

# THE EVALUATION OF THE INITIAL SHEAR MODULUS OF SELECTED COHESIVE SOILS

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**Abstract:** The paper concerns the evaluation of the initial stiffness of selected cohesive soils based on laboratory tests. The research materials used in this study were clayey soils taken from the area of the road embankment No. WD-18, on the 464th km of the S2 express-way, Konotopa-Airport route, Warsaw. The initial stiffness is represented here by the shear modulus ( $G_{\max}$ ) determined during resonant column tests. In the article, a number of literature empirical formulas for defining initial value of the shear modulus of soils being examined were adopted from the literature in order to analyze the data set. However, a large discrepancy between laboratory test results and the values of  $G_{\max}$  calculated from empirical relationships resulted in the rejection of these proposals. They are inaccurate and do not allow for an exact evaluation of soil stiffness for selected cohesive soils. Hence, the authors proposed their own empirical formula that enables the evaluation of the test soils'  $G_{\max}$  in an easy and uncomplicated way. This unique formula describes mathematically the effect of certain soil parameters, namely mean effective stress ( $p'$ ) and void ratio ( $e$ ), on the initial soil stiffness.

*Key words:* initial soil stiffness, cohesive soils, resonant column tests

## 1. INTRODUCTION

There are many kinds of ground vibrations existing in nature, caused by earthquakes, traffic loads, water waves, storms, machinery, wind power, construction operations, and so forth. In practical geotechnical engineering many problems are caused by the response of subsoil subjected to these vibrations. The shear modulus and damping properties are required for analysis and understanding the response of subsoil subjected to dynamic loads.

In the past few decades, many researchers and civil engineers have been concerned in practice with the prediction of settlements or transforms of earth constructions under cyclic loading in drained condition in the case of saturated and unsaturated soils. Several of the recently developed high speed transport systems, e.g., express-ways, high speed rail ways, airstrips, transmit dynamic loading to subsoil. Their serviceability is extremely sensitive to the dif-

ferential settlements. Other examples of situations, in which structures may be subjected to steady-state vibration, are construction and operation of factories equipped with precision machinery, wind power plants, bulwarks as well as pile penetration. They may induce a magnitude of vibration exceeding elastic range.

The settlements of subsoil in these cases are strongly related to the stiffness of soil. Understanding well the dynamic shear modulus of soil subjected to a long term dynamic loading is of great importance in order to know better the work capacity of soil.

Initially, Burland [1], later also Kriegel and Weisner [8], found that subsoil deformations under operational loads (in the range of 150–200 kPa) fall entirely into the range of small strains (from  $10^{-5}$  to  $10^{-3}$ ). In the range of moderate strains ( $5 \cdot 10^{-3}$  to  $5 \cdot 10^{-2}$ ) the values of the deformation modulus are subjected to relatively small changes and, at the same time, can be even dozens of times smaller than those obtained at

small strains ( $<10^{-5}$ ). Thereby, the settlements predicted without taking small strains into consideration can be drastically overestimated.

Small strain stiffness has become very important notion of soil mechanics recently. It refers to characteristic phenomenon for particular media relating to an abrupt drop in soil stiffness, which is initially high, when soil deformation increases in a range of small values [4]. Since the early 1980's, many laboratory and field tests have been performed world-wide to study soil behaviour when subjected to small strains and describe it mathematically. At the same time, the relevant research work has been carried out on various models and formulas defining overconsolidated soil behaviour under small strains.

Therefore, this paper is an attempt to determine the initial characteristics of stiffness of the selected cohesive soils based on the laboratory tests. The goal of the measurements was the evaluation of the dynamic shear modulus using resonant column apparatus and creation of the new statistical function that allows obtaining the value of the initial shear modulus ( $G_{\max}$  or  $G_0$ ) in an easy and uncomplicated way. Independently of that, the verification of some empirical formulas, selected from the literature, for determining the initial soil stiffness was performed. Having analysed the results, the authors ascertained the necessity of developing a new equation which will fit well the soils being tested.

## 2. THE SMALL-STRAIN SHEAR MODULUS

Hardin and Drnevich [6] argued that the critical parameter for many dynamic soil properties is the shear modulus ( $G$ ). To understand properly the nature of dynamic soil properties, the influence of many factors should be thoroughly investigated. A comprehensive general stress-strain relation for soil is extremely complicated because of the large number of parameters that affect the behaviour of soils [6]. Shear modulus is affected by various factors such as strain amplitude, confining pressure, void ratio, overconsolidation ratio, loading frequency, temperature, anisotropic stress, and so forth.

The degradation of the shear modulus under strain has been observed in soil dynamics since the 1970's. The dependence of the secant shear modulus on strain amplitude was illustrated for dynamic loading by a number of researchers using the resonant column test or improved triaxial tests [6], [7],

[12]. Nowadays, non-linear soil behaviour is a widely known and well-understood concept. In geotechnical practice, decision-making is usually based on simple calculations using a few easily accessible parameters from routine tests. A large amount of effort has been put to define the small-strain shear modulus and its reduction under strain. Only a few studies will be mentioned in this paper. Following the development of the resonant column test, Hardin and Black [5] demonstrated in 1966 the influence of void ratio ( $e$ ) and mean effective stress ( $p'$ ) on the maximum (elastic) shear modulus ( $G_0$ ), through an empirical equation of the form

$$G_0 = A \cdot F(e) \cdot (p')^m \quad (1)$$

where  $F(e)$  is a function of void ratio, and  $A$  and  $m$  are material constants. Hardin and Black proposed the following equation:  $F(e) = (e_g - e)^2 / (1 + e)$ , where different values of  $e_g$ ,  $A$  and  $m$  were suggested for sands of different angularity.

The small-strain stiffness of soils may be alternatively determined by measuring the velocity of shear waves through a triaxial sample using bender elements method [12]. Viggiani and Atkinson [13] proposed the following equation for the calculation of the small-strain shear modulus ( $G_0$ ), based on data from reconstituted samples of speswhite kaolin

$$\frac{G_0}{p_r} = A \cdot \left( \frac{p}{p_r} \right)^N \cdot (OCR)^M \quad (2)$$

where  $OCR$  is overconsolidation ratio and  $p_r$  is a reference pressure to make equation (2) dimensionally consistent ( $p_r$ , which influences the value of  $A$  normally taken to be 1 kPa or equal to atmospheric pressure). The values of  $N$  and  $M$  for kaolin were found in reference [13] and amounted to 0.653 and 0.196 respectively, giving an overall pressure dependence of  $p^{0.46}$ .

Many empirical formulas for the calculation of  $G_{\max}$  have been proposed in the literature. The authors of the article chosen few empirical functions on the basis of literature review in order to check if they fit the deformation parameters obtained from laboratory tests performed on cohesive soils under study. Selected formulas are presented in Table 1, where  $p' = \sigma'_m = \sigma'_0$  means effective stress,  $P_a$  is atmospheric pressure, equal to 98 kPa, and  $k$  is a parameter dependent on plasticity index as follows: PI = 1, 20, 40, 60, 80, 100 and then respectively  $k = 0, 0.18, 0.31, 0.41, 0.48, 0.5$ .

### 3. MATERIALS AND METHODS

Soil for laboratory tests, sampled in undisturbed state (standard tube samples), was collected from the region of the express-way S2 (around the road embankment No. WD-18) in the area of Warsaw, the capital of Poland. The investigated material belongs to a natural cohesive soil formation and is of Quaternary origin. Laboratory tests indicated that all the soil sampled can be classified as clayey sands, sandy clays and sandy silty clays [10] with low plasticity index, high bulk density, low porosity and high content of sand fraction. The index properties of test specimens are summarised in Table 2.

All tests were carried out in the GDS Resonant Column Apparatus (RCA) [3] at the Water-Centre Laboratory of the Warsaw University of Life Sciences. It is an example of Hardin–Drnevich resonant column using “fixed-free” configuration. This equipment is commonly used to study the dynamic deformation characteristics of soil. It is shown in Fig. 1 and its full description can be found in references [2], [11].

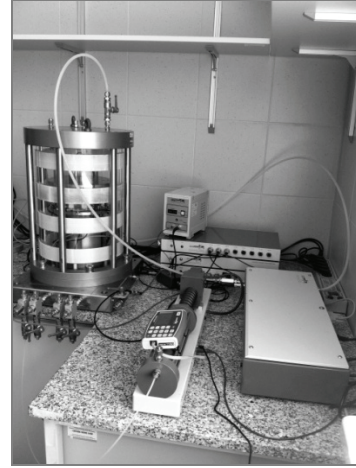


Fig. 1. Resonant Column Apparatus used for the tests

To keep the consistency of all the tests, they were performed under the following conditions: undisturbed cylindrical specimens with 70 mm in diameter and 140 mm in height were set up in the RCA cell, then saturated using back pressure method [9] in order to achieve full saturation, and afterwards consolidated.

Table 1. Selected empirical formulas for calculating  $G_{\max}$  on the basis of resonant column tests [2]

Form of the function	Authors	Kind of soil
$G_{\max} = \frac{625 \cdot OCR^k}{0.3 + 0.7e^2} \cdot \sqrt{Pa \cdot p'}$ (3)	Hardin (1978)	Overconsolidated cohesive soils
$G_{\max} = \frac{3270 \cdot (2.973 - e)^2}{1 + e} \cdot (\sigma'_m)^{0.5}$ (4)	Hardin & Black (1968)	Normally consolidated cohesive soils
$G_{\max} = \frac{1222 \cdot (2.97 - e)^2}{1 + e} \cdot \sqrt{Pa \cdot \sigma'_0}$ (5)	Hardin & Drnevich (1972)	Cohesive soils
$G_{\max} = 445 \cdot \frac{(4.4 - e)^2}{1 + e} \cdot (\sigma'_0)^{0.5}$ (6)	Marcuson & Wahls (1978)	Cohesive soils – clayey soils
$G_{\max} = 90 \cdot \frac{(7.32 - e)^2}{1 + e} \cdot (\sigma'_0)^{0.6}$ (7)	Kokusho et al. (1982)	Cohesive soils – clayey soils

Table 2. Index properties of tested soils

Sample name	Parameters						
	$\rho$ (kg/m <sup>3</sup> )	w (%)	w <sub>L</sub> (%)	w <sub>p</sub> (%)	I <sub>p</sub> (%)	e <sub>0</sub> (–)	p' (kPa)
RC-1	1.93	14.00	33.51	14.50	19.01	0.3800	45–100
RC-2	2.08	12.01	37.62	13.91	23.71	0.3100	10–30
RC-3	2.20	10.18	30.82	11.89	18.93	0.3267	45–180
RC-4	2.23	12.82	31.20	12.62	18.58	0.3137	45–315
RC-5	2.20	12.19	31.00	11.95	19.05	0.3573	135–225
RC-6	2.12	13.31	31.50	12.72	18.78	0.4332	225–315
RC-7	2.17	12.23	37.00	11.45	25.55	0.3855	45–315
RC-8	2.16	15.57	41.70	14.26	27.49	0.4346	45–315
RC-9	2.12	14.43	36.80	12.72	18.78	0.4475	45–315
RC-10	2.16	14.20	35.21	11.43	23.78	0.4191	45–315

The first effective consolidation pressure value was the same as the in-situ vertical effective stress ( $\sigma'_{v0}$ ) value. All samples were consolidated in stages under various mean effective stress ( $p'$ ) values. They are specified in Table 2. At the end of each consolidation stage, dynamic tests (resonant and damping) were conducted. In the present work, only resonant tests are discussed.

#### 4. TEST RESULTS AND DISCUSSION

##### *Effect of mean effective stress and void ratio*

The effects of effective stress on the small-strain shear modulus were studied extensively by the investigators in the past few decades. Confining pressure (or mean principal effective stress) is admittedly assumed as one of the two very important factors which significantly influence maximum shear modulus of sandy and clayey soils. Beside effective pressure, void ratio is recognized as the second meaningful parameter affecting soil stiffness.

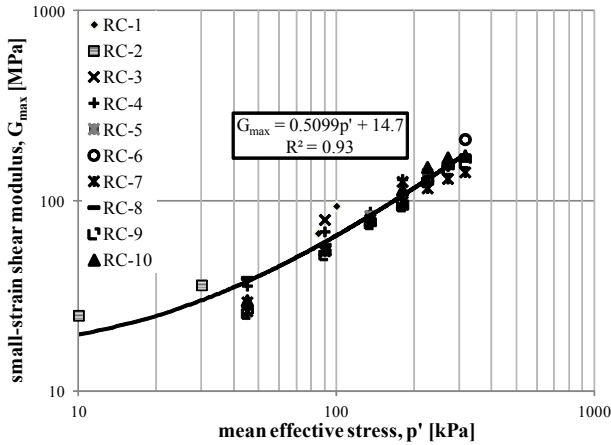


Fig. 2. The small-strain shear modulus from RC tests as a function of mean effective stress

In Fig. 2, the relationship between the calculated shear modulus and mean effective stress for tested cohesive soils is illustrated. Results shown in this graph confirm the positive impact of stress on the initial shear modulus. There is quite a small scatter of the test results. In all the cases analysed the values of  $G_{max}$  increase with the average effective stress. Many investigators of this topic have confirmed in their papers the increase of the small-strain shear modulus with pressure growth [1], [4]. This conclusion is true also in the light of the authors' results. Presentation of these two variables ( $G_{max}$  and  $p'$ ) on a logarithmic

scale allowed subsequently a common function describing all data to be found. The test results are well described by an increasing linear function  $G_{max} = 0.5099p' + 14.7$  (Fig. 2), showing a high coefficient of determination ( $R^2 = 0.93$ ). This equation explains at least 93% of the variation of  $G_{max}$ . This function matches the data set very closely.

In Fig. 3, the relationship between the initial shear modulus and void ratio of the samples under study is shown. Presented results clearly indicate that the small-strain shear modulus increases when void ratio decreases for most of the specimens, which is in agreement with the previous results in literature [1], [4].

The results also suggest that the absolute rate between the shear modulus and void ratio ( $dG/de$ ) increases with confining pressure. This is seen in the slopes of the different data sets shown in Fig. 4. Thus, it is confirmed that both factors ( $p'$  and  $e$ ) have equally strong influence on the maximum shear modulus. It is rather difficult to identify the individual contributions of  $p'$  and  $e$  to  $G_{max}$ .

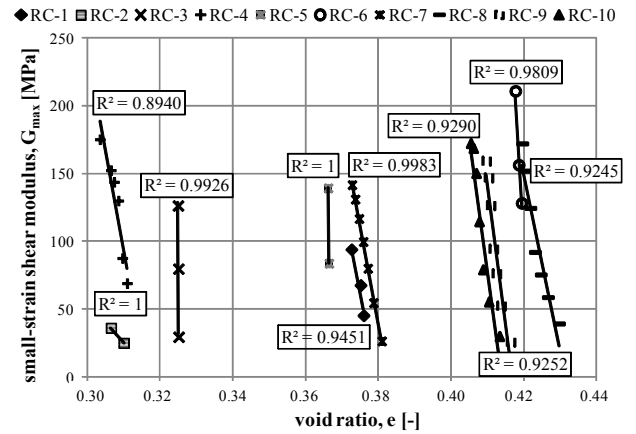


Fig. 3. The variability of the maximum shear modulus depending on void ratio from RC tests

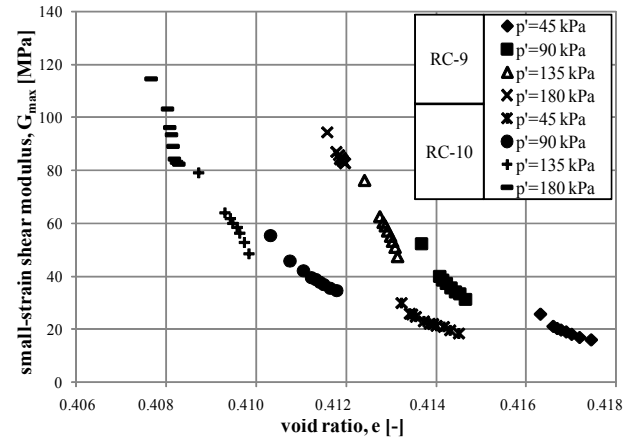


Fig. 4. The maximum shear modulus versus void ratio of the soil samples for various stress conditions

*Comparison of the selected empirical formulas to estimate the initial soil stiffness*

In Section 2 of this paper, a number of standard equations are listed, which have been adopted here to analyse the data from the tests. These empirical expressions have been proposed only for cohesive soils, both normally consolidated and overconsolidated. In Table 3, a summary is provided of the initial shear modulus values obtained from RC tests,

designated as  $G_{\text{estimated}}$ , and those calculated on the basis of formulas presented in Table 1, here designated as  $G_{\text{calculated}}$ . The explanatory parameters in the analysis are as follows: mean effective stress (symbol  $p'$ ,  $\sigma'_m$ ,  $\sigma'_0$ ), void ratio ( $e$ ) and overconsolidation ratio (OCR). Although not shown in Table 3, all the values of the mean effective stress used for the laboratory tests were taken into account during the calculations. The values of void ratio shown in Table 3 ( $e_{\text{avg}}$ ) are an average over all the values, which char-

Table 3. Verification of selected empirical formulas for calculation of  $G_{\text{max}}$

Equation adopted for the calculation of $G_{\text{max avg calculated}}$	$p'$ [kPa]	$e_{\text{avg}}$ [–]	$G_{\text{max avg estimated}}$ [kPa]	$G_{\text{max avg calculated}}$ [kPa]	$G_{\text{max avg estimated}}$ [MPa]	$G_{\text{max avg calculated}}$ [MPa]	Spread of $\Delta G_{\text{max avg}}$ [MPa]
3	45	0.37938	31117	103558	31	104	–93
	90	0.37752	61650	146811	62	147	
	135	0.38314	80367	178478	80	178	
	180	0.37525	109550	208239	110	208	
	225	0.39013	131683	228268	132	228	
	270	0.38921	151283	250364	151	250	
	315	0.38804	171950	270839	172	271	
4	45	0.37938	31117	105947	31	106	–97
	90	0.37752	61650	150227	62	150	
	135	0.38214	80367	182336	80	182	
	180	0.37525	109550	213162	110	213	
	225	0.39013	131683	232986	132	233	
	270	0.38921	151283	255578	151	256	
	315	0.38804	171950	276558	172	277	
5	45	0.37938	31117	39588	31	40	30
	90	0.37752	61650	56134	62	56	
	135	0.38314	80367	68132	80	68	
	180	0.37525	109550	79650	110	80	
	225	0.39013	131683	87057	132	87	
	270	0.38921	151283	95498	151	95	
	315	0.38804	171950	103338	172	103	
6	45	0.37938	31117	35055	31	35	38
	90	0.37752	61650	49683	62	50	
	135	0.38314	80367	60403	80	60	
	180	0.37525	109550	70454	110	70	
	225	0.39013	131683	77328	132	77	
	270	0.38921	151283	84804	151	85	
	315	0.38804	171950	91735	172	92	
7	45	0.37938	31117	30902	31	31	37
	90	0.37752	61650	46923	62	47	
	135	0.38314	80367	59484	80	59	
	180	0.37525	109550	71283	110	71	
	225	0.39013	131683	80258	132	80	
	270	0.38921	151283	89619	151	90	
	315	0.38804	171950	98424	172	98	

acterized the soil samples. A similar averaging procedure was carried out for  $G_{\max}$  values. The values of the shear modulus estimated directly from the experimental data ( $G_{\max \text{ avg estimated}}$ ) and the ones calculated using empirical relations found in the literature ( $G_{\max \text{ avg calculated}}$ ) were differentiated. The difference between the parameters  $G_{\text{estimated}}$  and  $G_{\text{calculated}}$ , defined as  $\Delta G_{\max} = G_{\max \text{ avg estimated}} - G_{\max \text{ avg calculated}}$ , was averaged and then presented as the results in Table 3. The last column in Table 3 shows, therefore, the average dispersion of the results of  $\Delta G_{\max}$  in relation to empirical formulas selected from the literature.

Based on the simple analysis, large discrepancies between the measured and the calculated values of the maximum shear modulus were found (Table 3). This motivated us to develop a more suitable empirical equation to describe the tests data.

### Equation proposal

Final selection of the best formula describing the deformation characteristics of test soils in the range of small strains was made after the following analysis: correlation analysis, simple and multiple linear regressions and nonlinear regression. Some nonlinear regression problems can be moved to a linear

domain by a suitable transformation of the model formulation and so did the authors of the article. The last phase of the development of the appropriate formula was non-linear estimation, where the derived formula is strongly non-linear. Therefore, to determine the value of the small-strain shear modulus one has to take into consideration two components, namely mean effective stress ( $p'$ ) and void ratio ( $e$ )

$$G_{\max} = p'^{0.853} \cdot e^{-0.261} \quad (8)$$

In Table 4, values of the coefficients as well as the basic fitting parameters for the proposed function, equation (8), are shown. In Fig. 5, a graphical illustration of the proposed equation is presented.

To examine the validity of the proposed empirical equation, some standard empirical equations (Table 1) are also used to verify their fit to the data set (Fig. 6). Based on Fig. 6, it can be observed that the authors' expression fits the results of the tests perfectly. Obviously, the authors are aware that their mathematical function requires further verification by performing more research on various cohesive soils. The authors remark, however, that their function is only valid within the cohesive soils studied.

Table 4. Laboratory fitting parameters of test soils for authors' equation

Form of the function	Coefficient		Standard error of the coefficient		Correlation coefficient	Mean relative error	Mean square relative error
	$a_1$	$a_2$	$a_1$	$a_2$	$R (-)$	$MRE (\%)$	$MSRD (\%)$
$G_{\max} = p'^{a_1} \cdot e^{a_2}$	0.85	-0.26	0.02	0.12	0.97	13.47	36.7

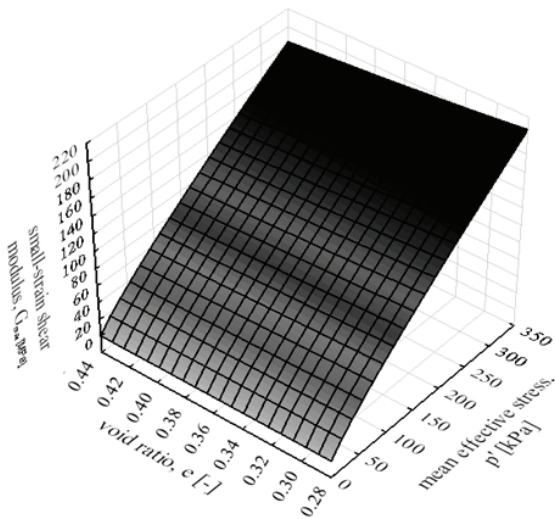


Fig. 5. Variation of the small-strain shear modulus with void ratio and mean effective stress for selected cohesive soil

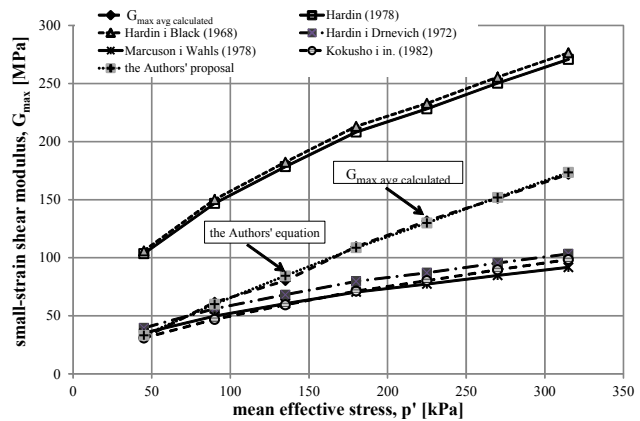


Fig. 6. Comparison of equation (8) with standard empirical equations from the literature



## 5. FINAL REMARKS

The small-strain shear modulus ( $G_{\max}$ ) of cohesive soils from Warsaw area was studied by means of resonant column tests. Completed laboratory studies and analysis of the test results allowed the following conclusions to be formulated.

There is a strong dependence of the initial stiffness of the soils tested on mean effective stress ( $p'$ ) and void ratio ( $e$ ). The authors' results illustrate that with the increase in mean effective stress and the decrease in void ratio, the initial stiffness of the soils examined increases, too.

In order to estimate the small-strain shear modulus of selected cohesive soils, taking into account the factors obtained from standard laboratory tests which affect this parameter, the authors do not recommend empirical formulas from the literature. Their results could not be adequately described by empirical formulas presented in the literature. Using the relationships from the literature causes significant differences between the measured and the calculated values of  $G_{\max}$ , in the range of 30 to –97 MPa. The existing expressions allow the small-strain stiffness of cohesive soils to be determined, but with mean relative errors superior to 20% (27.41% formula of Hardin 1978, 117.01% – Hardin & Black 1968, 121.90% – Hardin & Drnevich 1972, 32.66% – Marcuson & Wahls 1978 and 31.54% – Kokusho et al. 1982).

Based on the laboratory tests performed, a unique empirical relation,  $G_{\max} = p'^{0.853} \cdot e^{-0.261}$ , equation (8), was derived. This result allows the initial soil stiffness ( $G_{\max}$ ) to be determined with the knowledge of only two parameters measured in laboratory: mean effective stress ( $p'$ ) describing test conditions and void ratio ( $e$ ) describing physical property of soil. This formula makes it possible to find the initial soil stiffness more precisely, showing a superior accuracy in comparison with the equations presented in the literature. The authors attained a significant improvement of the scatter of the results, around 1 MPa. Mean relative error (MRE) made while applying equation (8) is 13.47%, which is about the half of the lowest MRE obtained using literature expressions listed in Table 1. The authors stress, however, that their equation, equation (8), was developed for clayey

soils with low plasticity index from the glaciations of Warta and Odra rivers. Their formula fits their data with high precision and they strongly encourage independent verification of their findings by other groups.

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