Articles

DEMYSTIFYING BENJAMIN FRANKLIN'S OTHER 8-SQUARE

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Abstract: *In this article, we reveal how Benjamin Franklin constructed his second* 8×8 *magic square. We also construct two new* 8×8 *Franklin squares.*

Keywords: Magic squares, Benjamin Franklin's 8-squares.

Introduction

The well-known squares in Figure 1 were constructed by Benjamin Franklin. The square F2 was introduced separately and hence is generally known as *the other 8-square*. The entries of the squares are from the set $\{1, 2, \ldots, n^2\}$, where $n = 8$ or $n = 16$. Every integer in this set occurs in the square exactly once. For these squares, the entries of every row and column add to a common sum called the *magic sum*. The 8×8 squares have magic sum 260 and the 16×16 square has magic sum 2056. In every half row and half column the entries add to half the magic sum. The entries of the main bend diagonals and all the bend diagonals parallel to it add to the magic sum. In addition, observe that every 2×2 sub-square in F1 and F2 adds to half the magic sum, and in F3 adds to one-quarter the magic sum. The property of the 2×2 sub-squares adding to a common sum and the property of bend diagonals adding to the magic sum are *continuous properties*. By continuous property we mean that if we imagine the square is the surface of a torus; i.e. opposite sides of the square are glued together, then the bend diagonals or the 2×2 sub-squares can be translated and still the corresponding sums hold. In fact, these squares have innumerable fascinating properties. See [1], [3], and [4] for a detailed study of these squares.

52	61	4	13	20	29	36	45
14	3	62	51	46	35	30	19
53	60	5	12	21	28	37	44
11	6	59	54	43	38	27	22
55	58	7	10	23	26	39	42
9	8	57	56	41	40	25	24
50	63	2	15	18	31	34	47
16	1	64	49	48	33	32	17

F1

17	47	30	36	21	43	26	40
32	34	19	45	28	38	23	41
33	31	46	20	37	27	42	24
48	18	35	29	44	22	39	25
49	15	62	4	53	11	58	8
64	2	51	13	60	6	55	9
1	63	14	52	5	59	10	56
16	50	3	61	12	54	7	57

F2

200	217	232	249	8	25	40	57	72	89	104	121	136	153	168	185
58	39	26	7	250	231	218	199	186	167	154	135	122	103	90	71
198	219	230	251	6	27	38	59	70	91	102	123	134	155	166	187
60	37	28	5	252	229	220	197	188	165	156	133	124	101	92	69
201	216	233	248	9	24	41	56	73	88	105	120	137	152	169	184
55	42	23	10	247	234	215	202	183	170	151	138	119	106	87	74
203	214	235	246	11	22	43	54	75	86	107	118	139	150	171	182
53	44	21	12	245	236	213	204	181	172	149	140	117	108	85	76
205	212	237	244	13	20	45	52	77	84	109	116	141	148	173	180
51	46	19	14	243	238	211	206	179	174	147	142	115	110	83	78
207	210	239	242	15	18	47	50	79	82	111	114	143	146	175	178
49	48	17	16	241	240	209	208	177	176	145	144	113	112	81	80
196	221	228	253	4	29	36	61	68	93	100	125	132	157	164	189
62	35	30	3	254	227	222	195	190	163	158	131	126	99	94	67
194	223	226	255	2	31	34	63	66	95	98	127	130	159	162	191
64	33	32	1	256	225	224	193	192	161	160	129	128	97	96	65

F3

Figure 1: Squares constructed by Benjamin Franklin.

3	61	16	50	7	57	12	54
32	34	19	45	28	38	23	41
33	31	46	20	37	27	42	24
62	4	49	15	58	8	53	11
35	29	48	18	39	25	44	22
64	2	51	13	60	6	55	9
1	63	14	52	5	59	10	56
30	36	17	47	26	40	21	43

N1

17	47	30	36	25	39	22	44
32	34	19	45	24	42	27	37
33	31	46	20	41	23	38	28
48	18	35	29	40	26	43	21
49	15	62	4	57	7	54	12
64	2	51	13	56	10	59	5
1	63	14	52	9	55	6	60
16	50	3	61	8	58	11	53

N2

44	61	4	21	12	29	36	53
22	3	62	43	54	35	30	11
45	60	5	20	13	28	37	52
19	6	59	46	51	38	27	14
47	58	7	18	15	26	39	50
17	8	57	48	49	40	25	16
42	63	2	23	10	31	34	55
24	1	64	41	56	33	32	9

N3

Table 1: Franklin squares constructed using Hilbert basis [1].

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29	35	18	48	21	43	26	40
20	46	31	33	28	38	23	41
45	19	34	32	37	27	42	24
36	30	47	17	44	22	39	25
61	3	50	16	53	11	58	8
52	14	63	1	60	6	55	9
13	51	2	64	5	59	10	56
4	62	15	49	12	54	7	57

B1

29	35	18	48	25	39	22	44
20	46	31	33	24	42	27	37
45	19	34	32	41	23	38	28
36	30	47	17	40	26	43	21
61	3	50	16	57	7	54	12
52	14	63	1	56	10	59	5
13	51	2	64	9	55	6	60
4	62	15	49	8	58	11	53

B2

Table 2: New Franklin squares.

From now on, row sum, column sum, or bend diagonal sum, etc. mean that we are adding the entries of those elements. Based on the descriptions of Benjamin Franklin, we define *Franklin squares* as follows (see [2]).

Definition 1 (Franklin Square)**.** *Consider an integer, $n = 2^r$ such that $r \geqslant 3$. Let the magic sum be denoted by M and $N = n^2 + 1$. We define an $n \times n$* Franklin square *to be a $n \times n$ matrix with the following properties:*

1. *Every integer from the set $\{1, 2, \ldots, n^2\}$ occurs exactly once in the square. Consequently,*
$$M = \frac{n}{2}N.$$
2. *All the the half rows, half columns add to one-half the magic sum. Consequently, all the rows and columns add to the magic sum.*
3. *All the bend diagonals add to the magic sum, continuously.*
4. *All the 2×2 sub-squares add to $4M/n = 2N$, continuously. Consequently, all the 4×4 sub-squares add to $8N$, and the four corner numbers with the four middle numbers add to $4N$.*

We call permutations of the entries of a Franklin square that preserve the defining properties of the Franklin squares *symmetry operations*, and two squares are called *isomorphic* if we can get one from the other by applying symmetry operations. Rotation and reflection are clearly symmetry operations. See [1] for more symmetry operations like exchanging specific rows or columns. In [1], we showed that F1 and F2 are not isomorphic to each other.

Constructing Franklin squares are demanding and only a handful such squares are known to date. We showed how to construct F1, F2, and F3 using methods from Algebra and Combinatorics in [1]. In the same article, for the first time since Benjamin Franklin, we constructed new Franklin squares N1, N2, and N3, given in Table 1. We proved that these squares were not isomorphic to each other nor to F1 or F2. In other words, these squares were really new. Those methods being computationally challenging are not suitable for higher order Franklin squares. Moreover, the constructions used computers and hence lacked the intrigue of Franklin's constructions. In [2], we followed Benjamin Franklin closely, and used elementary techniques to construct a Franklin square of every order. With these techniques we are able to construct F1 and F3, but not F2. In Section , we modify the methods in [2] to construct F2. In Section , we create new Franklin squares B1 and B2, given in Table 2, but this time using elementary techniques, in keeping with the true spirit of Benjamin Franklin.

Franklin's construction of his other 8-square

In this section, we show how to construct the Franklin square F2.

		4	13
14	3		
		5	12
11	6		
		7	10
9	8		
		2	15
16	1		

N−13	N−4	4	13
14	3	N−3	N−14
N−12	N−5	5	12
11	6	N−6	N−11
N−10	N−7	7	10
9	8	N−8	N−9
N−15	N−2	2	15
16	1	N−1	N−16

Figure 2: Construction of the left half of the Franklin square F1.

20	29		
		30	19
21	28		
		27	22
23	26		
		25	24
18	31		
		32	17

20	29	N−29	N−20
N−19	N−30	30	19
21	28	N−28	N−21
N−22	N−27	27	22
23	26	N−26	N−23
N−24	N−25	25	24
18	31	N−31	N−18
N−17	N−32	32	17

Figure 3: Construction of the right half of the Franklin square F1.

We present a quick recap of our construction of F1 in [2]. Note that $N = 65$ throughout this article. The 8×8 square was divided in to two halves: the left half consisting of first four columns and the right half consisting of the last four columns. The left half is constructed by first placing the numbers $1, 2, \ldots, 16$ in a specific pattern as shown in Figure 2. Then, the number $N - i$ is placed in the same row that contains i, where $i = 1, \ldots, 16$, such that i and $N - i$ are always equidistant from the middle of the left half. See Figure 2. Similarly, the second half is constructed by first placing numbers $17, 18, \ldots, 32$ in a specific pattern. Finally, the numbers $N - i$ is placed in the same row that contains i, where $i = 17, \ldots, 32$, such that i and $N - i$ are always equidistant from the middle of the right half. This procedure is explained in Figure 3. The algorithm for generating the pattern of $1, 2, \ldots, 32$ in F1 is given in [2]. We also generalized this procedure to construct a Franklin square for any given order in [2]. From now on, throughout the article, when we say *pattern of a square*, we mean the pattern of the numbers $1, 2, \ldots, 32$ in the square.

17		24		22		19	
32		25		27		30	
	31		26		28		29
	18		23		21		20
	15		10		12		13
	2		7		5		4
1		8		6		3	
16		9		11		14	

17	47	24	42	22	44	19	45
32	34	25	39	27	37	30	36
33	31	40	26	38	28	35	29
48	18	41	23	43	21	46	20
49	15	56	10	54	12	51	13
64	2	57	7	59	5	62	4
1	63	8	58	6	60	3	61
16	50	9	55	11	53	14	52

Figure 4: Construction of Franklin square S.

In our search for new Franklin squares, it is natural to swap rows for columns in the above method. Thus, we divide the 8×8 square in to two halves : the top half consisting of the first four rows and the bottom half consisting of the last four rows. We place the numbers $1, \ldots, 16$ in the bottom half, and the numbers $17, \ldots, 32$ in the top half in the pattern shown in Figure 4. This time we place the numbers $N - i$ in the same columns as i, equidistant from the middle of each part. The Franklin square S we get from this procedure is isomorphic to F1: check that we can derive this square from F1 by rotating it and then reflecting it. The square S is not interesting because it is essentially the same as F1. But the pattern of entering the numbers $1, 2, \ldots, 32$ in S is algorithmic since it is just a row version of the pattern of F1. That is, instead of filling two columns at a time, we fill two rows at a time. We modify the pattern of S to get a pattern for F2. After that, we place $N - i$ in the same column as i, for $i = 1, \ldots, 32$, equidistant from the center of each half, as usual. See Figure 5 for the construction of F2.

17		30		21		26	
32		19		28		23	
	31		20		27		24
	18		29		22		25
	15		4		11		8
	2		13		6		9
1		14		5		10	
16		3		12		7	

17	$N-18$	30	$N-29$	21	$N-22$	26	$N-25$
32	$N-31$	19	$N-20$	28	$N-27$	23	$N-24$
$N-32$	31	$N-19$	20	$N-28$	27	$N-23$	24
$N-17$	18	$N-30$	29	$N-21$	22	$N-26$	25
$N-16$	15	$N-3$	4	$N-12$	11	$N-7$	8
$N-1$	2	$N-14$	13	$N-5$	6	$N-10$	9
1	$N-2$	14	$N-13$	5	$N-6$	10	$N-9$
16	$N-15$	3	$N-4$	12	$N-11$	7	$N-8$

Figure 5: Constructing Franklin square F2.

The construction of F2 involved moving columns of in the $1, \ldots, 32$ pattern of the square S and then switching elements along the diagonals. This pattern is not easily generalized to higher orders. But in the next Section, we modify the patterns of known squares to create new squares.

Constructing new Franklin squares

In this section we construct the new Franklin squares B1 and B2 (see Table 2).

3		16		7		12	
32		19		28		23	
	31		20		27		24
	4		15		8		11
	29		18		25		22
	2		13		6		9
1		14		5		10	
30		17		26		21	

N1

17		30		25		22	
32		19		24		27	
	31		20		23		28
	18		29		26		21
	15		4		7		12
	2		13		10		5
1		14		9		6	
16		3		8		11	

N2

		4	21	12	29		
22	3					30	11
		5	20	13	28		
19	6					27	14
		7	18	15	26		
17	8					25	16
		2	23	10	31		
24	1					32	9

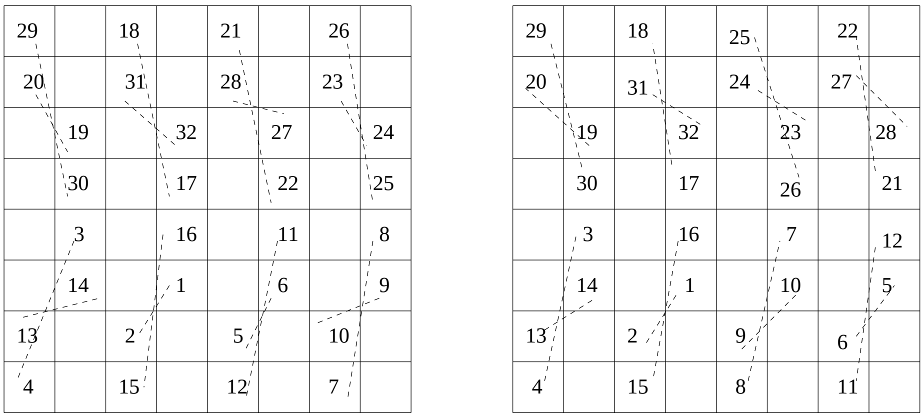
N3

Figure 6: The patterns of Franklin squares N1, N2, and N3.

To begin with, we look at the patterns of the numbers $1, 2, \ldots, 32$ appearing in the Franklin squares N1, N2, and N3. These are shown in Figure 6. We found that even in these squares that were constructed using Hilbert bases (see [1]), the strategy of finding the pattern of the Franklin square, and then placing i and $N - i$ in the same row or column, as the case may be, equidistant from the center of relevant half of the square, worked. Observe that the patterns of N1 and N2 are derived by modifying the pattern of F2. So in their constructions, we will place i and $N - i$ in the same columns. On the other hand, since, N3 is a permutation of the pattern of F1, we place i and $N - i$ in the same rows while constructing N3. Benjamin Franklin's patterns always restrict the entries

1, 2, . . . , 16 to one half of the square. Observe that N2 is the only square with this property.

Guided by these observations, we modified the pattern of F2 to build the new Franklin squares B1 and B2. See Figure 7 for the patterns of B1 and B2. Since the patterns were derived from F2, to construct them, we will place i and $N - i$ in the same columns. B1 and B2 are not isomorphic to each other or any other known squares because the columns were made different by construction.



B1 B2

Figure 7: The patterns of the Franklin squares B1 and B2.

Clearly, there are more Franklin squares. To find them, we need to find more patterns that will yield a Franklin square. So it is time to ask again "How many squares are there Mr. Franklin?"

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