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INFLUENCE OF INERTIA FORCES ON SOIL SETTLEMENT UNDER HARMONIC LOADING

BOGUMIŁ WRANA, NATALIA PIETRZAK

Cracow University of Technology.
E-mail: nati@silbud.pl; bwrana@interia.pl

Abstract: The paper deals with the comparison of Biot's model for saturated, porous soils with other simplified models used in dynamic analysis. The purpose of this paper is to determine some limits of validity of the various models. In order to do this a full set of governing, dynamic equations of Biot model and a series of simplifying models such as *u-p* simplification and quasi-static consolidation models are considered. These formulations are applied to a simple soil layer under periodic surface loading. A displacement of skeleton and a displacement of fluid are shown and compared with each model for various formulations.

1. INTRODUCTION

The response of saturated porous media under the dynamic load is of high interest in many fields ranging from geomechanics to biomechanics. Problems like transient phenomena during impact loading, earthquakes, water wave loading and consolidation are of significant interest in geomechanics. The nature of response of the saturated porous media depends not only on the nature of loading but also on the flow and deformation characteristics of the media. The response is said to be fully drained when the rate of loading is much smaller than the rate of pore fluid flow. The problem is said to be static if the steady-state pore fluid pressures depend only on the hydraulic conditions and are independent of the porous skeleton response leading to uncoupled flow and deformation problem. One-phase media for all the calculations are adequate then. On the other extreme, if the rate of loading is much faster than the rate of flow, the fluid follows the motion of the solid. This is an undrained condition, where the single-phase solution is also adequate.

Depending on the rate of loading and the characteristics of flow and deformation there are three idealizations possible:

– exact solution (fully dynamic or Biot's one [1])

– *u-p* formulation (partly dynamic [2])

– the consolidation equations (quasi-static [3])

In the first case, the coupled equations of flow and deformations are formulated including an acceleration of both the solid skeleton and the fluid. In the case of the *u-p* formulation the coupled equations consider only the acceleration of solid skeleton. The governing equations are represented only in terms of the solid displacement u and the pore fluid pressure p. When it comes to quasi-static case, all inertial terms are ignored.

2. EQUILIBRIUM AND MASS BALANCE RELATIONSHIP

The effective stress for saturated porous media is defined by

$$\sigma''_{ij} = \sigma_{ij} + \alpha\delta_{ij} p \,, \tag{1}$$

$$\boldsymbol{\sigma}'' = \boldsymbol{\sigma} + \alpha \mathbf{m}^T p \,, \tag{2}$$

where σ_{ij} is the total stress, σ''_{ij} is the effective stress and σ_{ij} is the Kronecker delta, p is the pore fluid pressure and $\mathbf{m}$ = {1, 1, 1, 0, 0, 0,}^{T}. In soils where the average material bulk modulus of the solid components of the skeleton $K_s \gg K_T$ (K_T is the tangential bulk modulus of an isotropic elastic material) the factor $\alpha = 1$ and as a result the equation simplifies to

$$\sigma'_{ij} = \sigma_{ij} + \delta_{ij} p \,, \tag{3}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma} + \mathbf{m}^T p \,, \tag{4}$$

and is responsible for the major part of the deformation.

Constitutive equations using the incremental definition can be written as

$$d\sigma'_{ij} = D_{ijkl}(d\varepsilon_{kl} - d\varepsilon^0_{kl}) \,, \tag{5}$$

$$d\boldsymbol{\sigma}'' = \mathbf{D}(d\boldsymbol{\varepsilon} - d\boldsymbol{\varepsilon}^0) \,, \tag{6}$$

where D_{ijkl} is the tangent coefficient matrix and ε^0_{kl} is the initial strain. We define strain increments as

$$d\boldsymbol{\varepsilon} = \{d\varepsilon_x, d\varepsilon_y, d\varepsilon_z, d\gamma_{xy}, d\gamma_{xz}, d\gamma_{yz}\}^T \,.$$

In plane strain the stress-strain relationship in terms of Lame's coefficients λ and G is written as

$$\sigma'_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2G\varepsilon_{ij} \tag{7}$$

where ε_{kk} is volumetric strain. In the above equation and for the rest of calculations we assume small strains. The strain ε_{ij} is then defined as

$$\varepsilon_{ij} = 0.5(u_{(i,j)} + u_{(j,i)}) \,. \tag{8}$$

In the two-phase model of soil there are three main equations in vectorial notation:

a) *The overall equilibrium or momentum balance relation* [2]

$$\mathbf{S}^T\boldsymbol{\sigma} - \rho\ddot{\mathbf{u}} - \rho_f(\dot{\mathbf{w}} + \mathbf{w}\nabla^T\mathbf{w}) + \rho\mathbf{b} = 0 \tag{9}$$

where

ρ_f – fluid density (water), $\rho = n * \rho_f + (1 - n)\rho_s$ – total density,

ρ_s – skeleton density,

b – body force per unit mass (generally gravity),

n – porosity,

w – the average Darcy velocity of percolating water,

S is the strain matrix

$$\mathbf{S} = \left[\frac{\partial}{\partial x}, 0;\ 0, \frac{\partial}{\partial y};\ \frac{\partial}{\partial y}, \frac{\partial}{\partial x} \right].$$

b) *The momentum balance of the fluid*

$$-\nabla p - \mathbf{R} - \rho_f \ddot{\mathbf{u}} - \frac{\rho_f(\dot{w} + w\nabla^T w)}{n} + \rho_f \mathbf{b} = 0 \tag{10}$$

where **R** represents viscous drag forces which, assuming the Darcy seepage law, can be written as **kR** = **w**. The permeability ***k*** is used with dimensions of [length]^3 [time]/[mass] which is different from the usual soil mechanics convention **k′** which has the dimension of velocity, i.e., [length]/[time]. Their values are related by $k = \dfrac{k'}{\rho_f g}$, where ρ_f and g are fluid density and gravitational acceleration at which the permeability is measured.

c) *The flow conservation equation*

$$\nabla^T \mathbf{w} + \alpha \mathbf{m}\dot{\varepsilon} + \frac{1}{Q}\dot{p} + n\frac{\dot{\rho}_f}{\rho_f} + \dot{s}_0 = 0 \tag{11}$$

where

$$\frac{1}{Q} = \frac{n}{K_f} + \frac{\alpha - n}{K_s} \cong \frac{n}{K_f} + \frac{1-n}{K_s}, \tag{12}$$

$n\dfrac{\dot{\rho}_f}{\rho_f}$ – change of density,

$\dot{s}_0$ – rate of volume expansion of the solid in the case of thermal change,

K_f – bulk modulus of the pore fluid.

The last equation is one accounting for mass balance of the flow. It balances the flow divergence by the augmented storage in the pores of a unit volume of soil occurring in time dt.

In the discussion below the convective term of the fluid acceleration $\mathbf{w}\nabla^T\mathbf{w} = 0$ and the acceleration itself $\dot{w} \approx 0$, because it is generally small. The body forces $\rho\mathbf{b} = 0$ and terms $\dot{\rho}_f = \dot{s}_0 \approx 0$ are also omitted. We use ρ_f, ρ instead of ρ_{11}, ρ_{12}, ρ_{22} like in typical Biot's formulation. This is because of the fact that we analyze the issue on a macroscopic scale, where the density can be defined as $\rho = n^*\rho_f + (1 - n)\rho_s$ [6]. As a result we obtain three equations written below in general form, which together with appropriate constitutive relations, define the behavior of the solid together with its pore pressure in both static and dynamic conditions. The so called Biot's equations are

$$\sigma_{ij,j} - \rho\ddot{u}_i - \rho_f \dot{w}_i = 0\,, \tag{13}$$

$$-p_{,i} - R_i - \rho_f \ddot{u}_1 - \frac{\rho_f \dot{w}_1}{n} = 0\,, \tag{14}$$

$$w_{i,j} + \alpha\dot{\varepsilon}_{11} + \frac{\dot{p}}{Q} = 0\,. \tag{15}$$

The unknown variables in this system are:
– pressure of the fluid (water), p,
– velocities of fluid flow, w_i,
– displacements of the solid matrix, u_i.

The boundary condition imposed on these variables will complete the problem. We have to define them for both solid and fluid phase. They are generally divided into two groups. For the solid phase we have the part of the boundary Γ_t on which we specify the total traction $t_i(t)$ (or in terms of the total stress $\sigma_{ij}n_j$ with n_i being the i-th component of the normal at the boundary) and we also have Γ_u where the displacement u_i is given. When it comes to the fluid phase the boundary conditions are again divided into two parts – Γ_p where the values of p are specified and Γ_w where the normal outflow w_n is prescribed (it would be a zero value for the normal outward velocity on an impermeable boundary). Having this in mind we can write

$$\Gamma = \Gamma_t \cup \Gamma_u\,, \tag{16a}$$

$$t = \sigma_{ij}n_j \quad \text{on} \quad \Gamma_t\,, \tag{16b}$$

$$u = \overline{u} \quad \text{on} \quad \Gamma_t\,. \tag{16c}$$

Further

$$\Gamma = \Gamma_p \cup \Gamma_w\,, \tag{16d}$$

$$p = \overline{p} \quad \text{on} \quad \Gamma_p \tag{16e}$$

on

$$\Gamma = \Gamma_w \tag{16f}$$

As we analyze harmonic excitation with a constant frequency as well as an amplitude, initial conditions are not considered.

3. DIFFERENT FORMULATIONS FOR ONE DIMENSIONAL PROBLEM

3.1. FULLY DYNAMIC IDEALIZATION – BIOT MODEL

For the one-dimensional case we assume $u_1 = u$, $u_2 = u_3 = 0$ and $1/Q = n/K_f$. For an isotropic material, D is given by and for small strains the stress relation is $\boldsymbol{\sigma} = \boldsymbol{\sigma}'' - p = D\varepsilon - p = D_{u,x} - p$, in the next relation we introduce $w = \dot{v}$ where v is the displacement of fluid.

From equation (15) we have relation $p = \dfrac{K_f}{n}(u_{,x} + v_{,x})$.

Substituting the above relation into a) and b) we obtain

$$\left((D + \frac{K_f}{n}\right)u_{,xx} + \frac{K_f}{n}v_{,xx} = \rho\ddot{u} + \rho_f\ddot{v}\,, \tag{17}$$

$$\frac{K_f}{n}(u_{,xx} + v_{,xx}) = \rho_f\ddot{u} + \frac{\rho_f}{n}\ddot{v} + \frac{\rho_f g w}{k}\,. \tag{18}$$

For a periodic applied surface load $q = Qe^{i\omega t}$ a periodic solution arises after the dissipation of the initial transient in the form $u = U(x)e^{i\omega t}$, $v = V(x)e^{i\omega t}$.

With the following notations

$$K = \frac{\frac{K_f}{n}}{D + \frac{K_f}{n}}, \quad \beta = \frac{\rho_f}{\rho}, \quad \alpha = \frac{\pi\beta L g}{k V_L} = \frac{\beta g \pi}{2\bar{f}k}, \quad z = \frac{x}{L}\,,$$

where

L – depth of soil layer,

$V_L = \sqrt{\dfrac{D + \frac{K_f}{n}}{\rho}}$ – compression wave velocity in porous soil,

$f = \dfrac{\omega}{2\pi}$ – frequency of excitation,

$\bar{f} = \dfrac{V_L}{4L}$ – reference frequency,

$\eta = \dfrac{f}{\bar{f}} = \dfrac{\omega L}{2\pi V_L}$ – frequency ratio as proportion of excitation to reference frequency,

one can obtain Biot's equations in the form

$$U_{,zz} + KV_{,zz} = -\eta^2\pi^2 U - \eta^2\pi^2\beta V\,, \tag{19}$$

$$KU_{,zz} + KV_{,zz} = -\eta^2\pi^2\beta U - \left[\frac{\eta^2\pi^2\beta}{n} - i\eta\alpha\right]V\,. \tag{20}$$

3.2. PARTLY DYNAMIC IDEALIZATION – U-P FORMULATION

In this case, the coupled equations of flow and deformation consider only the acceleration of solid skeleton and not that of pore fluid. It is called *u*-*p* formulation as in this case the governing equations can be represented only in terms of solid displacement u and pore fluid pressure p.

The reduced system in general form when omitting inertia forces of the fluid becomes

$$\sigma_{ij,j} - \rho\ddot{u}_i = 0\,, \tag{21}$$

$$-p_{,i} - R_i - \rho_f\ddot{u}_1 = 0\,, \tag{22}$$

$$w_{i,i} + \alpha\dot{\varepsilon}_{11} + \frac{\dot{p}}{Q} = 0\,. \tag{23}$$

The set of appropiate equations for one-dimensional problem (19), (20), after substituting new variables introduced in the previous section, becomes

$$U_{,xx} + KV_{,xx} = -\eta^2\pi^2 U\,, \tag{24}$$

$$KU_{,xx} + KV_{,xx} = -\eta^2\pi^2\beta U + i\eta\alpha V\,. \tag{25}$$

3.3. CONSOLIDATION PROBLEM

If we ignore all second time derivatives, in other words all inertial terms, the equations simplify to

$$\sigma_{ij,j} = 0\,, \tag{26}$$

$$-p_{,i} - R_i = 0\,, \tag{27}$$

$$w_{i,i} + \alpha\dot{\varepsilon}_{11} + \frac{\dot{p}}{Q} = 0\,. \tag{28}$$

The resulting system corresponds to quasi-static case, which is used frequently in soil mechanics. If the phenomenon is sufficiently slow the full set of equations is considerably reduced and after similar transformations as in the previous cases, we obtain

$$U_{,xx} + KV_{,xx} = 0\,, \tag{29}$$

$$KU_{,xx} + KV_{,xx} = i\eta\alpha V\,. \tag{30}$$

We solve all the aforementioned (see 3.1, 3.2, 3.3) differential equations analytically reducing them to two uncoupled differential equations of the fourth order.

4. BOUNDARY CONDITIONS

A one-dimensional soil layer subjected to a periodic surface force is considered as shown in Fig. 1. To complete the solution, boundary conditions must be applied at $z = 0$ and $z = 1$. These conditions are:

with $z = 0$: pore pressure $p = 0$, stress on external surface $\sigma = \overline{q}$,

with $z = 1$: displacement of skeleton $u = 0$, displacement of fluid $v = 0$.

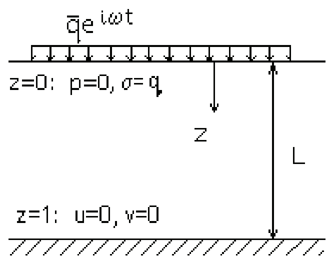


Fig. 1. Soil layer subjected to periodic loading

5. NUMERICAL EXAMPLE

Numerical analysis is performed with typical parameters for a wide range of sands: porosity $n = 1/3$, coefficient $K = 0.973$, and $\beta = 1/3$. The calculations of skeleton displacement – u and fluid displacement – v are carried out for three sets of models: FD-exact solution (Biot's model), PD – u-p model and QS – quasi-static model. All the calculations are done using the dimensionless η parameter which is the ratio of the excitation frequency to the comparative frequency which is 10 Hz in our case: $\eta = \frac{f}{\overline{f}} = \frac{f}{10\ \text{Hz}}$. The value of 10 Hz is a typical excitation for soil vibrators. There are generally considered five values of η: 0.1, 0.8, 1.0, 1.2, 5.0. These values comply with soil vibrators, earthquake or wave induced excitation inter alia.

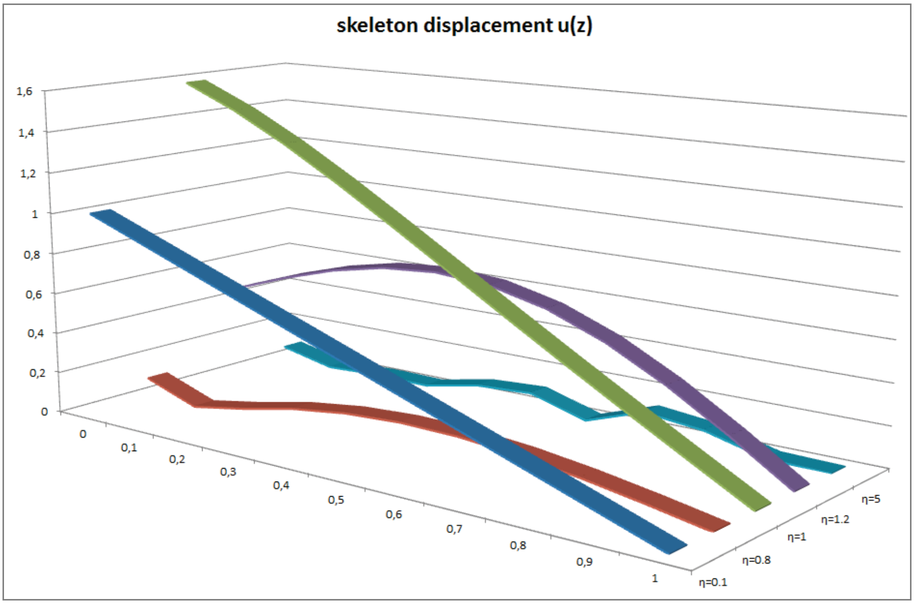


Fig. 2. Displacement of the soil skeleton for FD analysis for five different η

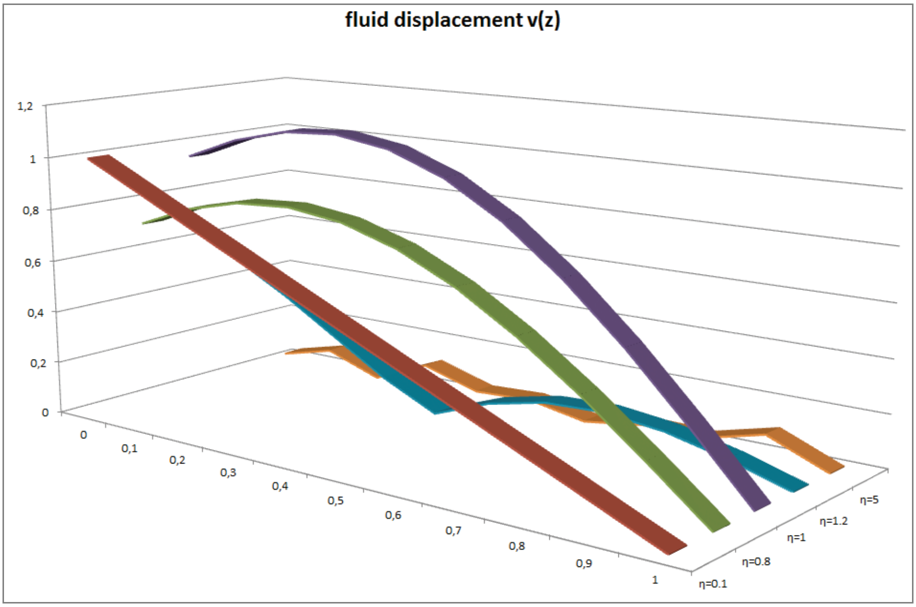


Fig. 3. Displacement of the fluid for FD analysis for five different η

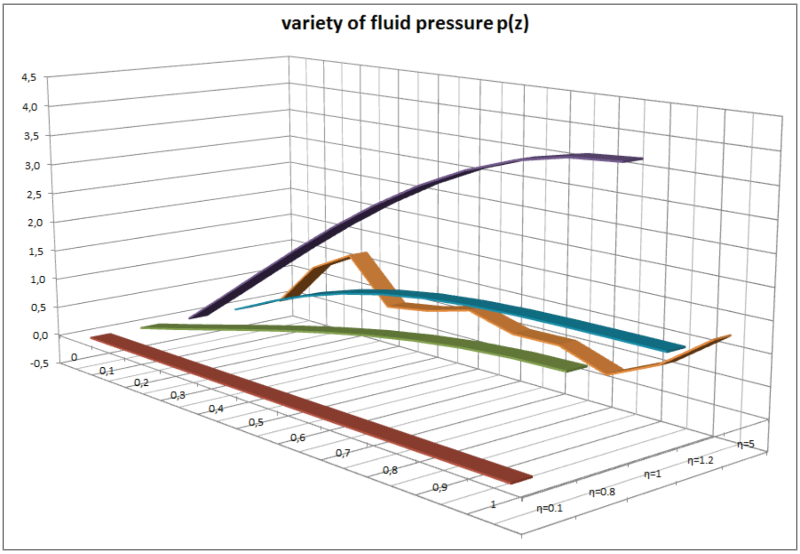
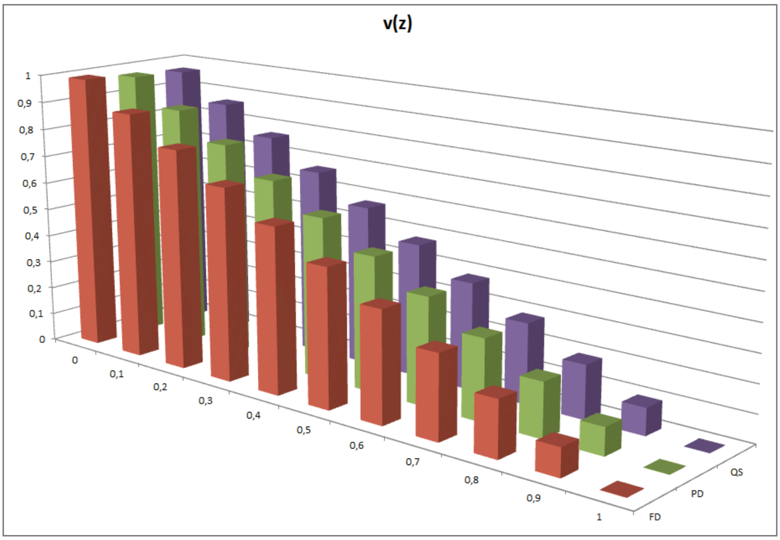
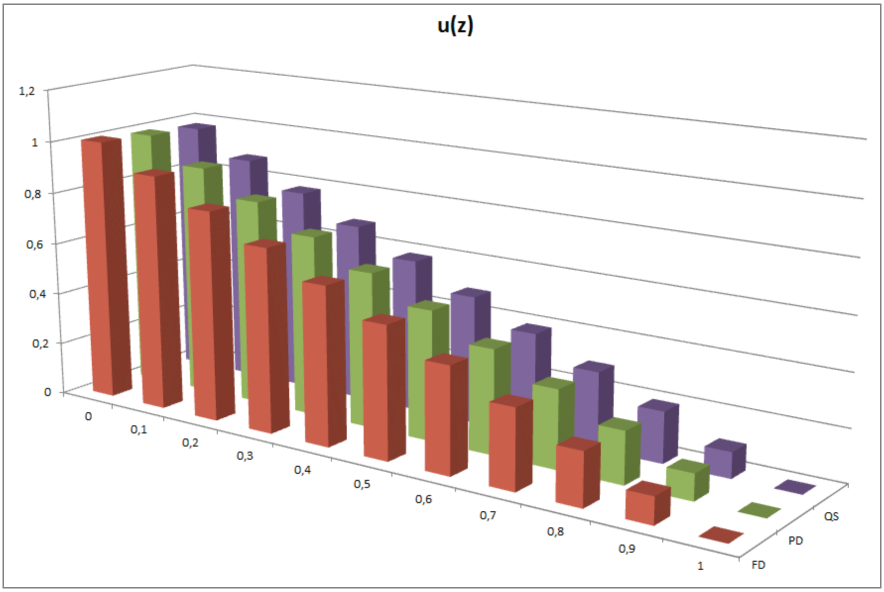


Fig. 4. General variation of the fluid pressure for FD analysis for five different η

The three figures above (Figs. 2 through 4) show the displacement of the skeleton and of the fluid and also the fluid pressure for Biot's model. There are varied values and shapes of the aforementioned unknowns for different η (or different excitation frequency in other words).

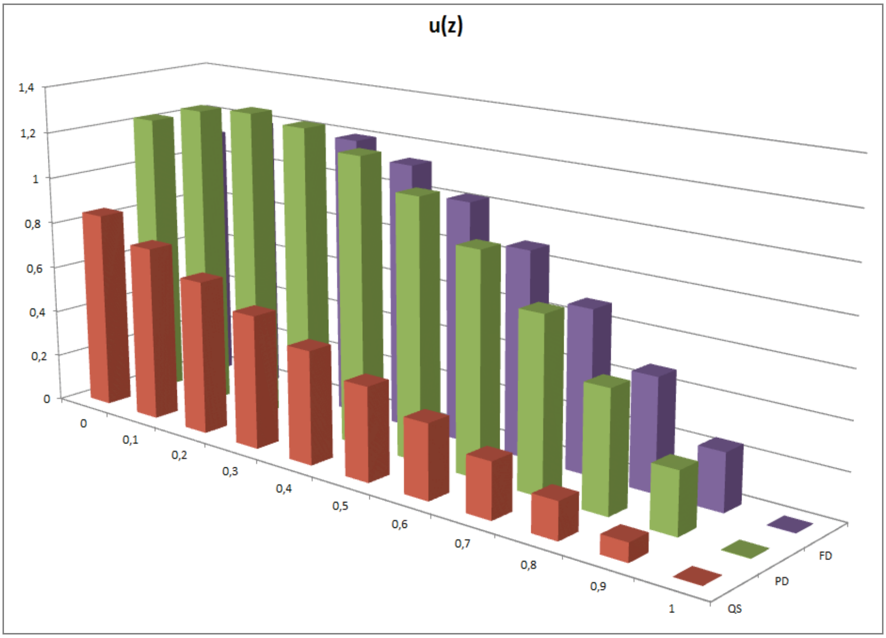
Figures 5 through 7, on the contrary, present a comparison of the skeleton and the fluid displacement for the formulations analysed and for three exemplary η.

a) 



b)

Fig. 5. Skeleton and fluid displacement for Biot's, *u-p* and consolidation model for $\eta = 0.1$



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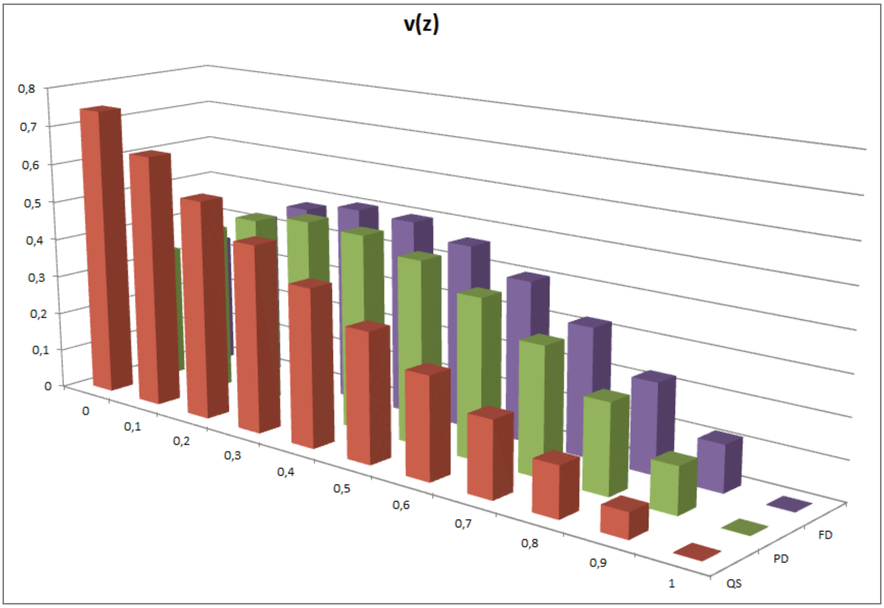
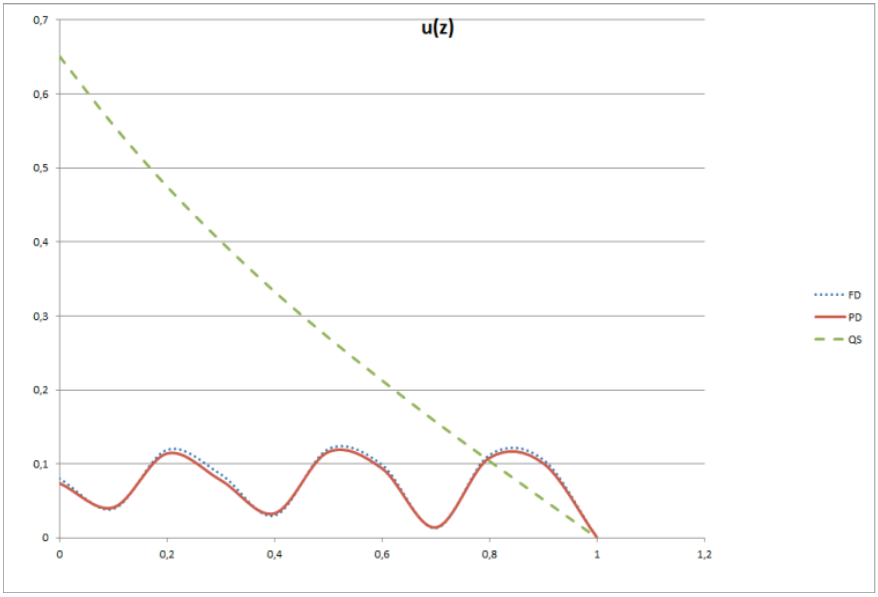


b)

Fig. 6. Skeleton and fluid displacement for Biot's, *u-p* and consolidation model for $\eta = 1$



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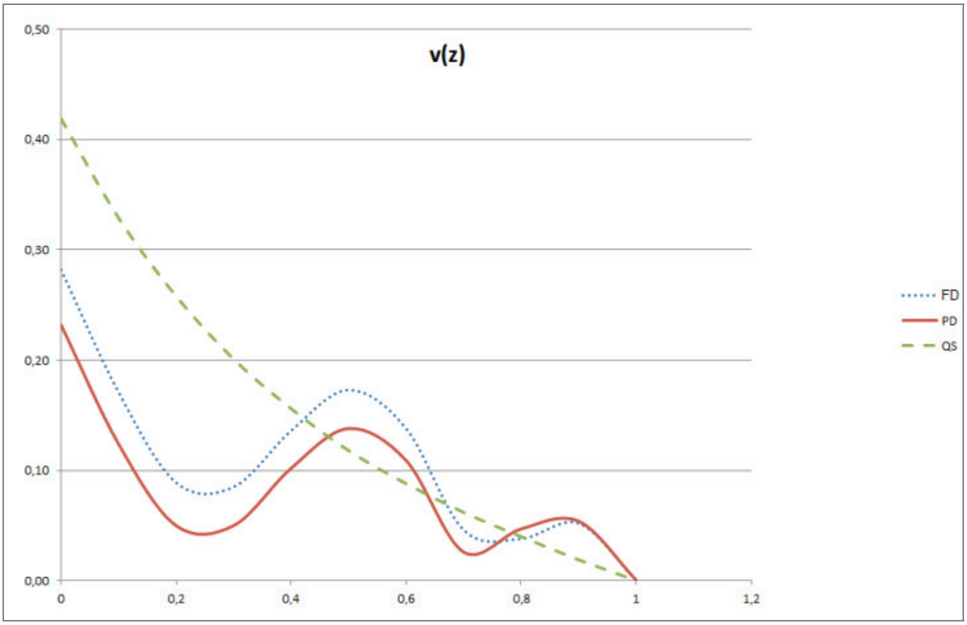
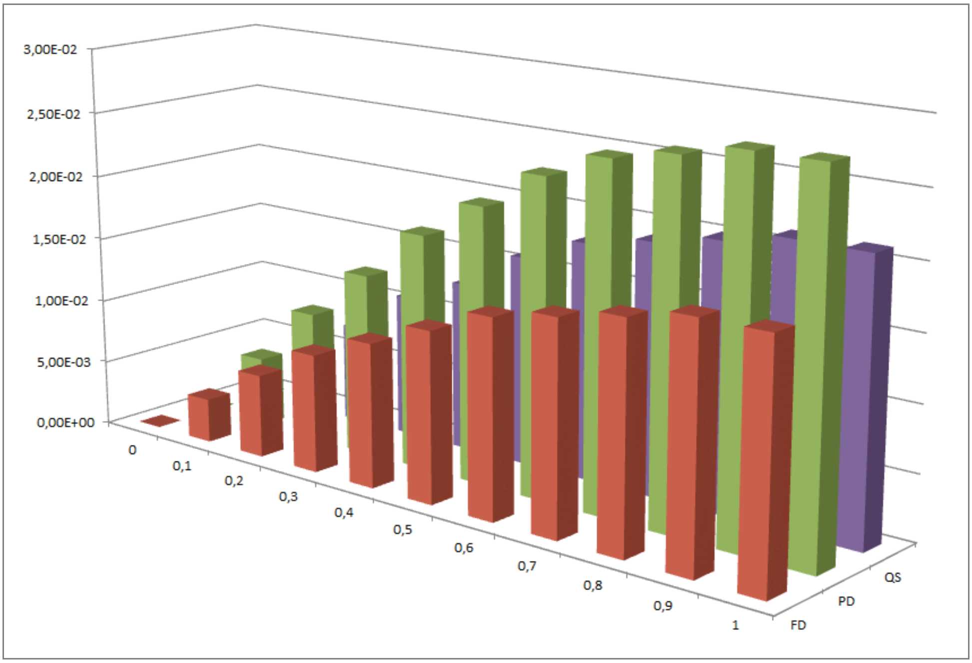


b)

Fig. 7. Skeleton and fluid displacement for Biot's, *u-p* and consolidation model for $\eta = 5$



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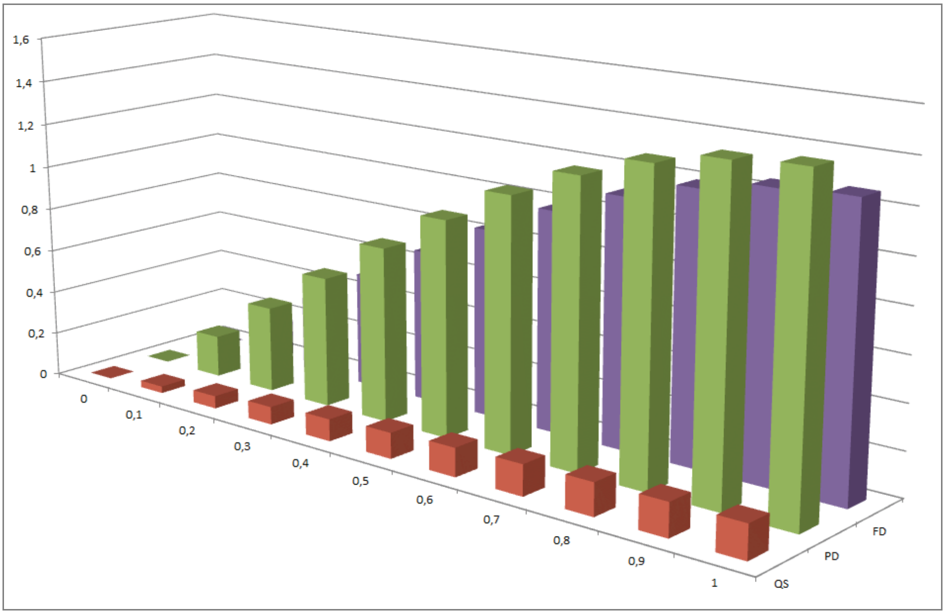

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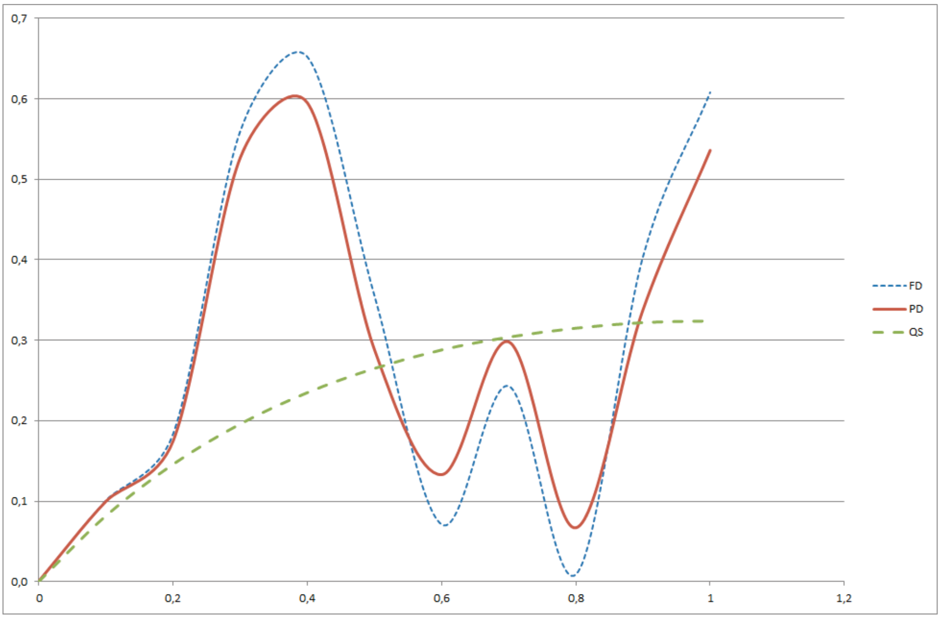

c)

Fig. 8. Fluid pressure for Biot's, *u-p* and consolidation model for: (a) $\eta = 0.1$, (b) $\eta = 1$, (c) $\eta = 5$

For $\eta = 5$ there is a different type of graph used, because it shows the oscillations and the violent changes much better than the column graph.

When it comes to the fluid pressure, the disparity between the models analysed is also bigger for higher frequencies as shown in Fig. 8 presenting the pressure as a function of depth.

6. CONCLUSION

The analytical study described helps us to determine the limits of applicability of the various assumptions in the particular case of a linear one-dimensional and periodic problem. An extrapolation of the conclusions can be made to other more realistic problems of soil mechanics giving some quantitative basis for the recommended analysis procedure and avoiding a priori assumptions.

The main question which is raised in this paper is when the PD or even QS model is sufficient and whether FD analysis is necessary to conduct. There are three main conclusions which can be made after calculations in the previous sections:

– for low excitation frequency ($\eta \leq 0.1$) the three models analysed give similar results, so there is no point in doing expanded calculations accompanying the PD or FD idealization;
– the influence of the fluid inertia forces is very low even for $\eta = 1$, so it is not an error to neglect them for $\eta \leq 1$ (PD idealization is sufficient);
– if the skeleton displacement is of main interest, it is possible to conduct the PD idealization (which neglects the fluid inertia forces) even for relatively high excitation frequency ($\eta = 5$).

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