Articles

A generalization of Trenkler's magic cubes formula

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Abstract: *A Magic Cube of order p is a $p \times p \times p$ cubical array with non-repeated entries from the set $\{1, 2, \ldots, p^3\}$ such that all rows, columns, pillars and space diagonals have the same sum. In this paper, we show that a formula introduced in* The Mathematical Gazette 84(2000)*, by M. Trenkler, for generating odd order magic cubes is a special case of a more general class of formulas. We derive sufficient conditions for the formulas in the new class to generate magic cubes, and we refer to the resulting class as regular magic cubes. We illustrate these ideas by deriving three new formulas that generate magic cubes of odd order that differ from each other and from the magic cubes generated with Trenkler's rule.*

Keywords: Magic cube, regular magic cube, magic cube formula, Trenkler's formula.

A Magic Cube of order p is a $p \times p \times p$ cubical array with non-repeated entries from the set $\{1, 2, \ldots, p^3\}$, such that all rows, columns, pillars and space diagonals have the same sum. This is a natural extension of the concept of a magic square of order p, defined as a square array of order p consisting of non-repeated entries from the set $\{1, 2, \ldots, p^2\}$ whose rows, columns and diagonals add up to the same sum. The sum of the elements in a magic cube of order p is $1 + 2 + \cdots + p^3 = p^3(p^3 + 1)/2$. Since these numbers are divided into p^2 rows each of which has the same sum, that sum (the magic constant) must be $p(p^3 + 1)/2$. The columns, pillars and space diagonals will also have the same sum. A magic cube is usually considered to be identical with the 47 magic cubes obtainable from it by performing rotations and/or reflections.

The following is an example of a magic cube:

$$
\begin{array}{ccc c ccc c ccc}
 & M_{:\,:\,1} & & \quad & & M_{:\,:\,2} & & \quad & & M_{:\,:\,3} & \\
20 & 6 & 16 & & 15 & 25 & 2 & & 7 & 11 & 24 \\
18 & 19 & 5 & & 1 & 14 & 27 & & 23 & 9 & 10 \\
4 & 17 & 21 & & 26 & 3 & 13 & & 12 & 22 & 8,
\end{array}
$$

where $M_{:\,:\,k} \stackrel{\text{def}}{=} \{m_{ijk} \colon 1 \leqslant i, j \leqslant p\}$.

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For centuries, magic squares and magic cubes have been sources of mathematical amusements and challenging open problems. They are classic examples of recreational mathematics topics which typically have a large number of enthusiasts, the majority of who are not professional mathematicians.

Different types of magic cubes, some methods for constructing them, and their history, can be found in the books by Andrews [1], Ball and Coxeter [2], Benson and Jacoby [4], the writings of Martin Gardner (cf. [6, 7]), the 1888 paper [3] by Barnard and the more recent papers [8, 9, 10, 11, 12].

In the sequel, the symbols $\mathbb{N}_p$ and $\mathbb{Z}_p$ will always designate, respectively, the sets $\mathbb{N}_p = \{1, 2, \ldots, p\}$ and $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$. Given integers a and b, a mod b and (a, b) will designate, respectively, the remainder when b divides a, and the greatest common factor of a and b.

Trenkler proved in [12] that the formula

$$\begin{aligned} m_{ijk} &= 1 + [(i - j + k - 1) \bmod p] + p[(i - j - k) \bmod p] \\ &\quad + p^2[(i + j + k - 2) \bmod p],\ i, j, k \in \mathbb{N}_p, \end{aligned} \tag{1}$$

yields a magic cube for all odd orders $p \geqslant 3$. In this paper, we show that a larger class of odd order magic cubes can be obtained from the formula

$$\begin{aligned} m_{ijk} &= 1 + [(a_1 i + b_1 j + c_1 k + d_1) \bmod p] \\ &+ p[(a_2 i + b_2 j + c_2 k + d_2) \bmod p] \\ &+ p^2[(a_3 i + b_3 j + c_3 k + d_3) \bmod p],\ i, j, k \in \mathbb{N}_p, \end{aligned} \tag{2}$$

where $p \geqslant 3$ is an odd number and the coefficients a_r, b_r, c_r, d_r are elements of $\mathbb{Z}_p$ for $r = 1, 2, 3$. The main purpose of the paper is to derive sufficient conditions that the coefficients must satisfy to yield a magic cube. The following results will be used to derive these conditions.

Lemma 1 (cf. [5])**.** *Let $\theta \in \mathbb{Z}_p$. Then the Diophantine equation*

$$\theta z \equiv b \pmod p$$

has one unique solution $z \in \mathbb{Z}_p$ associated with each integer b if and only if $(p, \theta) = 1$.

Proposition 2 ([13])**.** *Let p be a positive odd number, let q and z be any integers and let $\alpha = (p, q)$. Then the equation*

$$\sum_{i=1}^{p} [(qi + z) \bmod p] = p(p-1)/2$$

holds if and only if $z \bmod \alpha = (\alpha - 1)/2$.

Lemma 3. *Let $A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$ and let $\mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$ where a_i, b_i, $c_i \in \mathbb{Z}_p$ and $f_i \in \mathbb{Z}$ for $i = 1, 2, 3$. Let $\Delta = \det(A)$ and suppose that $(\Delta, p) = 1$. Then the system $A\mathbf{x} \equiv \mathbf{f} \pmod p$ has a unique solution $\mathbf{x} \in \mathbb{Z}_p^3$.*

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Proof. Let $\hat{A}$ be the adjugate matrix of A, defined as the transpose of the matrix of cofactors of A. It is well known that $A\hat{A} = \hat{A}A = \Delta I_3$, where I_3 is the identity matrix of order 3. An application of Lemma 1 shows that, since $(\Delta, p) = 1$, there exists a unique $\Delta' \in \mathbb{Z}_p$ such that $\Delta\Delta' \equiv 1 \pmod p$. It follows that $\Delta'\hat{A}$ is the inverse matrix of A, modulo p, and hence that the vector $\mathbf{x} = \Delta'\hat{A}\mathbf{f} \bmod p$ is the unique solution of the given system in $\mathbb{Z}_p^3$. □

We now give sufficient conditions for the coefficients in (2) to yield magic cubes.

Theorem 4. *Let $[a_r, b_r, c_r, d_r] \in \mathbb{Z}_p^4$, and let $\alpha_r = (p, a_r + b_r + c_r)$, $\beta_r = (p, a_r - b_r + c_r)$, $\delta_r = (p, -a_r + b_r + c_r)$ and $\theta_r = (p, a_r + b_r - c_r)$, for $r = 1, 2, 3$. Suppose that*

$$(\Delta, p) = 1 \tag{3}$$

and

$$\begin{aligned}
(p, a_r) = (p, b_r) = (p, c_r) &= 1, &(4)\\
d_r \bmod \alpha_r &= (\alpha_r - 1)/2, &(5)\\
(d_r + b_r) \bmod \beta_r &= (\beta_r - 1)/2, &(6)\\
(d_r + a_r) \bmod \delta_r &= (\delta_r - 1)/2, &(7)\\
(d_r + c_r) \bmod \theta_r &= (\theta_r - 1)/2, &(8)
\end{aligned}$$

for $r = 1, 2, 3$. Then the cubical array $M = (m_{ijk})$ defined in equation (2) is a magic cube.

Proof. Since $(p, a_r) = 1$, the equation

$$(b_r j + c_r k + d_r) \mod (p, a_r) = ((p, a_r) - 1)/2$$

holds trivially for all $j, k \in \mathbb{N}_p$. Therefore it follows from Proposition 2 that the following equation holds for $r = 1, 2, 3$ and $j, k \in \mathbb{N}_p$:

$$\sum_{i=1}^{p} (a_r i + b_r j + c_r k + d_r) \bmod p = p(p-1)/2.$$

Similarly, since $(p, b_r) = 1$, the equation

$$(a_r j + c_r k + d_r) \mod (p, b_r) = ((p, b_r) - 1)/2$$

holds trivially, and, since $(p, c_r) = 1$, the equation

$$(a_r j + b_r k + d_r) \mod (p, c_r) = ((p, c_r) - 1)/2$$

holds trivially, for all $j, k \in \mathbb{N}_p$. Therefore it follows from Proposition 2 that the following equation holds for $r = 1, 2, 3$ and $j, k \in \mathbb{N}_p$:

$$\sum_{i=1}^{p} (a_r j + b_r i + c_r k + d_r) \bmod p = p(p-1)/2,$$
$$\sum_{i=1}^{p} (a_r j + b_r k + c_r i + d_r) \bmod p = p(p-1)/2.$$

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It follows from Proposition 2, the definitions of α_r, β_r, δ_r , θ_r and the fact that conditions (5), (6), (7), (8) hold, that the following equations hold, respectively, for $r = 1, 2, 3$:

$$\sum_{i=1}^{p}[(a_r + b_r + c_r)i + d_r] \bmod p = p(p-1)/2,$$
$$\sum_{i=1}^{p}[(a_r - b_r + c_r)i + d_r + b_r] \bmod p = p(p-1)/2,$$
$$\sum_{i=1}^{p}[(-a_r + b_r + c_r)i + d_r + a_r] \bmod p = p(p-1)/2,$$
$$\sum_{i=1}^{p}[(a_r + b_r - c_r)i + d_r + c_r] \bmod p = p(p-1)/2.$$

By substituting these equations in (2) we see that (m_{ijk}) satisfies the $3p^2 + 4$ defining equations for a magic cube, namely:

$$\text{Rows}: \sum_{i=1}^{p} m_{ijk} = p(p^3+1)/2, \quad \forall k, j \in \mathbb{N}_p,$$

$$\text{Columns}: \sum_{i=1}^{p} m_{jik} = p(p^3+1)/2, \quad \forall k, j \in \ N_p,$$

$$\text{Pillars}: \sum_{i=1}^{p} m_{jki} = p(p^3+1)/2, \quad \forall k, j \in \mathbb{N}_p,$$

$$\text{Space Diagonals}: \sum_{i=1}^{p} m_{iii} = \sum_{i=1}^{p} m_{\bar{\imath}\bar{\imath}i} = \sum_{i=1}^{p} m_{ii\bar{\imath}} = \sum_{i=1}^{p} m_{\bar{\imath}ii} = p(p^3+1)/2,$$

where $\bar{\imath} \equiv p + 1 - i$.

To complete the proof we have to show that $M = \{m_{ijk} \colon i, j, k \in \mathbb{N}_p\}$ and $P = \{1, 2, \ldots, p^3\}$ coincide. It follows from the definition of the cubical array (m_{ijk}) in (2) that $m_{ijk} \in \mathbb{Z}$ and $1 \leqslant m_{ijk} \leqslant 1+(p-1)+p(p-1)+p^2(p-1) = p^3$ for all i, j, $k \in \mathbb{N}_p$. Therefore the inclusion $M \subseteq P$ holds. To prove the opposite inclusion, we let $z \in P$ be arbitrary. Then it follows from the remainder theorem that $z - 1$ can be written uniquely in the form $z - 1 = r + p^2 w$ where $0 \leqslant r \leqslant p^2 - 1$ and $0 \leqslant w \leqslant p - 1$. Since r can also be written uniquely in the form $r = u + pv$, it follows that z can be written uniquely in the form $z = 1 + u + pv + p^2 w$ where u, v, $w \in \mathbb{Z}_p$.

Since (3) holds, an application of Lemma 3 shows that the Diophantine system:

$$a_1(i-1) + b_1(j-1) + c_1(k-1) \equiv (u - d_1 - a_1 - b_1 - c_1) \pmod p$$
$$a_2(i-1) + b_2(j-1) + c_2(k-1) \equiv (v - d_2 - a_2 - b_2 - c_2) \pmod p$$
$$a_3(i-1) + b_3(j-1) + c_3(k-1) \equiv (w - d_3 - a_3 - b_3 - c_3) \pmod p$$

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has a unique solution $[i, j, k] \in \mathbb{N}_p^3$. Therefore, the equivalent linear system

$$\begin{aligned} a_1 i + b_1 j + c_1 k + d_1 &\equiv u \pmod p \\ a_2 i + b_2 j + c_2 k + d_2 &\equiv v \pmod p \\ a_3 i + b_3 j + c_3 k + d_3 &\equiv w \pmod p \end{aligned}$$

also has a unique solution $[i, j, k] \in \mathbb{N}_p^3$. It follows from equation (2) that $m_{ijk} = 1 + [(a_1 i + b_1 j + c_1 k + d_1) \bmod p] + p[(a_2 i + b_2 j + c_2 k + d_2) \bmod p] + p^2[(a_3 i + b_3 j + c_3 k + d_3) \bmod p] = 1 + u + pv + p^2 w = z \in M$. We conclude that $\{m_{ijk} : i, j, k \in \mathbb{N}_p\} = \{1, 2, \ldots, p^3\}$ and therefore (m_{ijk}) is a magic cube. □

As an application of this theorem, we can easily deduce the following result which was the subject of the paper [12].

Theorem 5 (Trenkler). *Equation (1) yields a magic cube for all odd values of $p \geqslant 3$.*

Proof. On rewriting the formula (1) with positive coefficients in the form

$$\begin{aligned} m_{ijk} \quad = \quad & 1 + [i + (p-1)j + k + p - 1) \bmod p] \\ & + p[(i + (p-1)j + (p-1)k \bmod p] \\ & + p^2[(i + j + k + p - 2) \bmod p], \end{aligned}$$

we see that it corresponds to the regular magic cube rule (2) with $[a_1, b_1, c_1, d_1] = [1, p-1, 1, p-1]$, $[a_2, b_2, c_2, d_2] = [1, p-1, p-1, 0]$ and $[a_3, b_3, c_3, d_3] = [1, 1, 1, p-2]$.

It is easy to verify, using the notation of Theorem 4, that $(\Delta, p) = (p^2 - 2p + 2, p) = (2, p) = 1$ and that $(p, a_r) = (p, b_r) = (p, c_r) = 1$ for $r = 1, 2, 3$. Therefore conditions (3) and (4) of Theorem 4 hold. The evaluations $\alpha_1 = (p, p+1) = 1$ and $\alpha_2 = (p, 2p-1) = 1$ show that condition (5) holds trivially for $r = 1$ and $r = 2$. Similarly, it follows from the evaluations $\beta_1 = (p, 1-p) = 1$, $\beta_2 = \beta_3 = (p, 1) = 1$ $\delta_1 = (p, p-1) = 1$, $\delta_2 = \delta_3 = (p, 1) = 1$, $\theta_1 = (p, p-1) = 1$, $\theta_2 = \theta_3 = (p, 1) = 1$, that conditions (6) and (8) holds trivially for $r = 1, 2, 3$. The evaluations $\delta_1 = (p, p-1) = 1$ and $\delta_2 = \delta_3 = (p, 1) = 1$ show that condition (7) holds trivially for $r = 1$ and $r = 2$. Finally, since $\delta_2 = (p, 2p-3) = (p, 3) = \alpha_3$, and

$$(p, 3) = \begin{cases} 3 \text{ if } p \mid 3 \\ 1 \text{ otherwise,} \end{cases}$$

it is not hard to verify, by considering these two cases, that condition (5) holds when $r = 3$ and condition (7) holds when $r = 2$. We conclude that Trenkler's cubical array (1) satisfies the hypotheses of Theorem 4 and is therefore a magic cube. □

Other magic cube formulas that obtainable from equation (2) and verifiable in

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a similar way with Theorem 4 include the equations

$$\begin{aligned}
m_{ijk} &= 1 + [(i+j+k+1) \bmod p] + p[(-i-j+k) \bmod p] \\
&\quad + p^2[(i-j-k) \bmod p], \\
m_{ijk} &= 1 + [(-2i+j-2k+1) \bmod p] + p[(i-j+k-1) \bmod p] \\
&\quad + p^2[(i-j-k) \bmod p], \\
m_{ijk} &= 1 + [(i+j+k-2) \bmod p] + p[(i-j-k) \bmod p] \\
&\quad + p^2[(i-j+k-1) \bmod p].
\end{aligned}$$

These equations yield magic cubes that are different from each other and from magic cubes that are obtainable with Trenkler's formula (1) for all odd values of $p \geqslant 3$. Such examples illustrate how the regular magic cube rule (2) extends Trenkler's equation (1) in a nontrivial manner.

In the sequel, given $\mathbf{a} = [a_1, a_2, a_3]$, $\mathbf{b} = [b_1, b_2, b_3]$, $\mathbf{c} = [c_1, c_2, c_3]$ and $\mathbf{d} = [d_1, d_2, d_3]$, $M_p(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ will designate the magic cube generated with equation (2). The next result shows that the magic cube $M_p(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ is uniquely defined by its coefficients $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$.

Theorem 6. *If $M_p(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) = M_p(\mathbf{a}', \mathbf{b}', \mathbf{c}', \mathbf{d}')$, and both are magic cubes, then $\mathbf{a} = \mathbf{a}'$, $\mathbf{b} = \mathbf{b}'$, $\mathbf{c} = \mathbf{c}'$ and $\mathbf{d} = \mathbf{d}'$,*

Proof. For all i, j, $k = 1, 2, \ldots, p$, we have

$$\begin{gathered}
[(a'_1 i + b'_1 j + c'_1 k + d'_1) \bmod p] + p[(a'_2 i + b'_2 j + c'_2 k + d'_2) \bmod p] \\
+ p^2[(a'_3 i + b'_3 j + c'_3 k + d'_3) \bmod p] = [(a_1 i + b_1 j + c_1 k + d_1) \bmod p] \\
+ p[(a_2 i + b_2 j + c_2 k + d_2) \bmod p] + p^2[(a_3 i + b_3 j + c_3 k + d_3) \bmod p].
\end{gathered}$$

Hence

$$\begin{aligned}
&[(a'_1 - a_1)i + (b'_1 - b_1)j + (c'_1 - c_1)k + (d'_1 - d_1)] \equiv 0 \pmod p \\
&[(a'_2 - a_2)i + (b'_2 - b_2)j + (c'_2 - c_2)k + (d'_2 - d_2)] \equiv 0 \pmod p \\
&[(a'_3 - a_3)i + (b'_3 - b_3)j + (c'_3 - c_3)k + (d'_3 - d_3)] \equiv 0 \pmod p.
\end{aligned}$$

On setting $i = j = k = p$, we obtain the relations $|d'_1 - d_1| \equiv 0 \pmod p$, $|d'_2 - d_2| \equiv 0 \pmod p$ and $|d'_3 - d_3| \equiv 0 \pmod p$, which imply (since $|d'_1 - d_1|$, $|d'_2 - d_2|$ and and $|d'_3 - d_3|$ are elements of $\mathbb{Z}_p$) that $d'_1 = d_1$, $d'_2 = d_2$ and $d'_3 = d_3$. On taking $i = 1$, $j = p$ and $k = p$, we obtain the relations $|a'_1 - a_1| \equiv 0 \pmod p$, $|a'_2 - a_2| \equiv 0 \pmod p$ and $|a'_3 - a_3| \equiv 0 \pmod p$, which imply that $a'_1 = a_1$, $a'_2 = a_2$ and $a'_3 = a_3$.

On setting $i = p$, $j = 1$ and $k = p$, we obtain the relations $|b'_1 - b_1| \equiv 0 \pmod p$, $|b'_2 - b_2| \equiv 0 \pmod p$ and $|b'_3 - b_3| \equiv 0 \pmod p$, which imply that $b'_1 = b_1$, $b'_2 = b_2$ and $b'_3 = b_3$.

On setting $i = p$, $j = p$ and $k = 1$, we obtain the relations $|c'_1 - c_1| \equiv 0 \pmod p$, $|c'_2 - c_2| \equiv 0 \pmod p$ and $|c'_3 - c_3| \equiv 0 \pmod p$, which imply $c'_1 = c_1$, $c'_2 = c_2$ and $c'_3 = c_3$. $\square$

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