. Set theory is more informative than arithmetic, and the whole of arithmetic more informative that its empirical part concerning small numbers. Thus, there is a tradeoff between informativeness and security, as discussed in previous sections.

This view of the fallibility of mathematics does not oppose the attitude called by Gödel *rationalistic optimism.* Gödel maintained that propositions being uncertain at a stage of evolution are likely to be either denied or affirmed due to creative insights; these would lead to new, more powerful, devices of deduction. For instance, he hoped that, in future, new set-theoretical axioms will be found to decide about the continuum hypothesis.

Hence, there is a perspective of a nonending and accelerating progress of mathematics. An ontological justification of that belief has been proposed by Barry Smith in the essay referred to above in §6. Thus we attain the idea which deserves to be called *fallibilistic rationalism.*

This idea throws light also on the question of whether is it possible to incessantly move forward the frontiers of natural science. In the face of the infinite complexity of the physical universe, as presumed by David Bohm and his followers, there must arise impassable limits of experimental physics. However, the question of the limits of theoretical physics seems to remain open, if we take into account an infinite progress of mathematics to provide physics with ever new devices for modeling physical reality. And this may result in new perspectives for natural science. Will this actually do? To learn the answer, we must wait for the future evolution of natural science based on the accelerating progress of mathematics.

Thus Seneca's famous maxim *errare humanum est* will be ever valid. However, we utter it not in the tone of resignation as our ancestors did, but being aware of its relevance to the accelerated progress of human cognition.

References are inserted into the text since most of them function as links to those pages in the Web which are relevant to the issue being discussed. In this way, the text gets incorporated into a big hypertext of relevant websites, hopefully, with a gain for those readers who take advantage of the electronic format of *Studies in Logic, Grammar and Rhetoric.* For instance, note 1 not only provides the bibliographic description of the item being recommended, but also directly links with the electronic version of that item (the reader should just mark with a mouse, and click, the URL address).

NOTES

[1] I regret that the obvious size limitations of this paper do not allow me to tell about hypercomputation as a fascinating case of information processing which is not utm-computational. To give the taste of the problem, let me refer to Hector Zenil's blog "Anima ex Machina", the post: http://www.mathrix.org/liquid/category/recreation entitled "Hypercomputation in A Computable Universe".

[2] See: Alan Turing, "Systems of Logic Based on Ordinals", *Proc. London Math. Soc.*, ser. 2, 45 (1939).

[3] See: G. J. Chaitin, *Algorithmic Information Theory*, Cambridge University Press, 1990 (2nd ed.), p. 62.

[4] An extensive account of Chatin's theory and its applications to the progress of sciences can be found in the book by Douglas S. Robertson *Phase Change: The Computer Revolution in Science and Mathematics*, Oxford University Press 2003. As for G. J. Chaitin's original texts, for present purposes his *Information, Randomness & Incompleteness: Papers on Algorithmic Information Theory* (World Scientific, Singapore 1990) would be very useful.

[5] This term appears in most recent discussions to take advantage of the explanatory merits of the idea of information with respect to the nature of the universe. See, e.g., Hector Zenil's polemics with Seth Lloyd in the former's blog "Anima ex Machina": http://www.mathrix.org/liquid/archives/tag/quantum-computer.

[6] A recent approach to the exponential growth of information is found in discussions inspired by Ray Kurzweil's bold predictions. See, e.g., the blog discussion entitled "Why so slow" at the page http://sciencehouse.wordpress.com/2008/06/10/why-so-slow/. Also "Big and Small" by R. D. Ekers at http://arxiv.org/pdf/1004.4279.pdf.

[7] Still in the first decades of the 20th century it was projected in the Vienna Circle to establish a logic of induction, able to grant such certainty to the natural sciences, as the logic of deduction does with respect to mathematics.

[8] See http://pl.wikipedia.org/wiki/Max_Planck, and http://en.wikipedia.org/wiki/Special_relativity.

[9] See http://en.wikipedia.org/wiki/Initial_singularity.

[10] Nicholas Rescher, *Satisfying Reason: Studies in the Theory of Knowledge* (Kluwer, Dordrecht 1995). See chapter 3. *Reason and Reality*, section 6. *The Burdens of Complexity*, p. 38.

[11] See the paper by Gordana Dodig-Crnkovic "Significance of Models of Computation, from Turing Model to Natural Computation", *Minds and Machines*, May 2011, volume 21, issue 2, pp. 301-32. Available with Springer if addressed: http://link.springer.com/article/10.1007/s11023-011-9235-1.

[12] Cp. http://www.mathrix.org/liquid/category/recreation – H. Zenil's post: "Meaningful Math Proofs and 'Math is not Calculation'".

[13] Available at https://mises.org/journals/jls/12_1/12_1_9.pdf. Published in: *Journal for Libertarian Studies*, 12(1) (Spring 1996), pp. 179-192. Center for Libertarian Studies.

[14] See http://dl.acm.org/citation.cfm?doid=2580723.2591012. Published in *Communications of the ACM*, April 2014, volume 57, issue 4, pp. 66-75. John Harrison belongs among the most renowned computer scientists in the field of automated theorem proving. Jeremy Avigad is a professor in the departments of philosophy and of mathematics at Carnegie Mellon University.

[15] More on this subject, see chapter 25 in George Boolos' book *Logic, Logic, and Logic*, Harvard University Press 1998. The proof of Gödel's 1936 theorem is given in: Samuel R. Buss, "On Gödel's Theorems on Lengths of Proofs I: Number of Lines and Speedups for Arithmetic", *Journal of Symbolic Logic*, 39, 1994, pp. 737-756.

[16] See http://logika.uwb.edu.pl/studies/index.php?page=search&vol=22, sections 1.1-1.5.