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EXPONENTIAL SEQUENCE IN THE OPERATIONAL CALCULUS MODEL FOR THE n^{TH}-ORDER FORWARD DIFFERENCE

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ABSTRACT

In the paper, there has been determined an exponential element in the discrete model of the non-classical Bittner operational calculus for the n^{th}-order forward difference.

Key words:

operational calculus, derivative, integrals, limit conditions, forward difference, exponential sequence.

Research article

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FOUNDATIONS OF THE BITTNER OPERATIONAL CALCULUS

The *Bittner operational calculus* [1–4] is a system

$$CO(L^0, L^1, S, T_q, s_q, Q), \tag{1}$$

where L^0 and L^1 are linear spaces (over the same scalar field Γ) such that $L^1 \subset L^0$. A linear operation $S : L^1 \longrightarrow L^0$ (denoted as $S \in \mathscr{L}(L^1, L^0)$), called a *derivative*, is a surjection. Moreover, Q is a set of indices q for the operations $T_q \in \mathscr{L}(L^0, L^1)$ and $s_q \in \mathscr{L}(L^1, L^1)$ such that $ST_q f = f, f \in L^0$ and $s_q x = x - T_q S x, x \in L^1$. T_q and s_q are called *integrals* and *limit conditions*, respectively. The kernel of S, i.e. $\operatorname{Ker} S$ is called a set of *constants* for the derivative S. It easy to check that the limit conditions s_q are projections of L^1 onto the subspace $\operatorname{Ker} S$.

The condition $L^1 \subset L^0$ enables iterating of integrals. In order to create derivatives of higher orders, using induction, we determine in turn a sequence of spaces $L^n, n \in \mathbb{N}$[1] in such a way that

$$L^n := \{x \in L^{n-1} : Sx \in L^{n-1}\}.$$

Then

$$\ldots \subset L^n \subset L^{n-1} \subset \ldots \subset L^1 \subset L^0$$

and

$$S^n(L^{m+n}) = L^m,$$

where

$$\mathscr{L}(L^n, L^0) \ni S^n := \underbrace{S \circ S \circ \ldots \circ S}_{n\text{–times}}, \quad m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\},\ n \in \mathbb{N}.$$

Assume that the only solution to an equation

$$Sx = \lambda x, \quad \lambda \in \Gamma \tag{2}$$

with a limit condition

$$s_q x = 0$$

is $x = 0$.

[1] $\mathbb{N}$ denotes a set of natural numbers.

If the equation (2) with a limit condition

$$s_q x = c, \quad c \in \operatorname{Ker} S \tag{3}$$

has a solution for $c \neq 0$, then we call it an *exponential element* (with a λ-exponent) and denote it as $\exp(\lambda, q, c)$.

It is easy to notice that the exponential element is uniquely determined and $\exp(0, q, c) = c$.

If we define the objects (1), then we mean a *representation* or a *model* of the operational calculus.

OPERATIONAL CALCULUS MODELS FOR THE FORWARD DIFFERENCE

Let $\Gamma := \mathbb{C}$ be a field of complexes and $L^0 = L^1 := C(\mathbb{N}_0, \mathbb{C})$ be a linear space of complex sequences $x = \{x(k)\}_{k\in\mathbb{N}_0}$[2] with a usual sequences addition and sequences multiplication by complex numbers.

In [1, 2, 4] Bittner considered a discrete representation of the operational calculus with a derivative understood as a *forward difference* Δ, i.e.

$$S_\Delta x \equiv \Delta x := \{x(k+1) - x(k)\},$$

to which there corresponds one integral

$$T_{\Delta,0} x := \begin{cases} 0 & \text{for} \quad k = 0 \\ \sum_{i=0}^{k-1} x(i) & \text{for} \quad k > 0 \end{cases}, \quad k \in \mathbb{N}_0$$

and one limit condition

$$s_{\Delta,0} x := \{x(0)\}.$$

Later there appeared, officially mentioned in [5], a model with integrals

$$T_{\Delta,k_0} x := \begin{cases} -\sum_{i=k}^{k_0-1} x(i) & \text{for} \quad k < k_0 \\ 0 & \text{for} \quad k = k_0 \\ \sum_{i=k_0}^{k-1} x(i) & \text{for} \quad k > k_0 \end{cases}, \quad k \in \mathbb{N}_0$$

[2] In the operational calculus, we differentiate between a symbol of a function and a symbol of a function value at a point. In particular, $\{x(k)\}$ means a sequence, while $x(k)$ – its value for a given $k \in \mathbb{N}_0$. This denotation is derived from J. Mikusiński [7]. In what follows, provided it does not cause ambiguity, we will skip the { } brackets.

and limit conditions

$$s_{\Delta,k_0}x := \{x(k_0)\},$$

where $k_0 \equiv q \in Q := \mathbb{N}_0$. This model was generalized in [6], where it was proved that to the so-called *forward difference with the base* $b = \{b(k)\}$

$$S_{\Delta_b}x \equiv \Delta_b x := \{x(k+1) - b(k)x(k)\}^3$$

there correspond the below integrals

$$T_{\Delta_b,k_0}x = \{e(k)\}T_{\Delta,k_0}\left\{\frac{x(k)}{e(k+1)}\right\}$$

and limit conditions

$$s_{\Delta_b,k_0}x = \left\{\frac{e(k)}{e(k_0)}\right\}s_{\Delta,k_0}\{x(k)\},$$

where $e(k) := \prod_{i=0}^{k-1} b(i), e(0) := 1.$

Another generalization of the models considered in this paper was done in [9]. It was shown that to the n^{th}*-order forward difference*

$$S_{\Delta_n}x \equiv \Delta_n x := \{x(k+n) - x(k)\}, \tag{4}$$

where n is a given natural number, there correspond integrals

$$T_{\Delta_n,k_0}x := \left\{\frac{1}{n}\sum_{j=0}^{n-1}\Big[\sum_{i=0}^{k-1}\varepsilon_j^{k-i}\,x(i) - \sum_{i=0}^{k_0-1}\varepsilon_j^{k-i}\,x(i)\,\Big]\right\} \tag{5}$$

and limit conditions

$$s_{\Delta_n,k_0}x := \left\{\frac{1}{n}\sum_{j=0}^{n-1}\sum_{i=k_0}^{k_0+n-1}\varepsilon_j^{k-i}\,x(i)\right\}, \tag{6}$$

where

$$\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{n-1}$$

are n^{th} roots of unity, i.e.

$$\varepsilon_j = \cos\frac{2j\pi}{n} + \mathrm{i}\,\sin\frac{2j\pi}{n}, \quad j \in \overline{0, n-1} := \{0, 1, \ldots, n-1\},$$

while ‘i’ denotes the imaginary unit.

---

[3] $\{b(k)\}$ is a sequence such that $b(k) \neq 0$ for each $k \in \mathbb{N}_0$, while $\{b(k)x(k)\}$ means a term-wise (Hadamard) multiplication of b, x sequences.

It was also shown that the operations

$$S_{\Delta_{b,n}}x \equiv \Delta_{b,n}x := \{x(k+n) - bx(k)\}^4 \tag{7}$$

and

$$T_{\Delta_{b,n},k_0}x := \{e(k)\}T_{\Delta_n,k_0}\left\{\frac{x(k)}{e(k+n)}\right\}, \tag{8}$$

$$s_{\Delta_{b,n},k_0}x := \{e(k)\}s_{\Delta_n,k_0}\left\{\frac{x(k)}{e(k)}\right\}, \tag{9}$$

where

$$\{e(k)\} := \{b^{\frac{k}{n}}\} \in \operatorname{Ker} S_{\Delta_{b,n}},$$

satisfy the fundamental operational calculus formulas, i.e.

$$S_{\Delta_{b,n}}T_{\Delta_{b,n},k_0}x = x, \quad T_{\Delta_{b,n},k_0}S_{\Delta_{b,n}}x = x - s_{\Delta_{b,n},k_0}x.$$

It is easy to verify that a solution to the Cauchy problem

$$\begin{matrix} x(k+1) - x(k) = \lambda x(k), \quad \lambda \in \mathbb{C} \setminus \{-1\} \\ x(k_0) = c_{k_0}, \quad c_{k_0} \in \mathbb{C} \setminus \{0\} \end{matrix} \iff \begin{matrix} S_\Delta x = \lambda x \\ s_{\Delta,k_0}x = c \equiv \{c_{k_0}\} \end{matrix} \tag{10}$$

is the sequence $x \equiv \{\exp_\Delta(\lambda, k_0, c)(k)\} = \{(1+\lambda)^{k-k_0}c_{k_0}\}$, which is an exponential element for the forward difference S_Δ (cf. [4]).

A generalization of (10) is an initial value problem

$$x(k+n) - bx(k) = \lambda x(k), \quad \lambda \in \mathbb{C} \setminus \{-b\} \tag{11}$$

$$x(k_0+\ell) = c_{k_0+\ell}, \quad c_{k_0+\ell} \in \mathbb{C}, \ell \in \overline{0, n-1}, \tag{12}$$

where $|c_{k_0}| + |c_{k_0+1}| + \ldots + |c_{k_0+n-1}| > 0$.

The above IVP defines an exponential element in the model with the forward difference (7).

EXPONENTIAL ELEMENT FOR A HIGHER ORDER FORWARD DIFFERENCE

We shall prove that the sequence

$$x = \{x(k)\} = \left\{\frac{1}{n}\sum_{j=0}^{n-1}\sum_{i=k_0}^{k_0+n-1} \varepsilon_j^{k-i}(b+\lambda)^{\frac{k-i}{n}}c_i\right\} \tag{13}$$

is a solution to the equation (11) and that it satisfies the initial conditions (12).

[4] $b \neq 0$ is a given complex number.

Since $\varepsilon_j^{k+n-i} = \varepsilon_j^{k-i}$ for $j \in \overline{0, n-1}$ and $i, k \in \mathbb{N}_0$, hence for each $k \in \mathbb{N}_0$ we obtain

$$x(k+n) = \frac{1}{n} \sum_{j=0}^{n-1} \sum_{i=k_0}^{k_0+n-1} \varepsilon_j^{k-i} (b+\lambda)(b+\lambda)^{\frac{k-i}{n}} c_i = (b+\lambda) x(k),$$

which means that (13) is a solution to (11).

As

$$\varepsilon_0^{k_0+\ell-i} + \varepsilon_1^{k_0+\ell-i} + \cdots + \varepsilon_{n-1}^{k_0+\ell-i} = 0$$

for $i \neq k_0 + \ell$, where $i \in \overline{k_0, k_0+n-1}$ and $\ell \in \overline{0, n-1}$, so

$$\begin{aligned} x(k_0+\ell) &= \frac{1}{n} \sum_{j=0}^{n-1} \sum_{i=k_0}^{k_0+n-1} \varepsilon_j^{k_0+\ell-i} (b+\lambda)^{\frac{k_0+\ell-i}{n}} c_i \\ &= \frac{1}{n} \sum_{j=0}^{n-1} \Bigg(c_{k_0+\ell} + \sum_{\substack{i=k_0 \\ i \neq k_0+\ell}}^{k_0+n-1} \varepsilon_j^{k_0+\ell-i} (b+\lambda)^{\frac{k_0+\ell-i}{n}} c_i \Bigg) \end{aligned}$$

$$= c_{k_0+\ell} + \frac{1}{n} \sum_{\substack{i=k_0 \\ i \neq k_0+\ell}}^{k_0+n-1} (\varepsilon_0^{k_0+\ell-i} + \varepsilon_1^{k_0+\ell-i} + \cdots + \varepsilon_{n-1}^{k_0+\ell-i})(b+\lambda)^{\frac{k_0+\ell-i}{n}} c_i = c_{k_0+\ell},$$

which signifies that (13) satisfies the initial conditions (12).

Hence, from (12), on the basis of (9) and (6), there follows a limit condition

$$s_{\Delta_{b,n}, k_0} x = \left\{ \frac{b^{\frac{k}{n}}}{n} \sum_{j=0}^{n-1} \sum_{i=k_0}^{k_0+n-1} \varepsilon_j^{k-i} b^{\frac{-i}{n}} c_i \right\} =: \{c(k)\} = c \in \operatorname{Ker} S_{\Delta_{b,n}}. \tag{14}$$

From (14), we get in turn the initial conditions (12), because

$$c(k_0+\ell) = \frac{b^{\frac{k_0+\ell}{n}}}{n} \sum_{j=0}^{n-1} \sum_{i=k_0}^{k_0+n-1} \varepsilon_j^{k_0+\ell-i} b^{\frac{-i}{n}} c_i = \frac{b^{\frac{k_0+\ell}{n}}}{n} \sum_{j=0}^{n-1} \Bigg(b^{-\frac{k_0+\ell}{n}} c_{k_0+\ell} + \sum_{\substack{i=k_0 \\ i \neq k_0+\ell}}^{k_0+n-1} \varepsilon_j^{k_0+\ell-i} b^{\frac{-i}{n}} c_i \Bigg)$$

$$= c_{k_0+\ell} + \frac{b^{\frac{k_0+\ell}{n}}}{n} \sum_{\substack{i=k_0 \\ i \neq k_0+\ell}}^{k_0+n-1} (\varepsilon_0^{k_0+\ell-i} + \varepsilon_1^{k_0+\ell-i} + \cdots + \varepsilon_{n-1}^{k_0+\ell-i}) b^{\frac{-i}{n}} c_i = c_{k_0+\ell}.$$

Therefore, the conditions (12),(14) are equivalent.

Thus, we have shown that the sequence $x \equiv \{\exp_{\Delta_{b,n}}(\lambda, k_0, c)(k)\}$ given by (13) is an exponential element in the operational calculus model with the derivative $S_{\Delta_{b,n}}$.

The sequence (13) is determined by the limit condition (14) or the initial conditions (12).

An arbitrary constant $c = \{c(k)\} \in \operatorname{Ker} S_{\Delta_{b,n}}$ shall be called a (b, n)-*periodic sequence*.

So, for a given (b, n)-periodic sequence we have

$$c(k + n) = bc(k), \quad k \in \mathbb{N}_0.$$

If we present the (b, n)-periodic sequence as

$$c(k) = b^{\frac{k}{n}} \tilde{c}(k), \quad k \in \mathbb{N}_0,$$

then $\{\tilde{c}(k)\}$ must be an n-periodic ($(1, n)$-periodic) sequence, i.e. $\{\tilde{c}(k)\} \in \operatorname{Ker} S_{\Delta_n}$.

Therefore, (b, n)-periodic sequences take the form of

$$c = \{c(k)\} = \{b^{\frac{k}{n}} (a_0 \varepsilon_0^k + a_1 \varepsilon_1^k + \cdots + a_{n-1} \varepsilon_{n-1}^k)\}, \tag{15}$$

where $a_0, a_1, \ldots, a_{n-1}$ are arbitrary complexes.

An exponential element $x = \{\exp_{\Delta_{b,n}}(\lambda, k_0, c)(k)\}$, where a constant c is of the form (15), constitutes a solution to the problem (11),(12) with initial conditions

$$x(k_0 + \ell) = c(k_0 + \ell) = b^{\frac{k_0+\ell}{n}} (a_0 \varepsilon_0^{k_0+\ell} + a_1 \varepsilon_1^{k_0+\ell} + \cdots + a_{n-1} \varepsilon_{n-1}^{k_0+\ell}), \quad \ell \in \overline{0, n-1}.$$

An exponential element is a $(b + \lambda, n)$-periodic sequence.

Example 1.

Let us consider an operational calculus model with the difference $\Delta_n \equiv \Delta_{1,n}$, where n is a given natural number. Let us also assume that with $k_0 = 0$, the initial conditions (12) take the form of

$$c_\ell = n, \quad \ell \in \overline{0, n-1}.$$

Then, the limit condition (14) is a constant sequence

$$c = \{n\},$$

which can be identified with the number n.

Hence, on the basis of (13), for $\lambda = 1$ and $n = 1, 2, 3$ we specifically get exponential sequences

$$\{\exp_{\Delta_1}(1, 0, 1)(k)\} = \{2^k\} = \{1, 2, 4, 8, 16, \ldots\},$$

$$\{\exp_{\Delta_2}(1, 0, 2)(k)\} = \left\{2^{\frac{k-1}{2}} \left(1 + \sqrt{2} + (-1)^k(-1 + \sqrt{2})\right)\right\} = \{2, 2, 4, 4, 8, 8, 16, 16, 32, 32, \ldots\},$$

$$
\begin{aligned}
\{\exp_{\Delta_3}(1,0,3)(k)\} &= \Big\{2^{\frac{k-2}{3}}\Big(1+\sqrt[3]{2}+\sqrt[3]{4}+(-1+\sqrt[3]{2})(1+2\sqrt[3]{2})\cos\frac{2k\pi}{3} \\
&\quad + \sqrt{3}(-1+\sqrt[3]{2})\sin\frac{2k\pi}{3}\Big)\Big\} \\
&= \{3,3,3,6,6,6,12,12,12,24,24,24,48,48,48,96,96,96,\ldots\},
\end{aligned}
$$

whose graphs are presented in fig. 1 below.

It is easy to notice that for any $n \in \mathbb{N}_0$ we have

$$\{\exp_{\Delta_n}(1,0,n)(k)\} = \left\{n \cdot 2^{\lfloor k/n \rfloor}\right\},$$

where $\lfloor r \rfloor$ denotes an integer part (floor) of the number $r \in \mathbb{R}$.

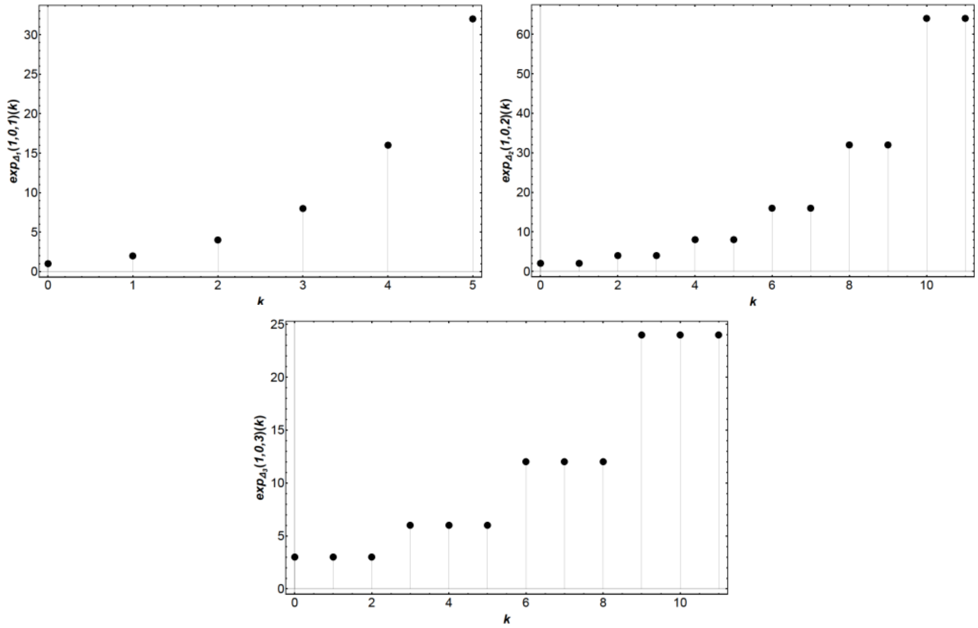

Fig. 1. Graphs of the exponential sequence $\{\exp_{\Delta_n}(1,0,n)(k)\}$ for $n = 1,2,3$

EXPONENTIAL ELEMENT IN A SPACE OF RESULTS

The problem (11),(12) can be presented as

$$S_{\Delta_{b,n}} x = \lambda x \tag{16}$$

$$s_{\Delta_{b,n},k_0} x = c, \tag{17}$$

where the constant $c \in \operatorname{Ker} S_{\Delta_{b,n}}$ has the form of (14).

The problem (16), (17) is equivalent to an integral equation

$$(I - \lambda T_{\Delta_{b,n},k_0})x = c, \tag{18}$$

where I means the identity operation defined on $L^0 = C(\mathbb{N}_0, \mathbb{C})$.

Since for $c = 0$ we get $x = 0$, then the operation $I - \lambda T_{\Delta_{b,n},k_0}$ is an injection. It is easy to check that $T_{\Delta_{b,n},k_0}$ is also an injection.

Let $\pi(L^0)$ be a multiplicative semigroup of injective endomorphisms of L^0, which is generated (for a given $k_0 \in \mathbb{N}_0$) by the operations $T_{\Delta_{b,n},k_0}$ and $I - \lambda T_{\Delta_{b,n},k_0}$ for any $\lambda \in \mathbb{C} \setminus \{-b\}$.

It is obvious that the semigroup $\pi(L^0)$ is commutative.

Let us consider ordered pairs

$$\xi := [x, U], \quad x \in L^0, \; U \in \pi(L^0)$$

and the equality relation

$$(\, [x, U] = [y, V] \,) \overset{\text{def}}{\Longleftrightarrow} (\, Vx = Uy \,), \quad x, y \in L^0, \; U, V \in \pi(L^0),$$

which is of equivalence type.

This relation divides the set of all considered pairs into equivalence classes, which are called *results* [2, 4].

A result is also an arbitrary representative ξ of a given class. For such a representative the fraction symbol

$$\xi = \frac{x}{U}$$

is used.

The set of results $\Xi(L^0, \pi(L^0))$, together with the operations

$$\frac{x}{U} + \frac{y}{V} := \frac{Vx + Uy}{UV}, \quad \gamma\left(\frac{x}{U}\right) := \frac{\gamma x}{U}, \quad x, y \in L^0, \; \gamma \in \Gamma, \; U, V \in \pi(L^0),$$

constitutes a linear space over the field Γ of complexes $\mathbb{C}$.

The sequences of L^0 can be treated as results, since

$$x \longmapsto \frac{Ux}{U}, \; x = \{x(k)\} \in L^0, \; U \in \pi(L^0)$$

is an isomorphism.

It is not difficult to verify that the results of the form

$$\frac{c}{T_{\Delta_{b,n},k_0}}$$

do not belong to L^0 for each $c \in \operatorname{Ker} S_{\Delta_{b,n}} \setminus \{0\}$ [4].

From this it follows that $\Xi(L^0, \pi(L^0)) \setminus L^0$ is a nonempty set.

The elements $\xi \in L^0$ and $\xi \in \Xi(L^0, \pi(L^0)) \setminus L^0$ are called a *regular result* and a *singular result*, respectively [8].

Let R be an endomorphism of L^0 commutative with the operations from the semigroup $\pi(L^0)$. The operation

$$\rho\left(\frac{x}{V}\right) := \frac{Rx}{UV}, \quad x \in L^0,\ U, V \in \pi(L^0)$$

is called an *operator* and denoted as $\rho \equiv \frac{R}{U}$ [2,4].

Thus, an operator is the endomorphism of the results' space $\Xi(L^0, \pi(L^0))$. The operator $\rho_0 := \frac{UR}{U}$, where $U \in \pi(L^0)$, is identified with the endomorphism R.

An operator given by the formula

$$p_{k_0} \equiv P_{\Delta_{b,n},k_0} := \frac{I}{T_{\Delta_{b,n},k_0}}$$

is called the *Heaviside operator* [4].

From (18) we obtain a form of an exponential element as a result

$$x = \frac{c}{I - \lambda T_{\Delta_{b,n},k_0}}.$$

It is a regular result, which can also be presented as

$$\{\exp_{\Delta_{b,n}}(\lambda, k_0, c)(k)\} = \frac{p_{k_0}}{p_{k_0} - \lambda I}\{c(k)\}. \tag{19}$$

Example 2.

Using *Mathematica*®, we shall solve the IVP

$$x(k+6) + 6x(k+3) - 16x(k) = 0, \quad k \in \mathbb{N}_0 \tag{20}$$

$$x(0) = 1,\ x(1) = 0,\ x(2) = 0,\ x(3) = 1,\ x(4) = 0, x(5) = -1. \tag{21}$$

A. Applying the following code

```
sol=RSolve[{y[k+6]+6y[k+3]-16y[k]==0,y[0]==1,y[1]==0,y[2]==0,
y[3]==1,y[4]==0,y[5]==-1},y[k],k];
x[k_]:=y[k]/.Flatten[sol];
FullSimplify[ComplexExpand[x[k]]]
Table[FullSimplify[x[k]],{k,0,20}]
```

we get a solution to the considered problem:

$$x(k) = \frac{1}{15}\cdot 2^{(k-11)/3}\Big[4\Big(9\cdot 2^{2/3} - 1 + \sqrt{3}\sin\frac{2k\pi}{3} + (1 + 18\cdot 2^{2/3})\cos\frac{2k\pi}{3}\Big)$$
$$+2^{2(k+1)/3}\Big(5\cdot(-1)^k + \sqrt{3}\sin\frac{k\pi}{3} + 7\cos\frac{k\pi}{3}\Big)\Big], \quad k \in \mathbb{N}_0 \tag{22}$$

as well as its consecutive terms for $k \in \overline{0, 20}$:

$$1, 0, 0, 1, 0, -1, 10, 0, 6, -44, 0, -52, 424, 0, 408, -3248, 0, -3280, 26272, 0, 26208.$$

B. We shall now determine a solution to the problem (20),(21) in a respective results' space using *Mathematica*® for all auxiliary calculations.

If we present the equation (20) as

$$(x(k+6) - 2x(k+3) + x(k)) + 8(x(k+3) - x(k)) - 9x(k) = 0, \quad k \in \mathbb{N}_0,$$

we can solve the problem in a model with the derivative

$$S\,x \equiv S_{\Delta_3}x = \{x(k+3) - x(k)\}.$$

Then, instead of (20), we get

$$S^2x + 8S\,x - 9x = 0, \tag{23}$$

where $x = \{x(k)\}$.

To the initial conditions (21), on the basis of (6) and with $k_0 = 0$, there correspond in turn the limit conditions

$$s_0x \equiv c = \left\{\frac{1}{3}(\varepsilon_0^k + \varepsilon_1^k + \varepsilon_2^k)\right\} = \{1, 0, 0, 1, 0, 0, \ldots\}$$
$$= \begin{cases} 1 & \text{for} \quad k = 3m \\ 0 & \text{for} \quad k \neq 3m \end{cases}, m \in \mathbb{N}_0, \tag{24}$$

$$s_0S\,x \equiv d = \left\{-\frac{1}{3}(\varepsilon_0^{k-2} + \varepsilon_1^{k-2} + \varepsilon_2^{k-2})\right\} = \{0, 0, -1, 0, 0, -1, \ldots\}$$
$$= \begin{cases} -1 & \text{for} \quad k = 3m+2 \\ 0 & \text{for} \quad k \neq 3m+2 \end{cases}, m \in \mathbb{N}_0, \tag{25}$$

where s_0 means $s_{\Delta_3,0}$ and

$$\varepsilon_0 = 1, \varepsilon_1 = -\frac{1}{2} + \frac{\mathrm{i}\sqrt{3}}{2}, \varepsilon_2 = -\frac{1}{2} - \frac{\mathrm{i}\sqrt{3}}{2}.$$

If T_0 means the integral (5) for $k_0 = 0$, then the problem (23)-(25) is equivalent to the equation

$$x + 8T_0x - 9T_0^2x = c + 8T_0c + T_0d,$$

that is

$$(I + 9T_0)(I - T_0)x = (I + 8T_0)c + T_0d. \qquad (26)$$

Let the results' space $\Xi(L^0, \pi(L^0))$ be determined by the space $L^0 = C(\mathbb{N}_0, \mathbb{C})$ and the semigroup $\pi(L^0)$ containing the operations

$$T_0, I + 9T_0, I - T_0.$$

In $\Xi(L^0, \pi(L^0))$ a solution to the equation (26) is the result

$$x = \frac{(I + 8T_0)c + T_0d}{(I + 9T_0)(I - T_0)},$$

that is

$$x = \frac{1}{10}\left(\frac{p_0}{p_0 + 9I}(c - d) + \frac{p_0}{p_0 - I}(9c + d)\right).$$

It is a regular result, because on the basis of (19) we obtain

$$x(k) = \frac{1}{10}\Big(\exp_{\Delta_3}(-9, 0, c - d)(k) + \exp_{\Delta_3}(1, 0, 9c + d)(k)\Big), \quad k \in \mathbb{N}_0.$$

Using the form (13) of the exponential element, we also have

$$\begin{aligned} x(k) = \frac{1}{120}\Big[&36 \cdot 2^{k/3}\left(1 + 2\cos\frac{2k\pi}{3}\right) + 2 \cdot 2^{(k+1)/3}\left(-1 + 2\sin\frac{(4k + 1)\pi}{6}\right) \\ &+(-2)^k\left(5 + 8\cos\frac{2k\pi}{3} - 2\sin\frac{(4k + 1)\pi}{6}\right)\Big], \quad k \in \mathbb{N}_0. \end{aligned} \qquad (27)$$

In *Mathematica*®, we can easily verify that the formulas (22) and (27) present the same solution to the problem (20),(21). Namely, after running the code

```
x1=…;
x2=…;
FullSimplify[x1==x2, Assumptions->Element[k,Integers] && k>=0]
```

where '…' mean the right sides of (22) and (27), respectively, we obtain the logical value True.

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CIĄG WYKŁADNICZY W MODELU RACHUNKU OPERATORÓW DLA RÓŻNICY PROGRESYWNEJ RZĘDU *n*

STRESZCZENIE

W artykule wyznaczono element wykładniczy w dyskretnym modelu nieklasycznego rachunku operatorów Bittnera dla różnicy progresywnej rzędu *n*.

Słowa kluczowe:
rachunek operatorów, pochodna, pierwotne warunki graniczne, różnica progresywna, ciąg wykładniczy.

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