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CROSS-COUPLED HEAT AND MOISTURE TRANSPORT PART 2/1: APPLICATIONS IN MECHANICS

ELECTRICAL ANALOGY – ANALYTICAL SOLUTION – EXPERIMENTS

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Abstract: Based on the theory laid down in the Part 1 [1] in this paper we are going to display applications in physics and especially in mechanics. After a short summary on coupled fields in mechanics, we generally analyze the analogies in mechanics. As the most important, the electrical analogy is discussed in details. By the electrical analogy based analytical solution we display the results of the coupled thermo-hygro problem with second sound. Finally, a few experiments are discussed, eg. experiments by electrical analogy, experiment on diffusivities, experiment on relaxation time, experiment on hydroglobe. This latter one leads to the engineering application of the problem.

KEYWORDS: thermo-hygro-mechanics, electrical analogy, second-sound phenomenon

The paper is devoted to Prof. Pavel Elesztos on the occasion of his 70th birthday and on the 40th anniversary of our friendship, that is supported by several co-authored publications and mutual visits in Bratislava and in Budapest.

1 Introduction

The Paper deals with the application of the cross-coupled heat and moisture transport theory [1] in physics, especially in mechanics. Also, the paper is built up around the electrical analogy (EA).

What is the importance of EA? First of all, it gives a deeper insight into the phenomenon. Second, the analytical solution worked out in electricity is applicable also in thermo-hygro-mechanics (THM). Third the numerical methods based on this analytical solution and widely used in electricity can be applied, too. And finally, the EA gives a good tool for experiments.

And all of these show the structure of this paper, i.e. we are going to deal with EA, analytical solution and experiments, first of all EA based experiments.

2 Frequent Coupled Fields in Mechanics

Some of the most often applied coupled fields of mechanics are collected in Table 1. Maybe most frequent is the thermo-elastic one.

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Table 1/a Coupled fields of mechanics and their relations

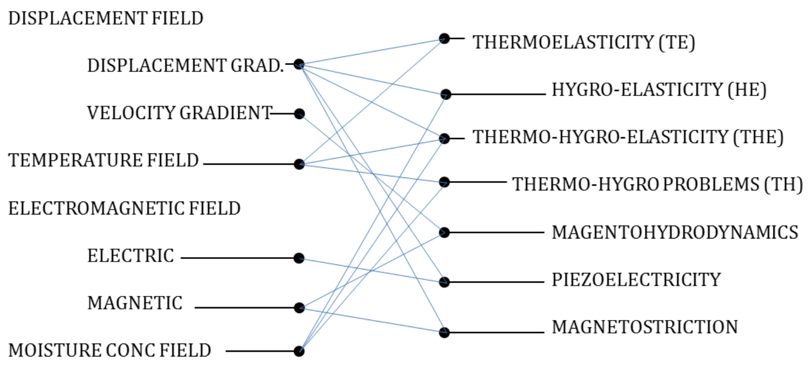

Table 1/b Coupled fields of mechanics and their relations in matrix multiplication form

$$\begin{bmatrix} TE \\ HE \\ THE \\ TH \\ MH \\ P \\ MS \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} D \\ V \\ T \\ E \\ Ma \\ Mo \end{bmatrix}$$

Coupled fields   Coupling matrix   Fields

Table 1/c Coupled fields of thermo-hygro-elasticity

FIELDS	**DISP. GRAD**	**TEMPERATURE**	**MOISTURE**
DISP. GRAD		**TE** **$\boxed{THE}$**	**HE**
TEMPERATURE			**TH**
MOISTURE			

In case of dynamical processes of solids, the coupling between displacement and thermal fields is not negligible (see Duhamel-Neumann eq.). For engineering applications of thermoelasticity the reader is referred to [2, 3]. This means that classical Fourier's law of heat conduction is no longer valid in such processes. Also, in hygroscopic materials, due to the Soret and Dufour cross coupling effects, heat and moisture transport is coupled, i.e. the moisture diffusion process is not described by classical Fick's law. Let us see first the basic equations of thermo-hygro-elastic materials.

The generalized displacement-temperature-moisture field equations for a homogeneous isotropic thermoelastic body with moisture take the form:

$$\mu u_{i,jj} + (\mu+\lambda)u_{k,ki} + \rho b_i - \beta T_{,i} - \beta_m m_{,i} = \rho \ddot{u}_i ; \tag{1}$$

$$D_T T_{,ii} + D_{mT} m_{,ii} - \dot{T} - \frac{\beta\, T_0}{\rho c}\dot{u}_{j,j} = 0\,; \tag{2}$$

$$D_m m_{,ii} + D_{Tm} T_{,ii} - \dot{m} - \frac{\beta_m m_0 D_m}{k_m}\dot{u}_{j,j} = 0\,. \tag{3}$$

Here u_i, T and m are the displacement, temperature difference and moisture concentration; μ, λ Lamé constants; ρ the density; b_i the body force; D_m, D_T and D_{Tm}, D_{mT} the moisture diffusivity, the temperature diffusivity and the cross coupled diffusivities, respectively; $\beta = (3\lambda + 2\mu)\alpha_t$ and $\beta_m = (3\lambda + 2\mu)\alpha_m$, where α_t and α_m are coefficients of thermal (CTE) and moisture expansions (CME); c, k_m, T_0 and m_0 are the specific heat, moisture conductivity, temperature and moisture concentration in the natural state; T and m are the relative temperature and moisture concentration above T_0 and m_0, that are, e.g. the environmental temperature and moisture concentration, the dot designates material time derivatives and the comma space derivatives.

If the temperature is neglected and we let $D_{mT} = \alpha_t = 0$, the equations (1) and (3) stand for the hygro-elastic (HE) case. If the moisture diffusion is neglected equations (1) and (2) describe the thermo-elastic (TE) phenomenon.

Let us deal in more detail form with the coupled thermal and moisture fields (TH). Taking into account the second sound phenomenon, the constitutive and conservation laws read (see [1], eqs. 4.13a, 4.12 and 4.14a):

$$J_i + \tau_{ij}\dot{J}_j = -\alpha_{ij}\nabla C_j\,, \tag{4}$$

$$\dot{C}_i = -\nabla J_i + \sigma_i\,, \tag{5}$$

where:

$$C_i = \begin{bmatrix} m \\ T \end{bmatrix},\quad J_i = \begin{bmatrix} f \\ q \end{bmatrix}$$

and f and q are the moisture and heat fluxes, τ_{ij} and α_{ij} are the relaxation time and conductivity matrices, σ_i is the source term. Eliminating J_i from (4) and (5), we obtain the governing equation of the problem:

$$\dot{C}_i + \tau_{ij}\ddot{C}_j = D_{ij}\nabla^2 C_j + \sigma_i + \tau_{ij}\dot{\sigma}_j\,. \tag{6}$$

In case of thermo-hygro-viscoelastic materials (THVE), the eqs. are slightly different. The time dependence of material parameters has to be taken into account.

3 Generally on Analogies in Mechanics

Dealing with the coupled fields of mechanics, the governing equations lead to complicated partial differential equations system due to material properties and couplings. The investigations often require experimental methods, including analogies.

Analogies in mechanics are known for a long time, e.g., between torsion and soap-film (Prandtl's) or sand heap (Nadai's) [4], or between heat conduction (Fourier's law) and moisture diffusion (Fick's law) [5]. There are others, e.g. between telegraphic and coupled thermoelastic phenomena [6,7], which have been mentioned only lately.

As another example, the parallel circuit electrical analogy may be mentioned for calculation of thermal conductivity [8]. There are many other analogies also. E.g. in Ref. [12] on experimental methods of mechanics a whole chapter is devoted to analogies and also the electrical analogy is discussed in connection with different problems. Starting from this one there are several other possibilities in coupled fields of mechanics. Based on the similarity between heat conduction and moisture diffusion the coupled hygroelastic [9], thermo-hygro [10], thermo-hygro-elastic [11] problems can be handled. In these cases, we have to take into account the Soret and Dufour effects.

On the other side, the method can be extended to processes taking into account the second sound phenomenon. In this case the electrical analogy provides a perfect opportunity to compare the results obtained by parabolic and hyperbolic systems. With time dependent electrical parameters it is possible to model viscoelastic materials, too.

As mentioned before, analogies are well-known and often used in mechanics. Beside the ones listed before one of the most often used is the analogy between mechanical and electrical vibrations. It is the basic example of this type of analogy. The reason for its frequent application is clear. The electrical circuit provides good opportunity for experiments and numerical methods worked out for this system are applicable in mechanics, too. On the other hand, the mechanical analogy helps the imagination of the physical process. It shows the symbiosis existing between mechanics and electricity.

Another very often mentioned analogy is between the heat conduction (Fourier's law) and moisture diffusion (Fick's law), e.g. references [10,11]. Beside the analogy the two processes are cross-coupled. Later we'll recall the equations of the problem but now it is worth to mention that in case of composite materials the relationship is triple: analogy, cross-coupling, and both the heat and moisture effects may cause degradation and lead to failure.

There is another, not very often used but known analogy between the coupled thermoelastic and telegraphic problems [7]. Recalling 1D displacement-temperature eqs. of a nonlinear dynamic coupled thermoelasticity

$$\frac{E}{\rho}u_{xx} - u_{tt} - \frac{E\alpha}{\rho}T_x = 0\,, \tag{7}$$

$$T_{xx} - f(T)T_t - g(T)u_{xt} = \frac{\gamma}{\kappa}u_x T_t\,, \tag{8}$$

and the 1D governing equations of the telegraphic phenomenon,

$$\frac{1}{LC}i_{xx} - i_{tt} + \frac{G}{LC}e_x = \frac{R}{L}i_t\,, \tag{9}$$

$$e_{xx} - RCe_t + L_{ixt} = RGe\,, \tag{10}$$

the analogy is obvious in spite of the difficulties in realization. The notations in eqs. (7-10) are as follows: u and T is the displacement and temperature difference above T_0 such that $T_1 = T_0 + T$ is the temperature of a natural state; E, ρ, α, c and κ are the Young modulus, density, CTE, specific heat, conductivity, $\gamma = (3\lambda + 2\mu)\alpha$, where λ and μ are Lame constants; i and e are current and voltage; R, L, G and C are resistance, inductance, electrical conductance, capacitance.

4 Electrical Analogy

The electrical background of our investigations is a circuit theory and Kirchhoff's equations. The best-known applications of these last ones are the telegraph equations. We are not going to recall it in details, only refer to the literature, e.g. [6]. The result shows the feasibility of electrical analogies in coupled fields in mechanics.

Coupled thermoelasticity and hygroelasticity

As the first application of electrical analogy, let us recall the eqs. (7-10) of the coupled thermoelastic and telegraphic problems [7]. Eqs. (9-10) are based on the elementary circuit shown in Fig.1. in which the current i and voltage e correspond to the displacement u and temperature T, respectively.

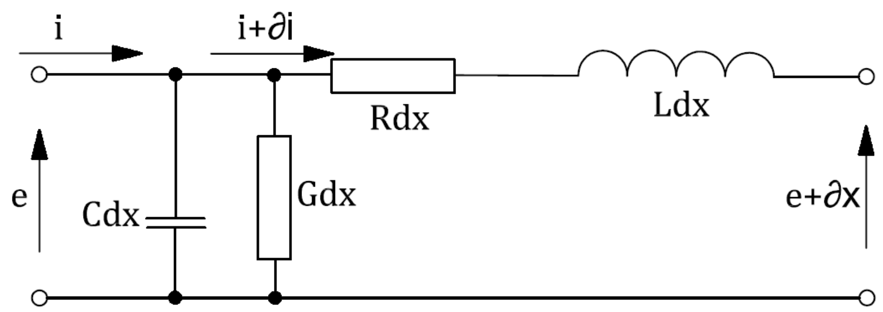

Fig. 1 Elementary circuit for coupled thermoelastic problem

A comparison of the equations (7)-(8) with (9)-(10) shows that there is an analogy if:

$$\frac{1}{LC}=\frac{E}{\rho};\ \frac{G}{LC}=\frac{E\alpha}{\rho};\ \frac{R}{L}\to 0;\ RC=f(e), \tag{11}$$

$$L=g(e);\ RG\to 0;\ \frac{\gamma}{\kappa}\to 0.$$

One can show that the elementary circuit scheme can be iterated in a laboratory in such a way that an electrical model of a long bar that satisfies eqs. (9)-(10) is recovered, and from (11) the physical properties of a long thermoelastic bar are obtained.

Heat conduction and moisture diffusion with second sound

Because of the similarity between the heat conduction and moisture diffusion, these two cases can be discussed simultaneously.

The basic eqs. of the elementary circuit shown in Fig. 2 will be examined. On the basis of Kirchhoff's equations, the equations of the elementary circuit are derived as follows:

$$\frac{\partial^2 e}{\partial x^2}=RC\frac{\partial e}{\partial t}+LC\frac{\partial^2 e}{\partial t^2}, \tag{12}$$

$$\frac{\partial^2 i}{\partial x^2}=RC\frac{\partial i}{\partial t}+LC\frac{\partial^2 i}{\partial t^2}, \tag{13}$$

Eq. (13) is analogous to the field equation:

$$\frac{\partial^2 C_i}{\partial x^2}=\frac{1}{D_i}\frac{\partial C_i}{\partial t}+\frac{\tau_i}{D_i}\frac{\partial^2 C_i}{\partial t^2} \tag{14}$$

that describes a modified heat conduction equation or a modified moisture diffusion equation, when C_i is identified with temperature T or moisture m, respectively. On this basis an experimental setup is offered. New notations: $(..)_x = \frac{\partial(..)}{\partial x}$. Considering the heat conduction problem of a finite length rod (see Fig. 3), the governing equation describing the heat propagation in the rod is as follows:

$$T_{xx} = \frac{1}{D_T} T_t + \frac{\tau_T}{D_T} T_{tt} \tag{15}$$

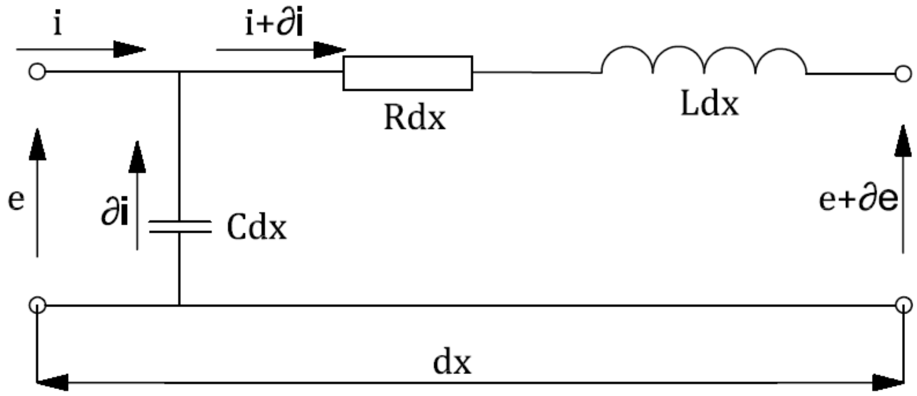

Fig. 2 Elementary circuit for heat conduction or moisture diffusion problem with second sound

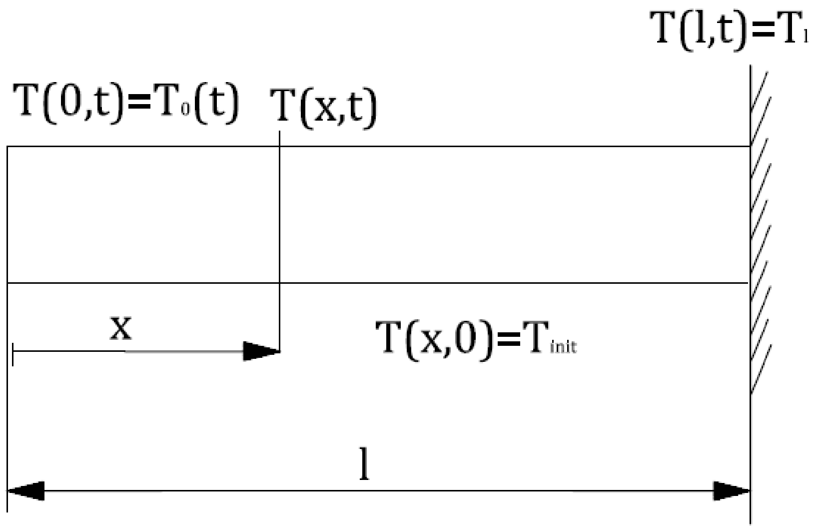

Fig. 3 Heat conduction for a long rod

The boundary and initial conditions correspond to heating one end of the rod to temperature $T_0(t)$ and keeping the other end at a constant temperature T_1, while the mantle is insulated. At the beginning there is a uniform temperature distribution $T(x,0) = T_{init}$.

Comparing equations (12) and (15) the following electrical setup is implemented (see Fig. 4), with the electrical elements chosen on the basis of equal coefficients in the (12) and (15):

$$\frac{1}{D_T} \to RC\,,\ \frac{\tau_T}{D_T} \to LC\,.$$

In this case the voltage $e(x,t)$ measured at discrete points as a function of time corresponds to the temperature $T(x,t)$. Of course, the conditions

$$T_0(t) \to e_0(t)\,,\ T_l = const \to e_l = const\,,$$
$$T_{init} = const \to e_{init} = const$$

must be fulfilled, too.

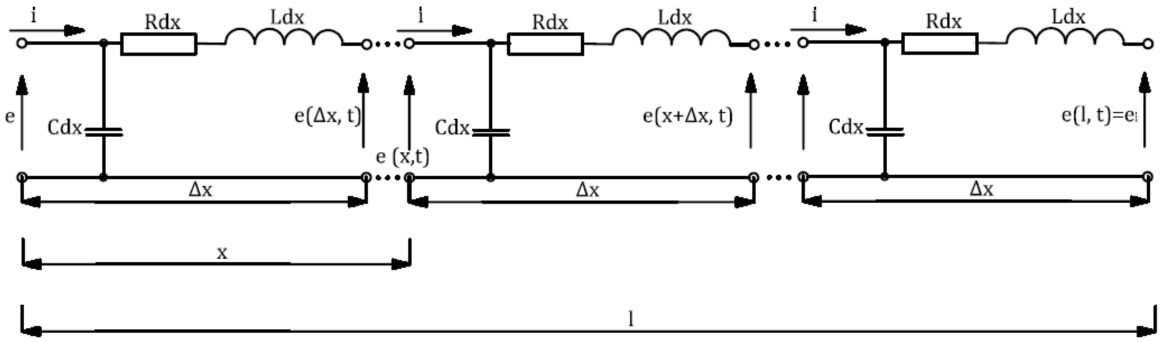

Fig. 4 Electrical model for for heat conduction or moisture diffusion problem with second sound

If $L = 0$, classical (parabolic) equation of heat conduction results:

$$LC = 0 \to \frac{\tau_t}{D_T} = 0 \to T_{xx} = \frac{1}{D_T} T_t\,. \tag{16}$$

The hyperbolic equation of heat conduction can be compared to the parabolic one.

5 Analytical Solution

The analytical solution method is built up according to the generalized thermoelastic equations and the governing equations of the telegraphic phenomenon [6].

The basic problems are the initial and boundary conditions, mainly the last ones. It takes the physics into the mathematical problem. That is why we start with the boundary conditions based on our previous works [13] and go on with the details of the solution.

Boundary conditions

The boundary condition for heat conduction in a solid immersed in a fluid is

$$\left.\frac{\partial T}{\partial n}\right|_S = -\frac{h}{k}(T_S - T_l)\,, \tag{17}$$

where h, T_s, T_l and k are the heat transfer coefficients, the temperature of solid and fluid on the surface and coefficient of heat conduction, resp. On the surface of a porous solid with moisture diffusion and bounded by a fluid, the boundary condition is the following (see Fig. 5). Let us equal the fluxes of diffusion and convection on the surface:

$$-D_m \frac{\partial m}{\partial n}\bigg|_S = h_m(m_S - m_l) \tag{18}$$

or

$$\frac{\partial m}{\partial n}\bigg|_S = -\frac{h_m}{D_m}(m_S - m_l)\,. \tag{19}$$

In normalized form:

$$\frac{\partial m^p}{\partial n}\bigg|_S = -\frac{h_m}{D_m}(m_S^p - m_l^p)\,. \tag{20}$$

The notations are according to the previous parts.

Boundary condition for cross-coupled heat and moisture diffusion can be obtained by following the consideration of the previous part. We can equal the fluxes $\underline{F}$ on both sides of the surface of a porous solid (see Fig. 6)

$$\underline{F} = -\underline{\underline{D}}\,\underline{\nabla}\big|_S = -\underline{\underline{H}}_h\underline{\Delta}\,, \tag{21}$$

where

$$\underline{F} = \begin{bmatrix} f \\ q \end{bmatrix},\ \underline{\underline{D}} = \begin{bmatrix} D_m & D_{mT} \\ D_{Tm} & k \end{bmatrix},\ \ \underline{\nabla} = \begin{bmatrix} \nabla m \\ \nabla T \end{bmatrix},\ \underline{\underline{H}}_h = \begin{bmatrix} h_m & h_{mT} \\ h_{Tm} & h \end{bmatrix},\ \underline{\Delta} = \begin{bmatrix} \Delta m \\ \Delta T \end{bmatrix} \tag{22}$$

are the diffusivity matrix, gradient vector, convectivity matrix and difference vector, respectively.

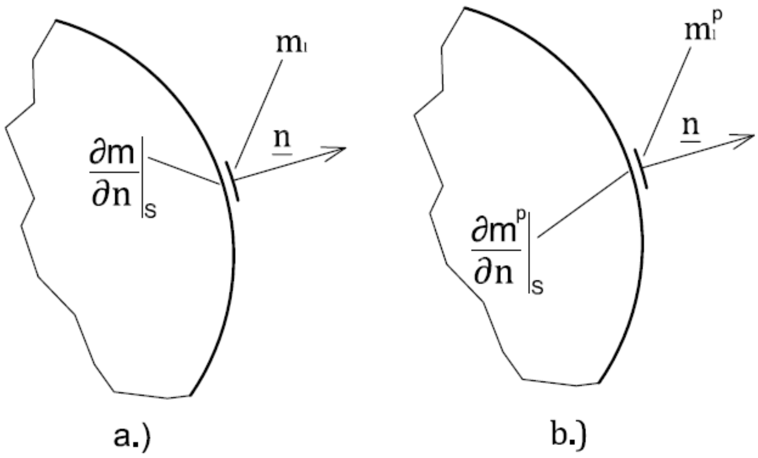

Fig. 5 Boundary condition for moisture diffusion
a.) moisture concentration b.) moisture potential

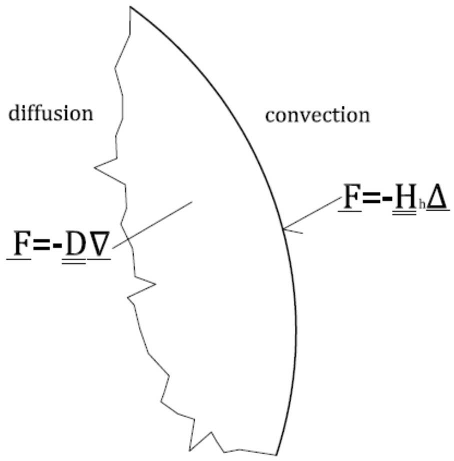

Fig. 6 Boundary condition for cross-coupled heat and moisture diffusion

In components Eqs. (21) take the form:

$$D_m \frac{\partial m}{\partial n}\bigg|_S + D_{mT} \frac{\partial T}{\partial n}\bigg|_S = h_m(m_S - m_l) + h_{mT}(T_S - T_l)\,, \tag{23}$$

$$D_{Tm} \frac{\partial m}{\partial n}\bigg|_S + k \frac{\partial T}{\partial n}\bigg|_S = h_{Tm}(m_S - m_l) + h(T_S - T_l)\,. \tag{24}$$

Coupled thermo-hygro problem with second sound

Let us start with the governing equation of the problem according to eq. (6), but neglecting the source term and using more simple notations:

$$\dot{C}_i + \tau_{ij}\ddot{C}_j = D_{ij}\nabla^2 C_j \tag{25}$$

Here:

$C_i = \begin{bmatrix} m \\ T \end{bmatrix}$ moisture temperature.

$\tau_{ij} = \begin{bmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{bmatrix}$ is the relaxation matrix with: τ_{11}, τ_{22} - temperature and moisture relaxation and τ_{12}, τ_{21} - crosscoupled relaxations.

$D_{ij} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}$ is the diffusivity matrix with: D_{11}, D_{22} - temperature and moisture diffusivities and D_{12}, D_{21} - crosscoupled diffusivities.

In a one-dimensional case eqs. (25) take the form:

$$\dot{m} + \tau_{11}\ddot{m} + \tau_{12}\ddot{T} = D_{11}m_{xx} + D_{12}T_{xx}\,, \tag{26}$$

$$\dot{T} + \tau_{12}\ddot{m} + \tau_{22}\ddot{T} = D_{21}m_{xx} + D_{22}T_{xx}\,. \tag{27}$$

Let us try to find the solution of the 1D problem in exponential form:

$$C_i = C_i^0 e^{j\omega t - \gamma x}\,,\ j = \sqrt{-1}\,. \tag{28}$$

Following the conventional way of the solution, we obtain:

$$\dot{C}_i = j\omega C_i\,,\ \ddot{C}_i = -\omega^2 C_i\,,\ C_{ix} = -\gamma C_i\,,\ C_{ixx} = \gamma^2 C_i\ , \tag{29}$$

$$j\omega I_{ij}C_j - \omega^2\tau_{ij}C_j = D_{ij}\gamma^2 C_j\,, \qquad I_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{30}$$

$$C_j \neq 0 \Rightarrow \det(\,j\omega I_{ij} - \omega^2\tau_{ij} - \gamma^2 D_{ij}) = 0\,, \tag{31}$$

$$\begin{vmatrix} j\omega - \omega^2\tau_{11} - \gamma^2 D_{11} & -\omega^2\tau_{12} - \gamma^2 D_{12} \\ -\omega^2\tau_{21} - \gamma^2 D_{21} & j\omega - \omega^2\tau_{22} - \gamma^2 D_{22} \end{vmatrix} = 0\,. \tag{32}$$

The analysis of equation (32) gives the different cases. The most general is the coupled heat/moisture diffusion (case A), it needs to solve a fourth order equation. Let us see the subcases (case B).

B11.) Simple heat conduction with second sound

It means the following parameters:

$$\tau_{11} = \tau_{21} = \tau_{12} = 0\,,\;\; D_{11} = D_{21} = D_{12} = 0\,. \tag{33}$$

By these the eq. (32) reads:

$$\begin{vmatrix} j\omega & 0 \\ 0 & j\omega - \omega^2\tau_{22} - \gamma^2 D_{22} \end{vmatrix} = 0\,, \tag{34}$$

and finally we obtain

$$\gamma^2 = j\frac{\omega}{D_{22}} - \omega^2\frac{\tau_{22}}{D_{22}}\,. \tag{35}$$

B12.) Simple heat conduction without second sound

The previous conditions and $\tau_{22} = 0$, and by this

$$\begin{vmatrix} j\omega & 0 \\ 0 & j\omega - \gamma^2 D_{22} \end{vmatrix} = 0\,, \tag{36}$$

and instead of eq. (35)

$$\gamma^2 = j\frac{\omega}{D_{22}}\,. \tag{37}$$

B21.) Simple moisture diffusion with second sound

It means the following parameters:

$$\tau_{22} = \tau_{21} = \tau_{12} = 0\,,\;\; D_{22} = D_{21} = D_{12} = 0\,. \tag{38}$$

By these similarly to eq. (35)

$$\gamma^2 = j\frac{\omega}{D_{11}} - \omega^2\frac{\tau_{11}}{D_{11}}\,. \tag{39}$$

B22.) Simple moisture diffusion without second sound

The previous conditions and $\tau_{11} = 0$ and by this, finally

$$\gamma^2 = j\frac{\omega}{D_{11}}\,. \tag{40}$$

B3.) Heat conduction and moisture diffusion without coupling

It means the following parameters:

$$\tau_{21} = \tau_{12} = 0 , \quad D_{21} = D_{12} = 0 . \tag{41}$$

By these the eq. (32) reads

$$\begin{vmatrix} j\omega - \omega^2\tau_{11} - \gamma^2 D_{11} & 0 \\ 0 & j\omega - \omega^2\tau_{22} - \gamma^2 D_{22} \end{vmatrix} = 0 , \tag{42}$$

or

$$(j\omega - \omega^2\tau_{11} - \gamma^2 D_{11})(j\omega - \omega^2\tau_{22} - \gamma^2 D_{22}) = 0 , \tag{43}$$

and it follows

$$(j\omega - \omega^2\tau_{11} - \gamma^2 D_{11}) = 0 , \tag{44}$$

and

$$(j\omega - \omega^2\tau_{22} - \gamma^2 D_{22}) = 0 . \tag{45}$$

Now let us calculate a temperature with second sound (or moisture) diffusion in details, according to previous sections (simple heat conduction and moisture diffusion with second sound). The governing equation with the conventional notations is the following (see eq. (15)):

$$T_{xx} = \frac{\tau_T}{D_T} T_{tt} + \frac{1}{D_T} T_t . \tag{46}$$

Repeating the conventional way of solution followed by eqs. (28-35) we obtain the propagation coefficient, γ (see eq. (35)) with slightly different notations:

$$\gamma^2 = \frac{\omega}{D_T}(-\omega\tau_T + j) . \tag{47}$$

Let us suppose γ in the following form:

$$\gamma = \alpha + j\beta , \tag{48}$$

where $\alpha > 0, \beta > 0$.

Then

$$T^+ = T_0^+ e^{j\omega t - (\alpha + j\beta)x} = T_0^+ e^{-\alpha x} e^{j(\omega t - \beta x)} . \tag{49}$$

By introducing the propagation velocity, v as:

$$v = \frac{\omega}{\beta} , \tag{50}$$

we obtain the thermal wave propagating in positive direction of the x-axis:

$$T^+ = T_0^+ e^{-\alpha x} e^{j\omega(t - \frac{x}{v})} . \tag{51}$$

By replacing x by $-x$ in (51) we obtain the thermal wave propagating in negative direction of the x-axis:

$$T^- = T_0^- e^{\alpha x} e^{j\omega(t+\frac{x}{v})} . \tag{52}$$

A general solution is the following:

$$T(x,T) = T^+ + T^- = T_0^+ e^{-\alpha x} e^{j\omega(t-\frac{x}{v})} + T_0^- e^{\alpha x} e^{j\omega(t+\frac{x}{v})} . \tag{53}$$

Let us calculate α and β by eqs. (35, 48):

$$\alpha + j\beta = \gamma = \sqrt{\frac{\omega}{D_T}\left(-\omega\tau_T + j\right)} . \tag{54}$$

From (54) we obtain:

$$\alpha^2 = \frac{\omega}{2D_T} \frac{1}{\omega\tau_T \pm \left((\omega\tau_T)^2 + 1\right)^{\frac{1}{2}}} , \tag{55}$$

$$\beta^2 = \frac{\omega^2\tau_T \pm \omega\left((\omega\tau_T)^2 + 1\right)^{\frac{1}{2}}}{2D_T} . \tag{56}$$

In the Table 2 the initial data and the parameters are collected. By these the numerical solution of the heat (or moisture) diffusion can be calculated. On the Fig.7 the results are displayed.

Table 2 Initial data and parameters to heat (or moisture) diffusion by analytical method

Initial data		**Parameters can be calculated**	
ω	frequency	α	damping coeff.
τ_T	(thermal) relaxation	β	phase angle (dimensoin: 1/m)
D_T	(thermal) diffusivity	$\gamma = \alpha + j\beta$	propagation coeff.
		$v = \frac{\omega}{\beta}$	propagation velocity
		$\Lambda = \frac{2\pi}{\beta}$	wavelength
		$T_w = \frac{2\pi}{\omega}$	period (wavelength in time)

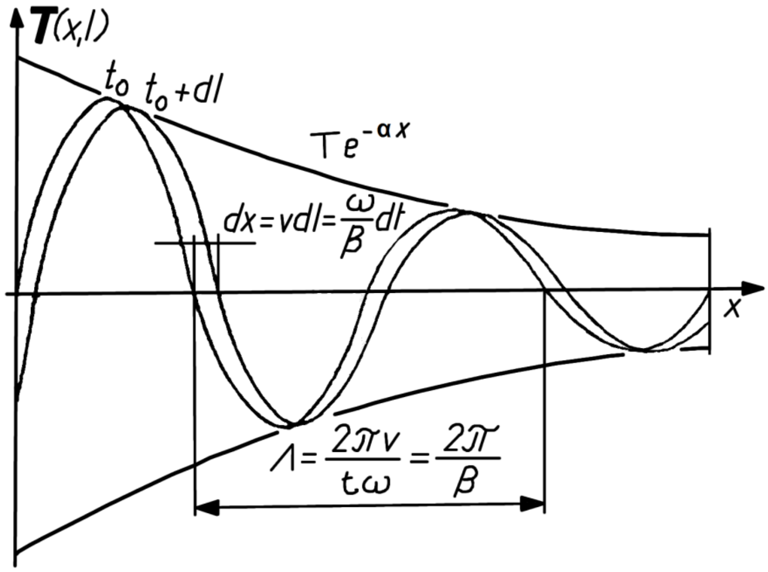

Fig. 7/a Temperature T (or moisture) vs. location, x

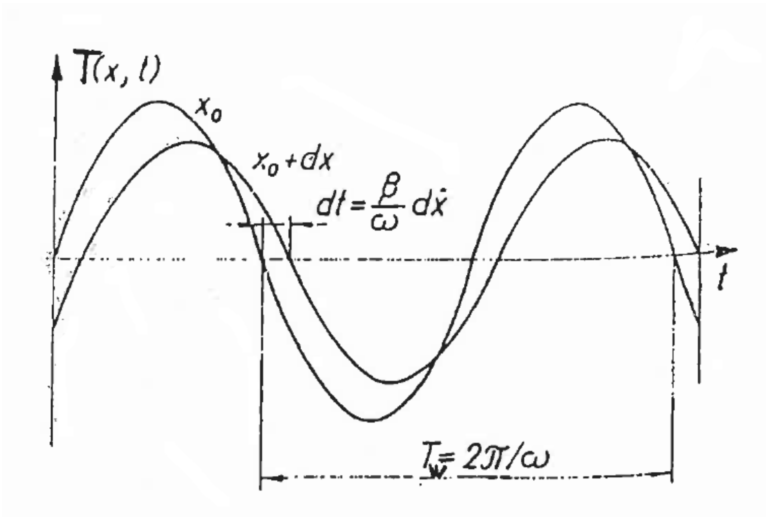

Fig. 7/b Temperature T (or moisture) vs. time, t

6 Experiments

Our experiments are to achieve different goals. One of them was to model the entire phenomenon. In this case the result is embodied by the functions of temperature, T(x,t) and moisture concentration, m(x,t). Choosing proper parameters, ie. different material and process properties, we may obtain information on the influence of these parameters.

An other purpose was to model parts of the process and gain the numerical values of the different parameters, such as relaxation times, τ and diffusivities, D, etc. In a summary, there are mainly two goals: check the entire theory and complete it e.g. by the material properties. According to all of these, this chapter deals with different kinds of experiments. We analyze the experimental possibility by EA to determine the relaxation time [7], diffusivities [14], and a pressure in a hydroglobe [15, 16]

Most of the details of the theoretical background and of the experiments are described in the previous sections.

Before dealing with some of the details of the above-mentioned experiments, we try to give a short summary on the philosophy of the experiments with emphasis on our field, i.e. on THM.

Generally on Experiments

Let us start with the Leonardo da Vinci thoughts: (i) There is no real science without a mathematical proof, and (ii) There is no real science without experiments. In our field the experiment is a special kind of experience. It is close to the practice, but not as close as e.g. the field measurement. The steps of the cognition on our field (THM) can be divided according to the Table 3. Of course, not all of these steps belong always to the proper branch of science, some of them may be left out[1].

Table 3 Possible steps of the cognition in THM

1.	Observation
2.	Theory
3.	Thought experiment: – EA (electrical analogy) – Analytical solution – Numerical solution
4.	Laboratory experiment, e.g.: – measurement by EA – generally by measurements
5.	Field experiment, e.g.: – measurements – observations
6.	Practice

Dealing with the experimental possibilities given by the EA, let's ponder again a little bit generally on the EA. In our case it offers triple-fold advantages:

- theoretical by giving deeper insight into the phenomenon,
- analytical solution,
- experimental possibility.

Some of the researchers have various opinion on the advantages mentioned above. E.g. they say that the experimental possibility given by the EA can be dropped, because it may be replaced by numerical calculations. Also, the analytical results after some hesitation may be dropped, because we already have got numerical results. Unfortunately, the theoretical possibility can't be replaced by any other tool since a theory is needed not only as a state of art but also as an instrument complementary to the experiment. If an experiment has no theoretical background, a proper evaluation of the results is impossible as said in the Einstein's joke about the flea's leg and ear.

Let us mention one more comment to the problem. Several years ago, on a conference somebody referred to the FEM calculations as experimental results. And the audience agreed, while I (the older author, A. Szekeres) was totally shocked.

[1] It may happen that no possibility exists to make experiment. Eg. for the religion as science there is no opportunity for experiments. Interesting example is Thornton Wilder's book entitled Bridge of King Saint Louis. In that story the priest takes the event as god's gift for human experiment. This idea together with the Da Vinci quotation was brought up first in a discussion by P.E. ca. 30 years ago.

Experiments by Electrical Analogy (EA)

As it has been mentioned before, the EA provides us with wide range of experimental possibilities. The cases dealt with in the chapter Electrical Analogy are collected in the Table 4. and completed with some comments.

We have to analyze the possible extensions of the EA for the following coupled phenomena:

- Thermo-hygro-elasticity (THE)
- Thermo-hygro-visco-elasticity (THVE)
- Cross-coupled thermo-hygro diffusion with and without Second Sound (SS)
- Convection of heat and moisture
- Cross-coupled convection of heat and moisture
- Coupled diffusion and convection of cross-coupled heat and moisture (general case, ie. C-C H&M T)

Table 4 Experiments by EA

Case	Comment
Thermoelasticity (TE)	See proper chapter and the Fig.1
Hygro-elasticity (HE)	Based on the analogy between Fourier and Fick's law.
Thermo-visco-ealsticity (TVE) and hygro-visco-elasticitiy (HVE)	Based on the previous cases taking into account the time dependence of the material properties.
Heat conduction with and without the second sound phenomennon	See the proper chapter, the Fig. 2 and Fig. 4 and the remarks on the SS problem (hyperbolic/parapolic type of eq.)
Moisture diffusion with and without the second sound phenomennon	See above.

Experiment on Relaxation Time

The eqs. (6, 15, 25, 34) and the Table 2. show the role and importance of the τ_T relaxation time. (For more details, please, see the proper chapter of [1] and [7].)

According to the investigation of the differential equation of heat conduction (16) obtained on the basis of Fourier's law the rate of wave propagation is infinite. This is obviously impossible.

To resolve this contradiction, the law of heat conduction can be modified. One possible form of the modified heat-conduction equation is the Cattaneo-Vernotte equation which can be written as:

$$T_{xx} = \frac{1}{D_T} T_t + \frac{\tau_T}{D_T} T_{tt}, \tag{57}$$

of which the velocity of propagation:

$$v_t = \sqrt{\frac{D_T}{\tau_T}}. \tag{58}$$

Here the question arises, how thermoelasticity of a long bar subject to a thermal shock is affected by this.

However, to investigate this problem, material property has to be known. Maurer [17] has determined the value of τ_T.

If the velocity of propagation v_T of the temperature disturbance could be determined, then, according to (58)

$$\tau_T = \frac{D_T}{v_T^2}, \tag{59}$$

where D_T is the so called temperature diffusivity, a known material property.

Accordingly, when measuring time t_2 required for the disturbance to propagate over distance l_2 in the setup shown in Fig. 8, the value of τ_T can be calculated with the following relationship:

$$\tau_T = \frac{D_T \cdot t_2^2}{l_2^2}. \tag{60}$$

With this setup, the time constant of the experimental setup had been reduced by ~1 order of magnitude and, as a result, the time constant has become about 10^{-2} sec.

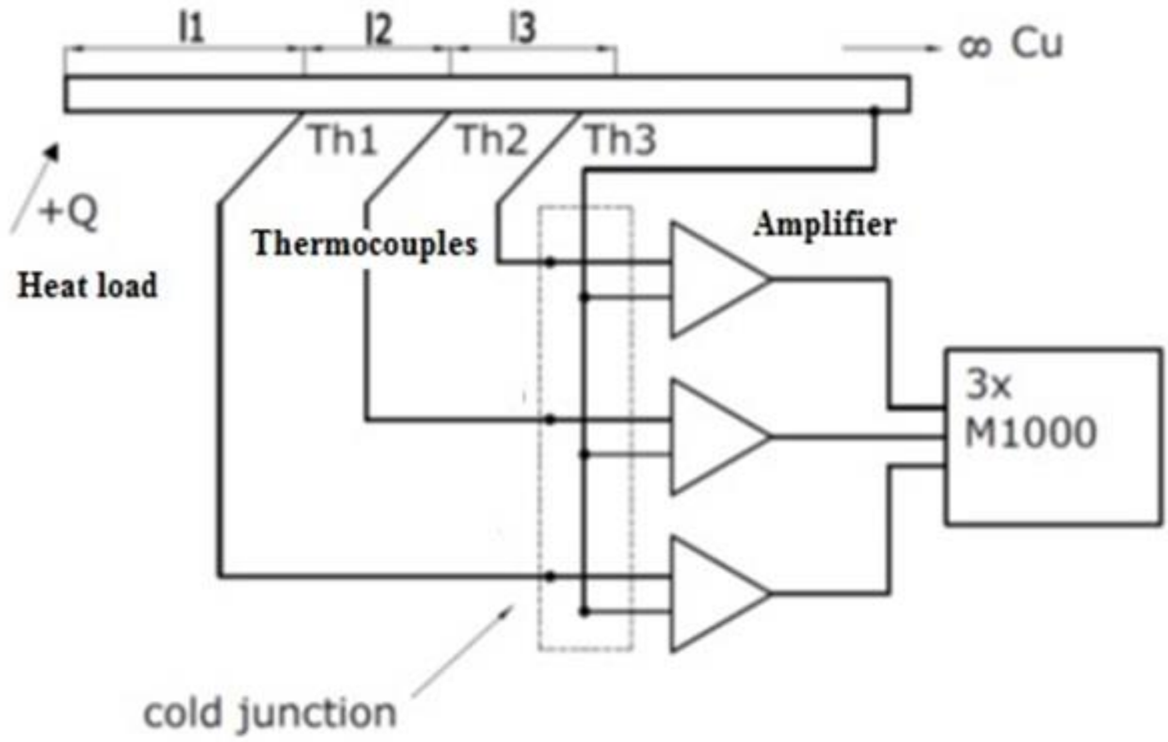

Fig. 8 Test setup for relaxation time

In spite of all the uncertainties, it can be seen from the measuring results (Fig. 9) that the value of τ_T is of the order of magnitude of 10^{-1} sec, which considerably differs from the value given in literature [17].

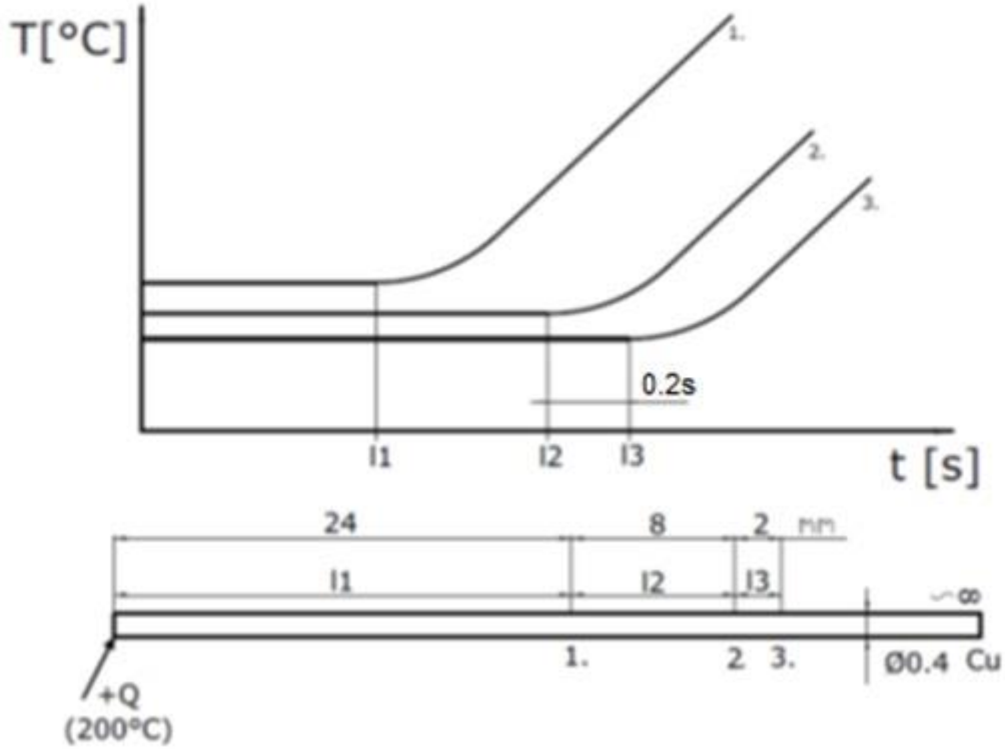

Fig. 9 Experimental results on relaxation time

This suggests at the same time that a considerable deviation of the results obtained with the modified law of heat conduction from those obtained using classical Fourier's law is expectable when applying this law to the dynamic problems of thermoelasticity.

Experiment on Diffusivities

Dealing with the tailoring of thermo-hygro-elastic composites four diffusivities arise, which are somehow related depending all of them on the microstructure of the material, but two of them are equal according to the Onsager's reciprocal relations.

The experiment on diffusivities was performed with a rod type specimen made of heat conducting and moisture absorbing composite matrix material [14]. The conditions modeled thermal shock, perfect insulation and different moisture content. The method was a combined one. Based on our theoretical works [1] we performed numerical calculations. Concerning the general form of the governing equation we refer to [10]. We get the final numerical values of parameters by comparison of experimental and numerical results.

There were two major questions concerning the experiment. First, how we are able to fulfill the ideal assumptions of the theoretical model, i.e. step loading and perfect insulation of test pieces. By doing some trials we found out a quite simple setup to do tests. Another question was to find a suitable material for experiments, a plastic which is at the same time appropriate heat conductor and a good water absorber. Polyamide seems to fulfill the requirements best, therefore the experiments were made with PA 6. The experimental setup is shown in Fig. 10.

The temperature and displacement responses were measured and calculated for different places and moisture contents. To estimate the coupled diffusivities and expansion coefficients, the values were compared. In spite of the characteristic qualitative results, the numerical separation of diffusivities and expansion coefficients was impossible.

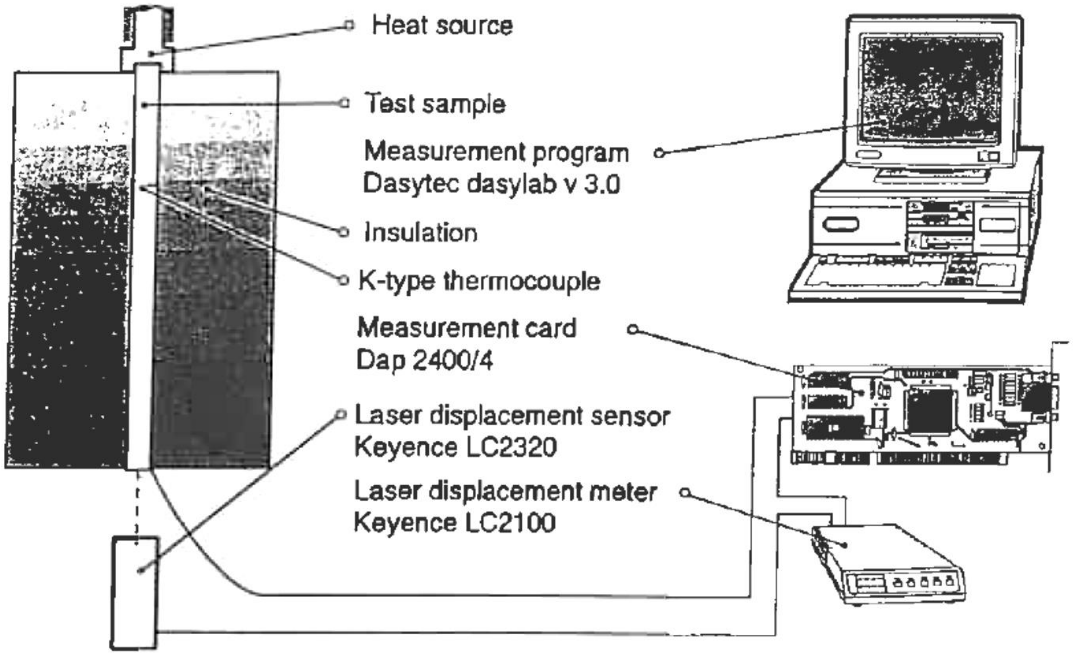

Fig. 10 Experimental setup for measure of diffusivity

Experiment on Hydroglobe

A hydroglobe (water container) can be modeled as a sphere [15, 16]. At low temperature there is a risk of freezing so that the container deteriorates. Let us take the model in Fig.11. Suppose that initially the container is filled up with water at 0 °C. Then, an ice layer at 0 °C, of thickness v_j is formed, so that no stress develops, as the filled-up condition permits a cubic expansion. Subsequently the ice layer is cooled, and – according to the supposed stationary state – there is a temperature distribution such that stress develops.

According to the negative coefficient of thermal expansion, an increase in diameter will come about, causing stresses. Imposing now the steel container on this ice layer of increased diameter, a compressive stress will develop between the two surfaces. The state of stress is a resultant of the two states. The change in the ice thickness:

$$\Delta v_j = u\left(\frac{d}{\mathbf{2}}\right) = \frac{C_j}{2E_j}\left[\left(1\text{-}2\mu_j\right)\left(\frac{d_j^{\ 2}}{4}\rho_{\mathbf{1}j} - 1\right) + \frac{d_j^{\mathbf{2}}}{4}\rho_j\frac{1+\mu_j}{2} + \frac{1\text{-}3\mu_j}{2}\right] + \alpha_j\frac{d}{2}\Delta t\,. \qquad (61)$$

Here u, and d are the displacement and the internal diameter of the container, E_j, μ_j, α_j, and d_j are the Young-modulus, Poisson-number, thermal expansion coefficient and internal diameter of the ice layer, $C_j = \frac{2\alpha_{\mathbf{1}j}}{1-\mu_j}\Delta t\rho_j r_b$, where $\alpha_{\mathbf{1}j} = E_j\alpha_j$ and $\rho_j = \frac{r_k}{r_k - r_b}$, r_k and r_b are the internal and external radii of the ice layer, Δt is the relative temperature.

From this overlapping the thin-walled steel container takes up:

$$w_1 = p\frac{d^2}{8vE}(1-\mu) = pw_1' \tag{62}$$

and the ice sphere:

$$w_2 = -p\frac{d}{2E_j}\left\{\frac{\left(\frac{d_j}{d}\right)^3 + 2}{2\left[1-\left(\frac{d_j}{d}\right)^3\right]}\left(1-\mu_j\right)-\mu_j\right\} = -pw_2' . \tag{63}$$

Here p is the contact pressure, v, E and μ are the thickness, Young-modulus and the Poisson-number of the container.

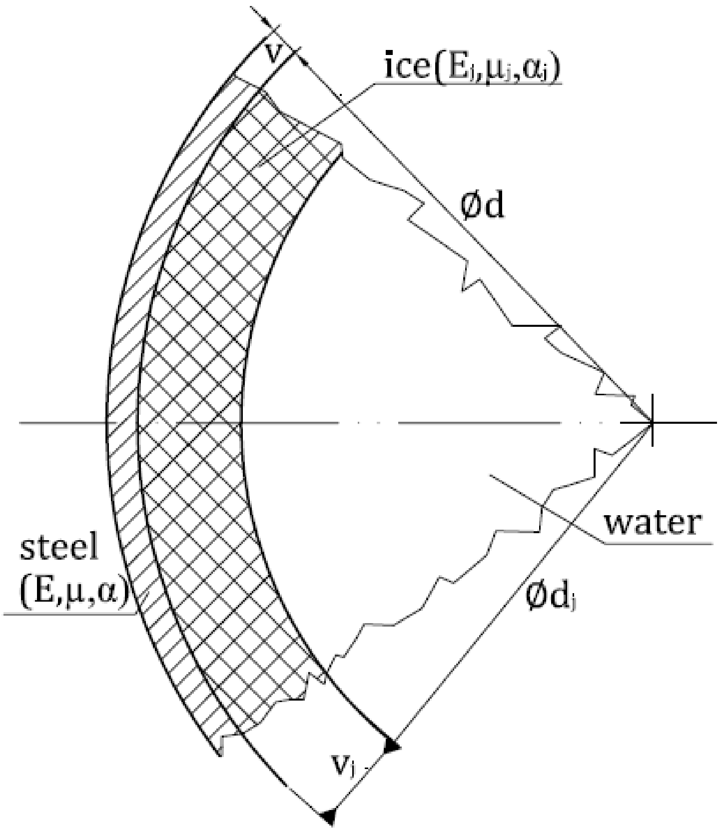

Fig. 11 Cross section of hydroglobe with ice layer

The pressure evolving between the two sphere is:

$$p = \frac{\Delta v_j}{w_1' - w_2'} \tag{64}$$

From the above the resultant stresses can be derived. According to calculations carried out with the above relationships, the ice sphere is in an ultimate stress state at a v_j value where no ultimate stresses develop as yet in the steel container. This method has been checked by ultrasonic and strain gauge tests.

To validation the above mechanical model, measurements were carried out by a scale model of the hydroglobe. The placements of the gauges were according to the Fig. 12. The character of the calculated strain-time curve and the measured strain-time curve were similar.

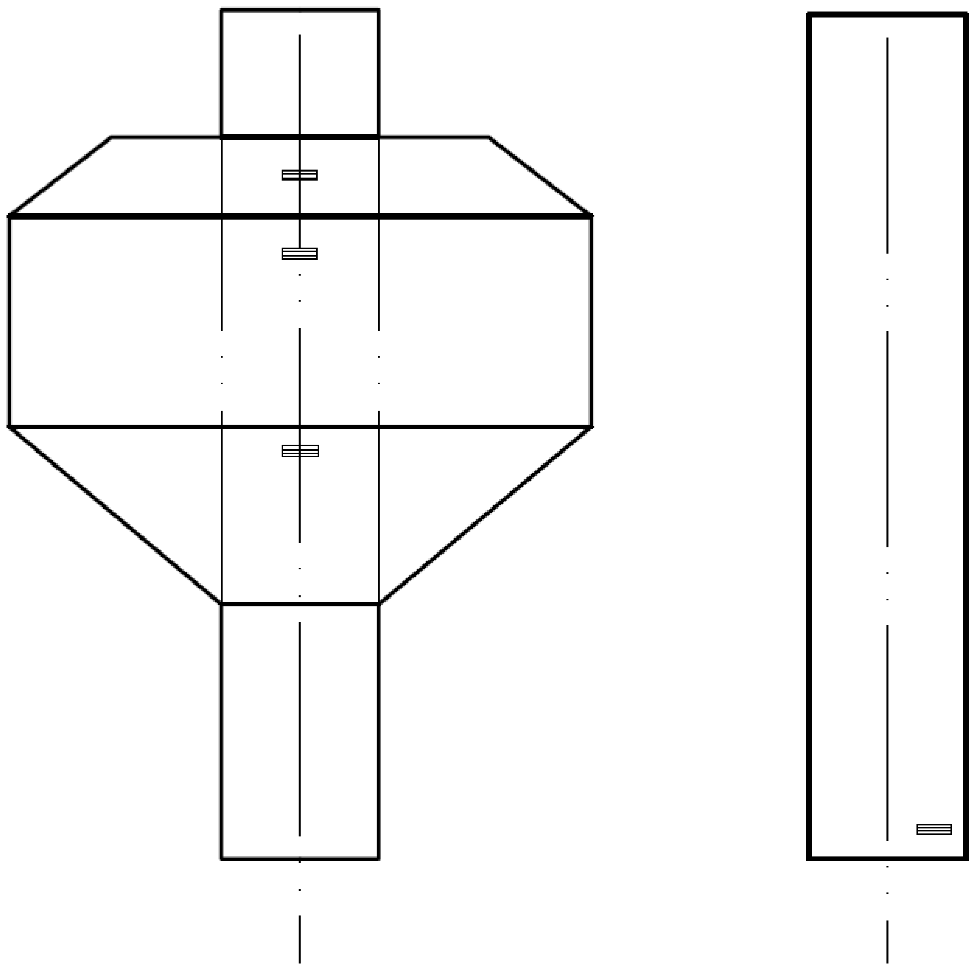

Fig. 12 Positions of strain gauges on hydroglobe

CONCLUSIONS

This is a continuation of Part I in which the one-dimensional dynamic coupled field theories such as: (i) a heat-moisture theory, (ii) a thermo-elasticity theory, and (iii) an electric current-voltage theory are discussed within an analogy between the governing equations for a pair of the coupled field theories; for example, it is shown that both the analytical and numerical methods of electric current-voltage theory can be applied to obtain the analytical and numerical solutions to problems of heat-moisture theory or thermo-elasticity theory. Also, it is shown that a number of experiments based on electric current-voltage theory could be used to find the material properties of heat-moisture theory or thermo-elasticity theory models. In particular, the experiments based on electric current-voltage theory are proposed to determine: (i) a relaxation time of a restricted model of thermo-elasticity theory in which the temperature propagates with a finite speed, (ii) a diffusivity of a moisture transport in a restricted model of heat-moisture theory, and (iii) the pressure formula for a spherical hydroglobe.

REFERENCES

[1] A. Szekeres. Cross-Coupled Heat and Moisture Transport Part 1 – Theory. *Journal of Thermal Stresses* **2012** (35), 1 – 21.

[2] P. Kraus, J. Úradníček, M. Musil, M. Bachratý. Effective Thermo-Structural Analysis for Disc Brake System Simulation. *Journal of Mechanical Engineering – Strojnícky časopis* **2018** (68), No. 2, 125 – 130. DOI: 10.2478/scjme-2018-0022

[3] M. Repka, J. Sládek, V. Sládek, M. Wünsche. Evaluation of Fracture Parameters for Cracks in Coupled Thermoelasticity for Functionally Graded Materials. *Journal of Mechanical Engineering – Strojnícky časopis* **2015** (65), No. 1, 57 – 76. DOI:10.1515/scjme-2016-0004

[4] A. Nadai. Theory of Flow and Fracture of Solids, McGraw-Hill Book Company, Inc. New York-Toronto-London, **1950**.

[5] A. Fick. Über Diffusion. *Ann. der Phys.* 94 **1855**, 59 – 86,

[6] K. Simonyi. Theoretical Electricity, Tankonyvkiado Budapest, **1973**.

[7] M. Szalontay, A. Szekeres. Experiments on Thermal Shock of Long Bars. Periodica Polytechnica. Mech. Eng. 1980 (24), No. 3, 243 – 251.

[8] A. Krach. Advani S. G., Influence of Void Shape, Void Volume and Matrix Anisotropy on Effective Thermal Conductivity of a Three-Phase Composite. *Journal of Composite Materials* **1996** (30), No. 8, 933 – 946.

[9] A. Szekeres. Composite Structures with Hygroscopic Element, Pres. held at CASS/CCMS Fifth Technical Conference, Oct. 13-15, , Virginia Tech, Blacksburg, VA, **1996**.

[10] A. Szekeres, J. Engelbrecht. Coupled Thermal and Moisture Fields with Application to Composites. *Periodica Polytechnica Ser. Mech. Eng.* 1997 (41), No. 2, 151 – 161.

[11] R. Heller, A. Szekeres. Thermodynamics of Complex Systems: Special Problems of Coupled Thermal and Moisture Fields and Application to Tailoring of Composites. *Periodica Polytechnica Ser. Chem. Eng.* **1998** (42), No. 1, 55 – 64.

[12] M. Hetenyi. Handbook of Experimental Stress Analysis, John Wiley & Sons, Inc., New York–London, **1960**.

[13] A. Szekeres. Analogy between heat and moisture Thermo-hygro-mechanical tailoring of composites by taking into account the second sound phenomenon. Computers and structures **2000** (76), 145 – 152.

[14] A. Szekeres, H. Koivurova. Theoretical and Experimental Investigation on the Diffusivities of Composites. *Periodica Polytechnica, Ser. Mech. Eng.* **1997** (41), No. 2, 143 – 150.

[15] A. Szekeres. Strength Analysis of a Thick-Walled Spheres in Spheroid Symmetrical Thermal State. *Periodica Polytechnica* **1973** (17), No. 1, 91 – 94.

[16] A. Szekeres. Festigkeitsuntersuchung bei niedriger Temperatur eingesetzter Hydroglobusse. *Periodica Polytechnica* **1979** (23), No. 1, 91 - 104.

[17] M. J. Maurer. Relaxation Model of Heat Conduction in Metal. *Journal of Applied Physics* **1969** (40), No. 13, 10 – 18.