Games and Puzzles

Reflections on the $n + k$ dragon kings problem

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Abstract: *A* dragon king *is a shogi piece that moves any number of squares vertically or horizontally or one square diagonally but does not move through or jump over other pieces. We construct infinite families of solutions to the $n + k$* dragon kings problem *of placing k pawns and $n + k$ mutually nonattacking dragon kings on an $n \times n$ board, including solutions symmetric with respect to quarter-turn or half-turn rotations, solutions symmetric with respect to one or two diagonal reflections, and solutions not symmetric with respect to any nontrivial rotation or reflection. We show that an $n + k$ dragon kings solution exists whenever $n \geqslant k + 5$ and that, given some extra conditions, symmetric solutions exist for $n \geqslant 2k + 5$.*

Keywords: shogi, combinatorics, symmetry, n-queens problem.

Introduction

There are many problems in the mathematics literature that involve placing chess pieces on a board under given constraints. For example, the classic *n queens problem* asks for placements of n queens on an $n \times n$ board so that no two queens are on the same row, column, or diagonal [1]. The *$n + k$ queens problem* asks for placements of k pawns and $n + k$ queens on an $n \times n$ board so that each pair of queens on the same row, column, or diagonal has at least one pawn between them [5]. The n queens problem has a solution for $n = 1$ and each $n \geqslant 4$ [1]. For $k = 1, 2, 3$, the $n + k$ queens problem has a solution for each $n \geqslant k + 5$. For $k \geqslant 4$, the $n + k$ queens problem is known to have solutions for $n > 25k$, but it is suspected that $25k$ is much larger than the true lower bound [5].

In this paper we consider a simpler variation of the $n + k$ queens problem where each queen is replaced by a "dragon king", which is a shogi piece that moves any number of squares vertically and horizontally (but not through or over other pieces) and one square diagonally; i.e., like a combination of the chess king and rook [2]. So we consider the $n+k$ *dragon kings problem*: On an $n \times n$ board, can we place k pawns and $n+k$ dragon kings on the board so that no two dragon kings "attack" each other (i.e., no dragon king is one move away from any other dragon king)? We show in Section 1 that for each $k \geqslant 0$ and $n \geqslant k+5$, we can construct a solution to the $n+k$ dragon kings problem.

Many placements of pieces on a square board are symmetric with respect to rotation or reflection. In Section 1 we explore symmetric solutions to the $n+k$ dragon kings problem, and show that, given some extra conditions, solutions in each of the possible symmetry classes exist for $n \geqslant 2k+5$.

We conclude in Section 2 with a discussion of open questions and other avenues for further study.

In this paper all boards are square, with rows numbered, from top to bottom, $0, 1, \ldots n-1$ and columns numbered, from left to right, $0, 1. \ldots n-1$. The square in row r and column c of a board is denoted as *square* (r, c). In the figures of this paper, dragon kings are represented by ☖, pawns are represented by ☗, and squares that we wish to emphasize are empty are marked by $\circ$.

1 Results

In [5], a connection was established between the $n+k$ queens problem and *alternating sign matrices (ASMs)*, which are matrices consisting of 0s, 1s, and -1s where the nonzero elements alternate in sign and the first and last nonzero element of each row and column is a 1 [3]. We note a similar connection for the $n+k$ dragon kings problem. Suppose we have an arrangement of $n+k$ mutually nonattacking dragon kings on an $n \times n$ board with k pawns. If there are no pawns in a row (or column), that row (column) can have at most one dragon king. With each additional pawn on a row (column), the capacity of that row (column) increases by at most one. With k pawns, we can place at most $n+k$ mutually nonattacking dragon kings. Furthermore, to get that capacity of $n+k$ dragon kings, each available segment of each row and column must be occupied by a dragon king. So, the pieces in each row must alternate between dragon king and pawn and the first and last piece in each row or column must be a dragon king. Take a solution and create an $n \times n$ matrix A so that the entry a_{ij} in the i^{th} row and j^{th} column is 1 if there is a dragon king in the i^{th} row and j^{th} column of our solution, -1 if there is a pawn in the i^{th} row and j^{th} column of our solution, and 0 if the i^{th} row and j^{th} column of our solution is empty. That matrix is an alternating sign matrix. Solutions to the $n+k$ dragon kings problem correspond to ASMs for which no two 1s are diagonally adjacent.

In [5, Theorem 2] it was proved that for $n > 1$, no arrangement of k pawns and $n+k$ mutually nonattacking queens is symmetric with respect to reflection. The argument made against vertical and horizontal reflection works for dragon kings as well as queens, and we repeat that argument here.

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Proposition 1. *(c.f. [5, Theorem 2]) No solution to the $n+k$ dragon kings problem (where $n > 1$) is symmetric with respect to vertical or horizontal reflection.*

Proof: Suppose we have a solution that is symmetric with respect to reflection across a vertical mirror. If n is even, then the dragon kings in the central columns must be adjacent and therefore attacking. If n is odd, then each square of the central column must be occupied; otherwise, the pieces closest to an empty square in the central column must be identical, contradicting the fact that the pieces in each row must alternate. The central column must have a dragon king on its first, third, and every other subsequent row. But then every square in the columns adjacent to the central column are attacked, and we cannot place a dragon king in those columns, contradicting the fact that each column must have at least one dragon king.

Switching each "row" and "column" in the previous argument gives us the argument against symmetry with respect to reflection across a horizontal mirror. ■

However, the argument against diagonal reflection fails. In Figure 4 we present an example of a $7+1$ dragon kings arrangement that is symmetric with respect to diagonal reflection.

We can therefore partition the set of $n+k$ dragon kings problem solutions into five symmetry classes:

1. *ordinary* arrangements that are not symmetric with respect to any (nontrivial) rotation or reflection. Figure 1 is ordinary.

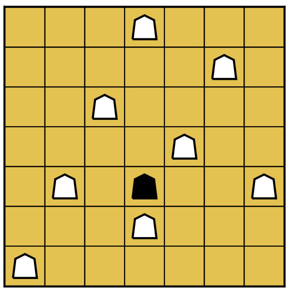

Figure 1: An ordinary $7+1$ dragon kings arrangement.

2. *centrosymmetric* arrangements that are symmetric with respect to half-turn rotation, but not quarter-turn rotations or any reflections. Figure 2 is centrosymmetric.

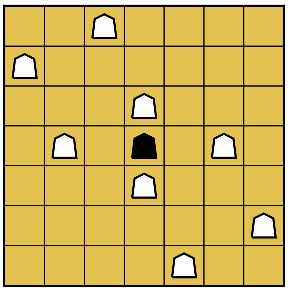

Figure 2: A centrosymmetric $7 + 1$ dragon kings arrangement.

3. *doubly centrosymmetric* arrangements that are symmetric with respect to quarter-turn rotations, but not with respect to reflections. Figure 3 is doubly centrosymmetric.

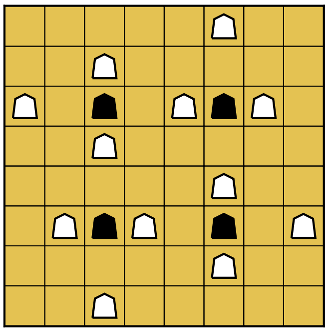

Figure 3: A doubly centrosymmetric $8 + 4$ dragon kings arrangement.

4. *monodiagonally symmetric* arrangements that are symmetric with respect to reflection across either the "main diagonal" that goes from the upper-left corner of the board to the lower-right corner or the "main antidiagonal" that goes from the lower-left corner of the board to the upper-right corner, but not both, and are also not symmetric with respect to nontrivial rotations. Figure 4 is monodiagonally symmetric.

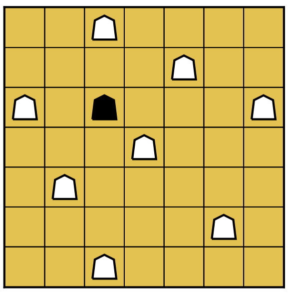

Figure 4: A monodiagonally symmetric $7 + 1$ dragon kings arrangement.

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5. *bidiagonally symmetric* arrangements that are symmetric with respect to reflection across both the main diagonal and main antidiagonal (and are therefore also symmetric with respect to half-turn rotations, but not quarter-turn rotations). Figure 5 is bidiagonally symmetric.

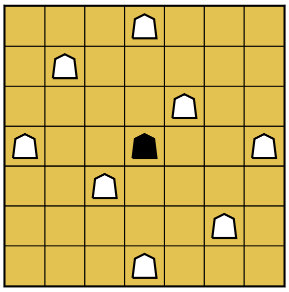

Figure 5: A bidiagonally symmetric $7+1$ dragon kings arrangement.

For each of these symmetry classes we construct families of solutions to the $n+k$ dragon kings problem and demonstrate that for each $k \geqslant 0$, (given extra conditions for the centrosymmetric, doubly centrosymmetric and bidiagonally symmetric classes) solutions exist for n large enough.

1.1 Ordinary solutions

We show by induction that there is at least one ordinary solution for each $k \geqslant 0$ and $n \geqslant k+5$. We start with two lemmas that will be used for the induction steps.

Lemma 2. *Given some $n \geqslant 5$ and $k \geqslant 0$, suppose we have a solution to the $n+k$ dragon kings problem with unoccupied squares $(0,0), (n-1,0), (n-1,n-2), (n-1,n-1), (n-2,n-1)$, and $(n-3,n-1)$. Then we can construct a solution to the $(n+1)+(k+1)$ dragon kings problem, with unoccupied squares $(0,0), (n,0), (n,n-1), (n,n), (n-1,n)$, and $(n-2,n)$.*

Proof: Add a new row n and column n to the given board and then place a pawn in square $(n-2,n-1)$ and dragon kings in squares $(n-2,n)$ and $(n,n-1)$, as illustrated in Figure 6. The new dragon kings do not attack each other or any of the previously placed dragon kings. We have an $(n+1)\times(n+1)$ board with $k+1$ pawns and $n+k+2$ mutually nonattacking dragon kings.

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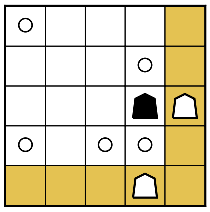

Figure 6: Illustration of first steps of the construction in Lemma 2. We add the shaded row and column to an ordinary arrangement and add two dragon kings and a pawn as shown.

Rotate the board a quarter-turn counterclockwise and relabel the squares so for each $0 \leqslant x, y \leqslant n$, square (x, y) becomes $(n-y, x)$, as illustrated in Figure 7. The empty squares that were $(0, n)$, $(0, 0)$, $(n-1, 0)$, $(n, 0)$, $(n, 1)$ and $(n, 2)$ are now labeled $(0, 0)$,$(n, 0)$,$(n, n-1)$, (n, n), $(n-1, n)$, and $(n-2, n)$, respectively. ■

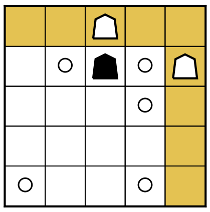

Figure 7: Board from Figure 6 after a quarter-turn rotation counterclockwise.

Lemma 3. *Given some $n \geqslant 1$ and $k \geqslant 0$, suppose we have an $n \times n$ board with an ordinary arrangement of k pawns and $n+k$ mutually nonattacking dragon kings. Then we can construct an ordinary arrangement of k pawns and $(n+1)+k$ mutually nonattacking dragon kings on an $(n+1) \times (n+1)$ board.*

Proof: First we show that at least two diagonally opposite corner squares are empty. Recall from the beginning of the section that our solution corresponds to an alternating sign matrix. In an alternating sign matrix, the first row (respectively, last row, first column, last column) has at most one 1 and no -1s [3]. So either the first or last entry in the first row (column) is a 0 and the corresponding square is empty. So at least two diagonally opposite corner squares must be empty.

Without loss of generality, suppose squares $(0, 0)$ and $(n-1, n-1)$ are empty. Then add a row n and column n and place a dragon king in the new square (n, n). We now have an $(n+1) \times (n+1)$ board with k pawns and $(n+1)+k$ mutually nonattacking dragon kings. Squares $(0, 0)$, $(0, n)$, and $(n, 0)$ are empty, so the arrangement is not symmetric with respect to quarter-turn or half-turn rotations or reflections across the diagonal from $(n, 0)$ to $(0, n)$. Also, the arrangement is not symmetric with respect to reflection across the diagonal from $(0, 0)$ to (n, n) since the upper-left $n \times n$ block is ordinary. ■

Proposition 4. *For $k \geqslant 0$ and $n \geqslant k+5$, we can construct an ordinary solution to the $n+k$ dragon kings problem.*

Proof: First we show by induction on k that for $k \geq 0$ and $n = k+5$, we can place k pawns and $n+k$ mutually nonattacking dragon kings on an $n \times n$ board. For the base case of $k = 0$, take a 5×5 board and place dragon kings on squares $(2,0)$,$(1,2)$,$(0,4)$,$(3,3)$, and $(4,1)$. These pieces do not attack each other, and squares $(0,0)$, $(4,0)$, $(4,3)$, $(4,4)$, $(3,4)$ and $(2,4)$ are empty. We use Lemma 2 for the induction step.

Next we show that these solutions are all ordinary. For $n \geqslant 5$, let B_n be the $n \times n$ solution generated in the previous paragraph. By inspection we can verify that B_5, B_6, and B_7 are ordinary. For $n \geqslant 8$, by induction we can show that the first and last row and column form a "hoop" with no pawns and four dragon kings on squares $(0, n-3)$, $(1, n-1)$, $(3,0)$, $(n-1,3)$. This hoop has no nontrivial rotational or reflective symmetries, so the arrangement on the whole board is ordinary.

Next we show by induction on $m \geqslant 0$ that for $n = k+5+m$ we can make an ordinary arrangement of k pawns and $n+k$ mutually nonattacking dragon kings on an $n \times n$ board. The base case is the previous paragraph. We use Lemma 3 for the induction step. ■.

The lower bound in Proposition 4 is not tight. For example, consider the ordinary $12+12$ dragon kings problem solution we get on an 12×12 board by placing pawns on squares $(1,4)$, $(3,7)$, $(4,2)$, $(5,6)$, $(5,9)$, $(6,1)$, $(6,3)$, $(7,10)$, $(8,2)$, $(8,5)$, $(9,8)$, and $(10,6)$ and dragon kings on squares $(0,4)$, $(1,2)$, $(1,7)$, $(2,5)$, $(3,3)$, $(3,9)$, $(4,1)$, $(4,6)$, $(5,4)$, $(5,8)$, $(5,10)$, $(6,0)$, $(6,2)$, $(6,6)$, $(7,9)$, $(7,11)$, $(8,1)$, $(8,3)$, $(8,7)$, $(9,5)$, $(9,10)$, $(10,2)$, $(10,8)$, and $(11,6)$. Using that example and arguments similar to those in the proof of Proposition 4, we can prove

Proposition 5. *For $k \geqslant 12$ and $n \geqslant k$, we can construct an ordinary solution to the $n+k$ dragon kings problem.*

1.2 Centrosymmetric solutions

We first note a restriction on the existence of centrosymmetric $n+k$ dragon kings problem solutions.

Proposition 6. *(c.f. [6, Proposition 2.1]) If n is even and k is odd, there are no centrosymmetric or bidiagonally symmetric $n+k$ dragon king arrangements.*

Proof. If n is even, the number of pawns in the left half must be equal to that in the right half. Therefore, the number of pawns must be even. ■

We next show that, if n and k are both even or n is odd, then for each $k \geqslant 0$ there are centrosymmetric $n+k$ dragon kings problem solutions for n sufficiently large.

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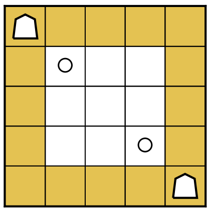

Figure 8: Illustration of the construction in Lemma 7. Given a centrosymmetric (or, respectively, monodiagonally symmetric, bidiagonally symmetric) solution in the unshaded region, we add two rows, two columns, and two dragon kings as shown to get a solution of the same symmetry class on a larger board.

Lemma 7. *Given a centrosymmetric (or, respectively, monodiagonally symmetric, bidiagonally symmetric) placement of p pawns and $n+k$ mutually nonattacking dragon kings on an $n \times n$ board, we can make a centrosymmetric (resp., monodiagonally symmetric, bidiagonally symmetric) placement of p pawns and $(n+2)+k$ mutually nonattacking dragon kings on an $(n+2) \times (n+2)$ board.*

Proof: On the $(n+2) \times (n+2)$ board, place the given arrangement in the middle $n \times n$ block. If squares $(1,1)$ and (n,n) are empty, place dragon kings on squares $(0,0)$ and $(n+1,n+1)$, as shown in Figure 8. Otherwise, place dragon kings on $(0,n+1)$ and $(n+1,0)$. In the hoop formed by the first and last row and column, the pieces are centrosymmetric and bidiagonally symmetric, and they do not attack each other or any dragon king in the middle $n \times n$ block. Since the middle block is centrosymmetric (resp., monodiagonally symmetric, bidiagonally symmetric), the entire arrangement is centrosymmetric (resp., monodiagonally symmetric, bidiagonally symmetric). ■

Lemma 8. *Given a centrosymmetric placement of p pawns and $n+k$ mutually nonattacking dragon kings on an $n \times n$ board, we can make a centrosymmetric placement of $p+2$ pawns and $(n+6)+k$ mutually nonattacking dragon kings on an $(n+4) \times (n+4)$ board.*

Proof: On the $(n+4) \times (n+4)$ board, place pawns on squares $(1,2)$ and $(n+2,n+1)$ and dragon kings on squares $(0,2)$, $(1,0)$, $(1,n+2)$, $(n+2,1)$, $(n+2,n+3)$, and $(n+3,n+1)$. Then, rotating if necessary, place the given arrangement in the middle $n \times n$ block of the $(n+4) \times (n+4)$ board so that squares $(2,n+1)$ and $(n+1,2)$ are empty. See Figure 9 for an illustration. Since the pieces in the hoop formed by the first two and last two rows and columns is centrosymmetric and attacks no piece in the center block, we conclude that the arrangement on the whole board satisfies the claimed properties. ■

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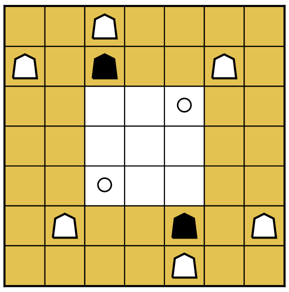

Figure 9: Illustration of the construction in Lemma 8. Given a centrosymmetric solution in the unshaded region, we add four rows, four columns, two pawns, and six dragon kings as shown to get a centrosymmetric solution on a larger board.

Lemma 9. *Suppose we have a centrosymmetric (or, respectively, doubly centrosymmetric, bidiagonally symmetric) $n + k$ dragon kings problem solution, where n is even. Then we can construct a centrosymmetric (resp., doubly centrosymmetric, bidiagonally symmetric) $(n + 1) + k$ dragon kings problem solution.*

Proof: Suppose $n = 2m$. The central four squares of the given $n \times n$ board $((m-1, m-1)$, $(m, m-1)$, $(m-1, m)$ and $(m, m))$ do not have any dragon kings in them, for if one of the squares had a dragon king, then another of the squares would have a dragon king by symmetry and those pieces would be mutually attacking.

On an $(n+1) \times (n+1)$ board, copy the upper-left (and respectively, lower-left, upper-right, lower-right) $m \times m$ block of the given arrangement onto the upper-left (resp.,lower-left, upper-right, lower-right) $m \times m$ block of the $(n + 1) \times (n + 1)$ board. Then place a dragon king on square (m, m) of the $(n + 1) \times (n + 1)$ board. This dragon king is on its own row and column and is not diagonally adjacent to any dragon king. We can check that the resulting arrangement (as illustrated in Figure 10) is an $(n+1)+k$ dragon kings solution in the claimed symmetry class. ■

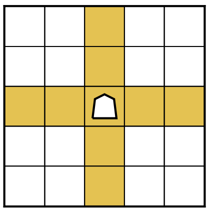

Figure 10: Illustration of Lemma 9. Given a centrosymmetric, doubly centrosymmetric, or bidiagonally symmetric solution on a $2m \times 2m$ board, we add a new central row and column and place a dragon king in the new central square to obtain a solution of the same symmetry class on a $(2m+1) \times (2m+1)$ board.

Proposition 10. *If k is even, there is a centrosymmetric $n+k$ dragon kings problem solution for all $n \geqslant 2k+6$. If k is odd, there is a centrosymmetric $n+k$ dragon kings problem solution for all odd $n \geqslant 2k+5$.*

Proof: First we show by induction on k that if k is even, a centrosymmetric $n+k$ dragon kings problem solution exists for $n = 2k+6$. For the base case, we use the 6×6 board with dragon kings on squares $(0,1)$, $(1,3)$, $(2,5)$, $(3,0)$, $(4,2)$, and $(5,4)$. Then we use Lemma 8 for the inductive step. To show that a centrosymmetric solution exists for even $n = 2k+6+2m$, we use Lemma 7 for an induction on m. To show that a centrosymmetric solution exists for odd $n = 2k+7+2m$, apply Lemma 9 to the solutions generated in the previous sentence.

Now suppose k is even. We show that a centrosymmetric solution exists for odd $n = 2k+5$ by using as a base case ($k = 1$) the 7×7 board with a pawn on square $(3,3)$ and dragon kings on squares $(0,1)$, $(1,3)$, $(2,6)$, $(3,2)$, $(3,4)$, $(4,0)$, $(5,3)$, and $(6,5)$ and Lemma 8 for the inductive step. To show that a centrosymmetric solution exists for odd $n = 2k+5+2m$, we use Lemma 7 for another induction on m. ■.

The bounds in Proposition 10 are not tight, as we can see with two examples:

1. the centrosymmetric $8+4$ dragon kings solution with pawns on squares $(2,2)$, $(3,6)$, $(4,1)$, and $(5,5)$ and dragon kings on squares $(0,2)$, $(1,6)$, $(2,1)$, $(2,3)$, $(3,5)$, $(3,7)$, $(4,0)$, $(4,2)$, $(5,4)$, $(5,6)$, $(6,1)$, and $(7,5)$, and
2. the centrosymmetric $9+5$ dragon kings solution with pawns on squares $(2,5)$, $(3,2)$, $(4,4)$, $(5,6)$, and $(6,3)$ and dragon kings on squares $(0,5)$, $(1,2)$, $(2,4)$, $(2,6)$, $(3,1)$, $(3,8)$, $(4,3)$, $(4,5)$, $(5,0)$, $(5,7)$, $(6,2)$, $(6,4)$, $(7,6)$, and $(8,3)$.

Using these examples and arguments similar to those in Proposition 10, we can prove

Proposition 11. *If $k \geqslant 4$ is even, there is a centrosymmetric $n+k$ dragon kings problem solution for all $n \geqslant 2k$. If $k \geqslant 5$ is odd, there is a centrosymmetric $n+k$ dragon kings problem solution for all odd $n \geqslant 2k-1$.*

1.3 Monodiagonally symmetric solutions

In this subsection we show that for each $k \geqslant 0$ we can construct a monodiagonally symmetric $n+k$ dragon kings problem solution for n large enough.

Lemma 12. *Given a monodiagonally symmetric placement of p pawns and $n+k$ mutually nonattacking dragon kings on an $n \times n$ board with empty corners, we can construct a monodiagonally symmetric placement of $p+2$ pawns on $n+6$ mutually nonattacking dragon kings on an $(n+4) \times (n+4)$ board with empty corners.*

Proof: Without loss of generality, suppose the given arrangement is symmetric with respect to reflection across the main diagonal but is not symmetric with respect to reflection across the main antidiagonal. On an $(n + 4) \times (n + 4)$ board, place pawns on squares $(1, n + 1)$ and $(n + 1, 1)$ and dragon kings on squares $(0, n+1)$, $(1, 1)$, $(1, n+3)$, $(n+1, 0)$, $(n+2, n+2)$, and $(n+3, 1)$. In the central $n \times n$ block place the given arrangement. See Figure 11 for an illustration. The pieces in the hoop formed by the first two and last two rows and columns is symmetric with respect to reflection across the main diagonal but not the main antidiagonal, and the pieces attack nothing in the central block. Also the corners are empty. We conclude that the resulting arrangement on the full board satisfies the desired conditions. ■

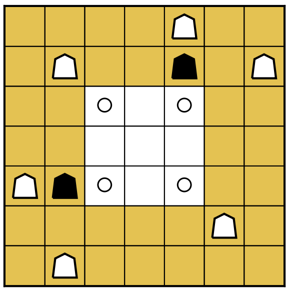

Figure 11: Illustration of Lemma 12. Given a monodiagonally symmetric solution in the central unshaded region, we add four rows, four columns, two pawns, and six dragon kings as shown to get a solution with the same symmetry on a larger board.

Proposition 13. *For every $k \geqslant 0$ there is a monodiagonally symmetric $n + k$ dragon kings problem solution for every $n \geqslant 2k + 5$.*

Proof: We first show that the statement is true for $n = 2k + 5$. For $k = 0$, consider the 5×5 board with dragon kings on squares $(0, 3)$, $(1, 1)$, $(2, 4)$, $(3, 0)$, and $(4, 2)$. For $k = 1$, consider the 7×7 board with a pawn on square $(2, 2)$ and dragon kings on squares $(0, 2)$, $(1, 4)$, $(2, 0)$, $(2, 6)$, $(3, 3)$, $(4, 1)$, $(5, 5)$, and $(6, 2)$. Then we use Lemma 12 for an induction on k.

Next we show that the statement is true for $n = 2k + 6$. For $k = 0$, take the 6×6 board with dragon kings on squares $(0, 3)$, $(1, 5)$, $(2, 2)$, $(3, 0)$, $(4, 4)$ and $(5, 1)$. For $k = 1$, take the 8×8 board with a pawn on square $(5, 5)$ and dragon kings on squares $(0, 5)$, $(1, 1)$, $(2, 4)$, $(3, 6)$, $(4, 2)$, $(5, 0)$, $(5, 7)$, $(6, 3)$, and $(7, 5)$. We use Lemma 12 for another induction on k.

To show the assertion is true for $n = 2k+5+m$, use the solutions generated in the previous two paragraphs as base cases and use Lemma 7 for induction on m. ■

The bound in Proposition 13 is not tight, as we can see in the monodiagonally symmetric $9 + 3$ dragon kings problem solution with pawns on squares $(4, 4)$, $(5, 6)$, and $(6, 5)$ and dragon kings on squares $(0, 4)$, $(1, 7)$, $(2, 5)$, $(3, 3)$, $(4, 0)$, $(4, 6)$, $(5, 2)$, $(5, 8)$, $(6, 4)$, $(6, 6)$, $(7, 1)$, and $(8, 5)$.

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1.4 Doubly centrosymmetric solutions

We note some necessary conditions for doubly centrosymmetric $n+k$ dragon kings problem solutions.

Proposition 14. *(c.f. [6, Proposition 2.3]) Let $n \geqslant 1$ and $k \geqslant 0$ be integers for which there is a doubly centrosymmetric $n+k$ dragon kings arrangement. Then n and k must satisfy one of the following conditions:*

1. *$n \equiv 0 \pmod 4$ and $k \equiv 0 \pmod 4$*
2. *$n \equiv 1 \pmod 4$ and $k \equiv 0 \pmod 4$*
3. *$n \equiv 3 \pmod 4$ and $k \equiv 1 \pmod 4$*

Proof: Since a doubly centrosymmetric arrangement is invariant under rotations of any integer multiple of quarter-turn rotations, if such an arrangement has a piece at square (a, b), it must also have pieces of the same type at squares $(b, n-1-a)$, $(n-1-a, n-1-b)$, $0 \leqslant a, b < n$, and $(n-1-b, a)$. Unless $(a, b) = (\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil)$ and n is odd, the four squares listed above are distinct. So, the number of dragon kings is congruent to either 0 or 1 (mod 4), and the number of pawns is also congruent to either 0 or 1 (mod 4).

The number of pawns in an $n+k$ dragon kings arrangement is k, so we must have $k \equiv 0 \pmod 4$ or $k \equiv 1 \pmod 4$. The number of dragon kings in an $n+k$ dragon kings arrangement is $n+k$, so either $n+k \equiv 0 \pmod 4$ or $n+k \equiv 1 \pmod 4$. If n is even, clearly $n+k \equiv 0 (\mathrm{mod}\ 4)$ and $k \equiv 0 \pmod 4$, so $n \equiv 0 (\mathrm{mod}\ 4)$, and we have condition 1. If n is odd, and the middle square is empty, we have an even number of pawns and an even number of dragon kings, which leads to a contradiction with the parity of n. If n is odd and the middle square has a dragon king, then $n+k \equiv 1 \pmod 4$ and $k \equiv 0 \pmod 4$, so $n \equiv 1 \pmod 4$ and we have condition 2. If n is odd and the middle square has a pawn, then $k \equiv 1 \pmod 4$ and $n+k \equiv 0 \pmod 4$, so $n \equiv 3 (\mathrm{mod}\ 4)$ and we have condition 3. ■

If n and k follow the restrictions above, and n is sufficently large, then we can construct a doubly centrosymmetric $n+k$ dragon kings problem solution.

Lemma 15. *Given a doubly centrosymmetric placement of p pawns and $n+k$ mutually nonattacking dragon kings on an $n \times n$ board, we can make a doubly centrosymmetric placement of p pawns and $(n+4)+k$ mutually nonattacking dragon kings on an $(n+4) \times (n+4)$ board.*

Proof: On the $(n+4) \times (n+4)$ board, place dragon kings on squares $(0, n+2)$, $(1, 0)$, $(n+2, n+3)$, and $(n+3, 1)$. Then place the given arrangement on the $n \times n$ board obtained by removing the first two and last two rows and columns. We can check that the resulting arrangement, as illustrated in Figure 12, is a doubly centrosymmetric placement of k pawns and $(n+4)+k$ mutually nonattacking dragon kings on an $(n+4) \times (n+4)$ board. ■

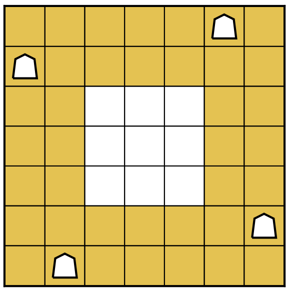

Figure 12: Illustration of the construction in Lemma 15. Given a doubly centrosymmetric solution in the central unshaded region, we add four rows, four columns, and four dragon kings as shown to get a doubly centrosymmetric solution on a larger board.

Lemma 16. *Given a doubly centrosymmetric placement of p pawns and $n+k$ mutually nonattacking dragon kings on an $n \times n$ board, we can make a doubly centrosymmetric placement of $p+4$ pawns and $(n+8)+(k+4)$ mutually nonattacking dragon kings on an $(n+8) \times (n+8)$ board.*

Proof: On the $(n+8) \times (n+8)$ board, place pawns on squares $(2,2)$, $(2, n+5)$, $(n+5, 2)$ and $(n+5, n+5)$ and dragon kings on squares $(0,2)$, $(1, n+5)$, $(2,1)$, $(2,3)$, $(2, n+7)$, $(3, n+5)$, $(n+4, 2)$, $(n+5, 0)$, $(n+5, n+4)$, $(n+5, n+6)$, $(n+6, 2)$, and $(n+7, n+5)$. Then place the given arrangement on the $n \times n$ board obtained by removing the first four and last four rows and columns. We can check that the resulting arrangement (illustrated in Figure 13) satisfies the desired conditions. ■

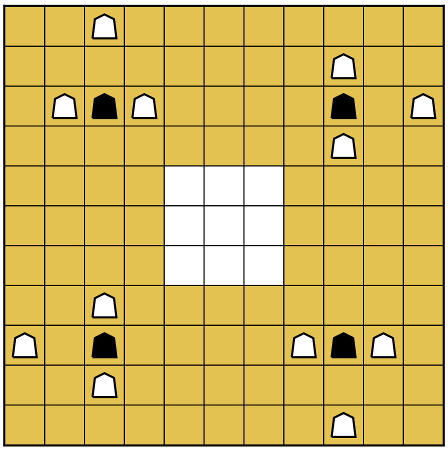

Figure 13: Illustration of the construction in Lemma 16. Given a doubly centrosymmetric solution in the central unshaded region, we add eight rows, eight columns, four pawns, and twelve dragon kings as shown to get a doubly centrosymmetric solution on a larger board.

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Proposition 17. *There is a doubly centrosymmetric $n+k$ dragon kings problem solution if n and k satisfy one of the following:*

1. $n \equiv 0$ *(mod 4),* $n \geqslant 4$*, and* $k = 0$
2. $n \equiv 0$ *(mod 4),* $k \equiv 0$ *(mod 4),* $k \geqslant 4$*, and* $n \geqslant 2k$
3. $n \equiv 1$ *(mod 4),* $k \equiv 0$ *(mod 4), and* $n \geqslant 2k+1$
4. $n \equiv 3$ *(mod 4),* $k = 1$*, and* $n \geqslant 11$
5. $n \equiv 3$ *(mod 4),* $k \equiv 1$ *(mod 4),* $k \geqslant 5$*, and* $n \geqslant 2k+1$

Proof: We consider the cases in order.

1. For $n = 4$ and $k = 0$ we use the 4×4 board with dragon kings on squares $(0, 2)$, $(1, 0)$, $(2, 3)$, and $(3, 1)$. We finish this case by induction using Lemma 15

2. Let $k = 4i$ for $i \geqslant 1$. We show there is a solution for $n = 2k = 8i$. For $i = 1$, we use the 8×8 board shown in Figure 3 with pawns on squares $(2, 2)$, $(2, 5)$, $(5, 2)$, and $(5, 5)$ and dragon kings on squares $(0, 5)$, $(1, 2)$, $(2, 0)$, $(2, 4)$, $(2, 6)$, $(3, 2)$, $(4, 5)$, $(5, 1)$, $(5, 3)$, $(5, 7)$, $(6, 5)$, and $(7, 2)$. For $i > 1$ we use Lemma 16 for an induction. We complete this case by inductively constructing solutions for $n = 8i + 4m$ for $m \geqslant 0$ using Lemma 15

3. For $n = 1$ and $k = 0$ we use the 1×1 board with a dragon king placed on the only square. For all other values of n and k in this case, take the solutions generated in the previous two cases and apply Lemma 9.

4. In this case, for $n = 11$ consider the 11×11 board with a pawn on square $(5, 5)$ and dragon kings on squares $(0, 8)$, $(1, 6)$, $(2, 0)$, $(3, 5)$, $(4, 1)$, $(5, 3)$, $(5, 7)$, $(6, 9)$, $(7, 5)$, $(8, 10)$, $(9, 4)$, and $(10, 2)$. For $n = 11 + 4j$ use Lemma 15 inductively.

5. In this case, $n = 4i + 3$ for some $i \geqslant 1$ and $k = 4j + 1$ for some $j \geqslant 1$. We first show that a solution exists for $n = 2k+1 = 8j+3$. For $j = 1$ consider the 11×11 board with pawns on squares $(3, 3)$, $(3, 7)$, $(5, 5)$, $(7, 3)$, and $(7, 7)$ and dragon kings on squares $(0, 4)$, $(1, 7)$, $(2, 3)$, $(3, 1)$, $(3, 5)$, $(3, 8)$, $(4, 10)$, $(5, 3)$, $(5, 7)$, $(6, 0)$, $(7, 2)$, $(7, 5)$, $(7, 9)$, $(8, 7)$, $(9, 3)$, and $(10, 6)$. For larger j we use induction with Lemma 16 for the inductive step. We complete the case by showing that a solution exists for $n = 2k + 1 + 4m$ by induction on m using Lemma 15. ■

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1.5 Bidiagonally symmetric solutions

Recall that Proposition 6 restricts the values of n and k for which there are bidiagonally symmetric solutions. We show that for each $k \geqslant 0$, if n is sufficiently large and n and k satisfy the necessary conditions given in Proposition 6, then we can construct a bidiagonally symmetric $n+k$ dragon kings solution.

Lemma 18. *Given a bidiagonally symmetric placement of p pawns and $n+k$ mutually nonattacking dragon kings on an $n \times n$ board (with $n \geqslant 3$), we can make a bidiagonally symmetric placement of $p+4$ pawns and $(n+8)+(k+4)$ mutually nonattacking dragon kings on an $(n+8) \times (n+8)$ board.*

Proof: On the $(n+8) \times (n+8)$ board place pawns on squares $(2,5)$, $(5,2)$, $(n+2, n+5)$, and $(n+5, n+2)$ and dragon kings on squares $(0,2)$, $(1,5)$, $(2,0)$, $(2, n+5)$, $(3,3)$, $(5,1)$, $(n+2, n+6)$, $(n+4, n+4)$, $(n+5, 2)$, $(n+5, n+7)$, $(n+6, n+2)$, and $(n+7, n+5)$. Then place the given arrangement, or its horizontal reflection, on the central $n \times n$ block so that squares $(4,4)$ and $(n+3, n+3)$ remain empty. We can check that the resulting arrangement (as illustrated in Figure 14) satisfies the desired properties. ■

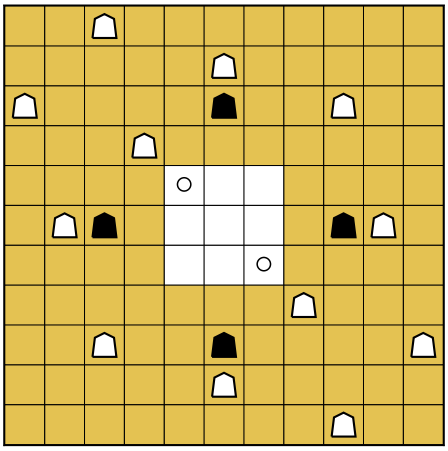

Figure 14: Illustration of the construction in Lemma 18. Given a bidiagonally symmetric solution in the unshaded central region, we add eight rows, eight columns, four pawns, and twelve dragon kings as shown to get a bidiagonally symmetric solution on a larger board.

Proposition 19. *For k even and $n \geqslant 2k+6$, there is a bidiagonally symmetric $n+k$ dragon kings problem solution. For n and k odd and $n \geqslant 2k+5$, there is a bidiagonally symmetric $n+k$ dragon kings problem solution.*

Proof: Suppose first that k is even. We show there is a bidiagonally symmetric $n+k$ dragon kings problem solution for $n = 2k+6$. For $k = 0$, consider the 6×6 board with dragon kings on squares $(0,2)$, $(1,4)$, $(2,0)$, $(3,5)$, $(4,1)$, and $(5,3)$. For $k = 2$, consider the 10×10 board with pawns in squares $(3,6)$ and $(6,3)$ and dragon kings in squares $(0,3)$, $(1,6)$, $(2,4)$, $(3,0)$, $(3,8)$, $(4,2)$, $(5,7)$,$(6,1)$, $(6,9)$, $(7,5)$, $(8,3)$, and $(9,6)$. We use Lemma 18 for the inductive step.

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Next we apply Lemma 7 to show there is a solution for $n = 2k+6+2m$, where $m \geqslant 0$. To construct solutions for $n = 2k+7+2m$, apply Lemma 9 inductively.

Now suppose k is odd. We next show there is a solution for $n = 2k + 5$. For $k = 1$, take the 7×7 board with a pawn in square $(3, 3)$ and dragon kings in squares $(0, 3)$, $(1, 1)$, $(2, 4)$, $(3, 0)$, $(3, 6)$, $(4, 2)$, $(5, 5)$, $(6, 3)$. For $k = 3$, take the 11×11 board with pawns in squares $(3, 7)$, $(5, 5)$ and $(7, 3)$ and dragon kings in squares $(0, 10)$,$(1, 7)$, $(2, 5)$, $(3, 3)$, $(3, 9)$, $(4, 6)$, $(5, 2)$, $(5, 8)$, $(6, 4)$, $(7, 1)$, $(7, 7)$, $(8, 5)$, $(9, 3)$, and $(10, 0)$. We again use Lemma 18 for the inductive step.

We again apply Lemma 7 to show there is a solution for $n = 2k+5+2m$, where $m \geqslant 0$. ■

$n \backslash k$	0	1	2	3	4	5	6
4	2	0	0	0	0	0	0
5	14	0	0	0	0	0	0
6	90	32	0	0	0	0	0
7	646	762	124	0	0	0	0
8	5242	14412	9056	1688	94	0	0
9	47622	250326	380776	216678	48374	3540	0
10	479306	4252504	12538132	16006424	9629406	2790292	389100

Table 1: Number of $n+k$ dragon kings problem solutions for $4 \leqslant n \leqslant 10$ and $0 \leqslant k \leqslant 6$

2 Conclusion and Open Problems

We have shown that ordinary solutions to the $n+k$ dragon kings problem exist for $n \geqslant k+5$. If k is even or n is odd, centrosymmetric solutions and bidiagonally symmetric solutions exist for $n \geqslant 2k+5$. Monodiagonally symmetric solutions exist for $n \geqslant 2k+5$. If $k \geqslant 4$, doubly centrosymmetric solutions exist under any of the following three conditions:

1. $n \equiv 0 \pmod 4$, $k \equiv 0 \pmod 4$, and $n \geqslant 2k$
2. $n \equiv 1 \pmod 4$, $k \equiv 0 \pmod 4$, and $n \geqslant 2k+1$
3. $n \equiv 3 \pmod 4$, $k \equiv 1 \pmod 4$, and $n \geqslant 2k+1$

There are many unanswered questions provoked by the results in this paper, such as

1. Many of the lower bounds we have found are not tight. How much can we tighten the bounds? Furthermore, can we determine all the pairs (n, k) for which there are solutions in each of the symmetry classes to the $n+k$ dragon kings problem?

2. Given n and k, how many solutions are there to the $n + k$ dragon kings problem?

   We know the number of solutions to the $n + 0$ dragon kings problem (originally called the "n kings problem" by Kaplansky [7]); it is given by sequence A002464 of the On-Line Encyclopedia of Integer Sequences [8]. Taking Kosters' ASM algorithm for counting solutions of the $n+k$ queens problem [6, Figure 6] and adapting it to the $n + k$ dragon kings problem, we obtain the numbers in Table 1.

3. What happens if we consider different types of board, such as cylinders, tori, or three-dimensional boards?

4. What solution patterns can we construct if we replace the dragon king with another piece, such as the k-step dragon king. defined in [4], which moves k squares diagonally or any number of squares vertically or horizontally?

References

[1] Bell, J., Stevens, B. "A survey of known results and research areas for n-queens", *Discrete Math*, 309, 1–31, 2009.

[2] Bodlaender, H., Duniho, F. "Shogi: Japanese chess", 2017.
http://www.chessvariants.com/shogi.html

[3] Brualdi, R.A., Kiernan, K.P., Meyer, S.A., Schroeder, M.W. "Patterns of alternating sign matrices", *Linear Algebra and its Applications*, 438(10), 3967–3990, 2013.

[4] Chatham, D. "Independence and domination on shogiboard graphs", *Recreational Mathematics Magazine*, 4(8), 25–37, 2017.

[5] Chatham, R.D. "Reflections on the $N + k$ Queens Problem", *College Mathematics Journal*, 40(3), 204–210, 2009.

[6] Chatham, R.D., Doyle, M., Jeffers, R.J., Kosters, W.A., Skaggs, R.D., Ward, J.A. "Centrosymmetric solutions to chessboard separation problems", *Bulletin of the Institute of Combinatorics and its Applications*, 65, 6–26, 2012.

[7] Kaplansky, I. "Symbolic solution of certain problems in permutations", *Bulletin of the American Mathematical Society* 50, 906–914, 1944.

[8] Sloane, N.J.A. Sequence A002464 in *The On-Line Encyclopedia of Integer Sequences*, 2017.
https://oeis.org