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Games and Puzzles

ETERNAL PICARIA

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Abstract: *Picaria is a traditional board game, played by the Zuni tribe of the American Southwest and other parts of the world, such as a rural Southwest region in Sweden. It is related to the popular children's game of Tic-tac-toe, but the 2 players have only 3 stones each, and in the second phase of the game, pieces are slided, along specified move edges, in attempts to create the three-in-a-row. We provide a rigorous solution, and prove that the game is a draw; moreover our solution gives insights to strategies that players can use.*

Keywords: Abstract strategy game, Alignment game, Board game, Cyclic game, Loopy game, Luffarschack, Play-proof, Tapatan, Three-in-a-row, Three men's morris, Zuni tribe.

Introduction

Picaria is a traditional board game, played by the Zuni tribe of the American Southwest and other parts of the world, such as in a rural Southwest region in Sweden[1]. The game is related to the popular children's game of Tic-tac-toe, but it is even more related to other three-in-a-row games such as Three men's morris, Tapatan, Nine Holes, Achi, Tant Fant and Shisma. These latter games are sometimes played in two phases, the first phase being placement of stones, and the second part being sliding of stones along prescribed "move edges". In either case, the possibility of infinite play puts Picaria in a different class than Tic-tac-toe.

The "blockade" games of Pong Hau K'i, from China, and Mu Torere, played by the Māori people from the east coast of New Zealand's North Island, are also related (in these games, if a player cannot move, he loses); with quite few

[*]Supported by the Killam trust.

[1]The "first" author played this game as a child with his grandparents in the village Rångedala close to the Swedish city Borås, and the game was called "luffarschack".

positions, only 16 and 46 respectively, these games are solved [2] by depicting the position graphs.

In Picaria, there are two players who alternate turns, and the goal is to be the first player to place 3 game pieces of a kind in a row, vertically, horizontally or diagonally. Each player has their own type of pieces, say, to use the convention of Tic-tac-toe, X and O. In our study, we assume that player X starts. The players alternate turn to (in phase 1) place their stones in an open space in a 3 by 3 grid. When each piece has been played in the first phase (and assuming a non-loss so far), then player X begins the second phase by sliding one of the three Xs to an empty adjacent node; then O slides a stone, and so on. Adjacency here means a neighboring node, horizontally, vertically or diagonally. A game position is declared a draw if periodicity of a pattern is forced by one of the players. In this paper we give a constructive proof that the opening position (the empty board) is a draw[2].

Picaria was described in the literature for the first time in 1907 by the ethnographer Stewart Culin [1]. The original board of Picaria is displayed in Figure 1. The players place stones on the vertices of the graph and slide along the edges. In this paper we play the game in an equivalent manner using instead a Tic-tac-toe board.

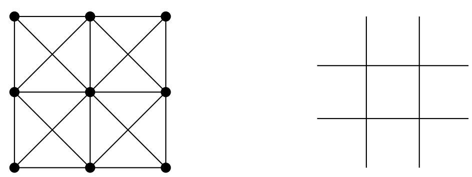
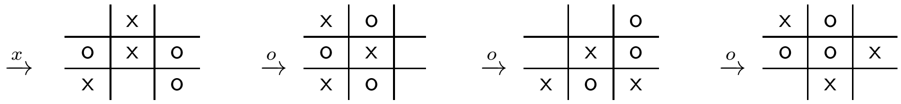


Figure 1: The original Picaria board and a Tic-tac-toe board.

Before all stones are on the board, the number of positions coincide with those of Tic-tac-toe. Subsequently there are 456 positions modulo symmetries (see Section 3 for details) and many of those positions could be revisited during the course of a game, so notably play is very different from Tic-tac-toe. Even though a computer could be used to solve Picaria, this would not provide a full understanding on how to play a successful strategy. In this paper we give the explicit *strategies* of optimal play, which means that perfect information players will not play to draw if they can win and they will not play to lose if they can avoid loss. The latter idea will be useful at particular stages of our play proofs. In optimal play, if a player revisits a position, then the game is a

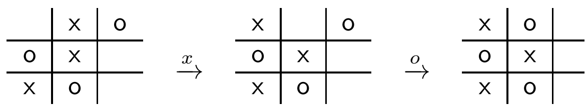
[2]The historical popularity of the game is probably due to the fact that it is very easy to make a single mistake and then there is a human player winner. As a children's game, by the author's experience, the outcome is rarely a draw. After many plays though, over several years, there is a kind of certainty that both players should be able to draw, and here we show how. By this play heuristics we believe that any proof would be non-trivial.

draw. It turns out that both players are able to draw from the initial position. As a consequence, efficient strategies to win by for example using *Fork*-, *Trap-, Race-, or Zugzwang-positions* (as illustrated), will *not* occur in optimal play



Fork Trap Race Zugzwang

Note here that a game position is depicted as a game board together with a flag for who just moved. Often however we omit the move-flag, because it is clear by the context which play position we discuss (for example in the placement phase of the game). For example, play just before the Trap-position involves a bad move by X



Trap

Such play does not belong to X's strategy, and similar ideas are commonly applied inside proofs.

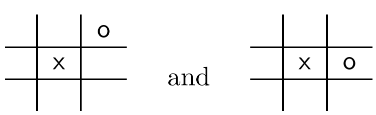
A Rigorous Play-analysis of Picaria

In our convention X is the first player. We show that player O can prevent X from winning, by forcing X to play a periodic sequence of moves. That would prove that the game is a draw if X can prevent O from winning too. We begin by proving that it is easy for X to avoid loss.

The second player cannot win Picaria

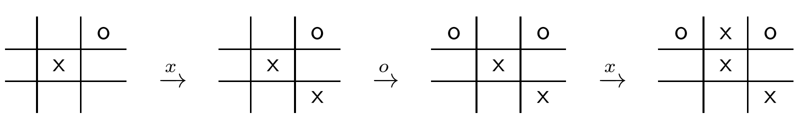
Theorem 1. *Player O cannot win.*

Proof. Player X starts by playing in the center and then there are two cases to consider



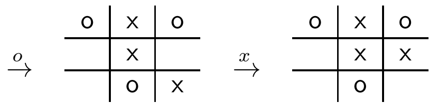
(i) (ii)

For game (i)



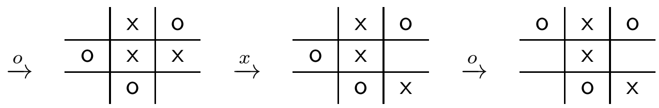
(i)

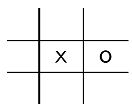
DOI 10.1515/rmm–2017–0016

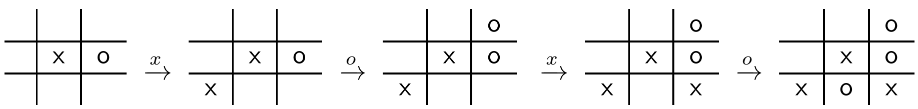


(A)

Thus, X can force a return to the game (A), depicted above, by



On the other hand, game (ii)  is losing for player O, since



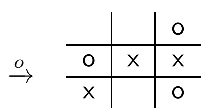
(ii)

and this is a Race-position, from which player X wins in two moves.

□

The first player cannot win playing from a Loop position

Next we analyse a special configuration which is quite recurring in the game, called a *Loop position*:



Loop

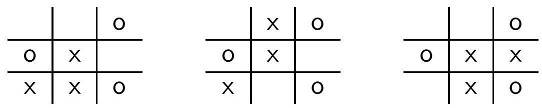
Any other symmetric position (a rotation or reflection; see for example game (A) in the proof of Theorem 1) is also called a Loop position[3]. The reason for this will be clear in the next result, Theorem 5, where we show that O prevents X from winning the game by means of a periodic sequence of moves. Consider that player X *holds the center*. This restricts the possibilities for X in that only the two outer stones can be moves. In particular if X starts from the Loop position, then player X cannot win, which constitutes our next lemma. When Picaria is played by human (non-optimal) players, a player holding the center often appears to enjoy a certain advantage. The next results also revolve around the idea of a player either holding or leaving the center.

Lemma 2. *If player X is to move and it refuses to leave the center starting from a Loop position, then player O can force a return to this position.*

[3]We use loop in the sense of something reappearing, although in graph theoretical terms it would be more correct to call this a cycle.

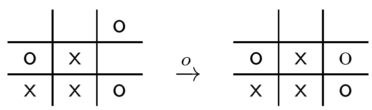
DOI 10.1515/rmm–2017–0016

Proof. There are only three possible moves from a Loop-position for X, since it holds the center.



(A) (B) (C)

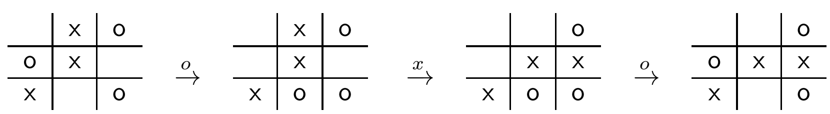
For game (A), player X gets trapped by



(A)

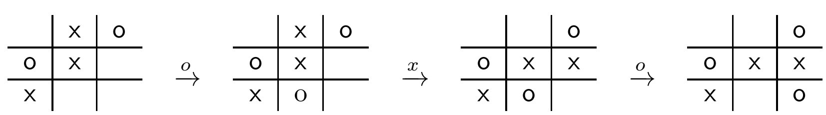
So X would have no option but give up the center and clearly loses the game. Therefore, the game (A) does not belong to X's strategy.

Now, player O can force either of the following sequences



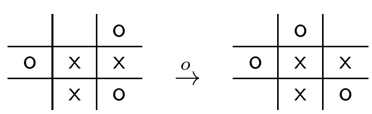
(B)

or



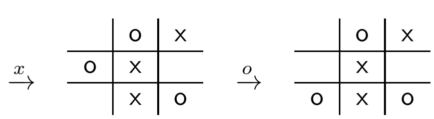
(B)

which is the initial (Loop) position. Note that in the second diagram X does not move into the Trap position. Finally,



(C)

Now by symmetry X moves to



which returns to the initial Loop-position. □

The next lemma concerns a "dual" result for a Loop-position.

Lemma 3. *If O is to move from* $\begin{array}{c|c|c} & & x \\ \hline x & o & o \\ \hline o & & x \end{array}$ *then O can force a Loop.*

Proof. Player O begins with

$$\begin{array}{c|c|c} & & \mathsf{X} \\ \hline \mathsf{X} & \mathsf{O} & \mathsf{O} \\ \hline \mathsf{O} & & \mathsf{X} \end{array} \xrightarrow{o} \begin{array}{c|c|c} & \mathsf{O} & \mathsf{X} \\ \hline \mathsf{X} & \mathsf{O} & \\ \hline \mathsf{O} & & \mathsf{X} \end{array}$$

Now X has two defense possibilities

$$\xrightarrow{x} \underset{\text{(A)}}{\begin{array}{c|c|c} & \mathsf{O} & \mathsf{X} \\ \hline & \mathsf{O} & \\ \hline \mathsf{O} & \mathsf{X} & \mathsf{X} \end{array}} \quad \text{or} \quad \xrightarrow{x} \underset{\text{(B)}}{\begin{array}{c|c|c} & \mathsf{O} & \mathsf{X} \\ \hline \mathsf{X} & \mathsf{O} & \\ \hline \mathsf{O} & \mathsf{X} & \end{array}}$$

For game (A)

$$\underset{\text{(A)}}{\begin{array}{c|c|c} & \mathsf{O} & \mathsf{X} \\ \hline & \mathsf{O} & \\ \hline \mathsf{O} & \mathsf{X} & \mathsf{X} \end{array}} \xrightarrow{o} \begin{array}{c|c|c} & & \mathsf{X} \\ \hline & \mathsf{O} & \mathsf{O} \\ \hline \mathsf{O} & \mathsf{X} & \mathsf{X} \end{array} \xrightarrow{x} \begin{array}{c|c|c} & & \mathsf{X} \\ \hline \mathsf{X} & \mathsf{O} & \mathsf{O} \\ \hline \mathsf{O} & & \mathsf{X} \end{array}$$ which is the original position.

Similarly, for game (B)

$$\underset{\text{(B)}}{\begin{array}{c|c|c} & \mathsf{O} & \mathsf{X} \\ \hline \mathsf{X} & \mathsf{O} & \\ \hline \mathsf{O} & \mathsf{X} & \end{array}} \xrightarrow{o} \begin{array}{c|c|c} & & \mathsf{X} \\ \hline \mathsf{X} & \mathsf{O} & \mathsf{O} \\ \hline \mathsf{O} & \mathsf{X} & \end{array} \xrightarrow{x} \begin{array}{c|c|c} & & \mathsf{X} \\ \hline \mathsf{X} & \mathsf{O} & \mathsf{O} \\ \hline \mathsf{O} & & \mathsf{X} \end{array}$$

which is again the original position. □

Lemma 4. *If X is to move from Loop position, and it leaves the center, then X cannot win.*

Proof. Suppose first that X leaves the center in the first move. There are the following possibilities

$$\underset{\text{(A)}}{\begin{array}{c|c|c} & \mathsf{X} & \mathsf{O} \\ \hline \mathsf{O} & & \mathsf{X} \\ \hline \mathsf{X} & & \mathsf{O} \end{array}} \qquad \underset{\text{(B)}}{\begin{array}{c|c|c} & & \mathsf{O} \\ \hline \mathsf{O} & & \mathsf{X} \\ \hline \mathsf{X} & \mathsf{X} & \mathsf{O} \end{array}} \qquad \underset{\text{(C)}}{\begin{array}{c|c|c} \mathsf{X} & & \mathsf{O} \\ \hline \mathsf{O} & & \mathsf{X} \\ \hline \mathsf{X} & & \mathsf{O} \end{array}}$$

For game (A), O can force the sequence

$$\underset{\text{(A)}}{\begin{array}{c|c|c} & \mathsf{X} & \mathsf{O} \\ \hline \mathsf{O} & & \mathsf{X} \\ \hline \mathsf{X} & & \mathsf{O} \end{array}} \xrightarrow{o}$$

$$\begin{array}{c|c|c} & \mathsf{X} & \\ \hline \mathsf{O} & \mathsf{O} & \mathsf{X} \\ \hline \mathsf{X} & & \mathsf{O} \end{array} \xrightarrow{x} \begin{array}{c|c|c} \mathsf{X} & & \\ \hline \mathsf{O} & \mathsf{O} & \mathsf{X} \\ \hline \mathsf{X} & & \mathsf{O} \end{array}$$

Now by Lemma 3, player X cannot win. If X creates game (B), then it Loses since O can do

$$\underset{\text{(B)}}{\begin{array}{c|c|c} & & \mathsf{O} \\ \hline \mathsf{O} & & \mathsf{X} \\ \hline \mathsf{X} & \mathsf{X} & \mathsf{O} \end{array}} \xrightarrow{o} \begin{array}{c|c|c} & & \\ \hline \mathsf{O} & \mathsf{O} & \mathsf{X} \\ \hline \mathsf{X} & \mathsf{X} & \mathsf{O} \end{array} \xrightarrow{x} \begin{array}{c|c|c} & & \mathsf{X} \\ \hline \mathsf{O} & \mathsf{O} & \\ \hline \mathsf{X} & \mathsf{X} & \mathsf{O} \end{array} \xrightarrow{o} \begin{array}{c|c|c} & & \mathsf{X} \\ \hline \mathsf{O} & \mathsf{O} & \mathsf{O} \\ \hline \mathsf{X} & \mathsf{X} & \end{array}$$

So game (B) does not happen. For game (C), player O moves

$$\underset{\text{(C)}}{\begin{array}{c|c|c} \mathsf{X} & & \mathsf{O} \\ \hline \mathsf{O} & & \mathsf{X} \\ \hline \mathsf{X} & & \mathsf{O} \end{array}} \xrightarrow{o} \begin{array}{c|c|c} \mathsf{X} & & \mathsf{O} \\ \hline & \mathsf{O} & \mathsf{X} \\ \hline \mathsf{X} & & \mathsf{O} \end{array}$$

If player X moves $\xrightarrow{x} \begin{array}{c|c|c} & & \mathsf{O} \\ \hline \mathsf{X} & \mathsf{O} & \mathsf{X} \\ \hline \mathsf{X} & & \mathsf{O} \end{array} \xrightarrow{o} \begin{array}{c|c|c} & \mathsf{O} & \\ \hline \mathsf{X} & \mathsf{O} & \mathsf{X} \\ \hline \mathsf{X} & & \mathsf{O} \end{array}$

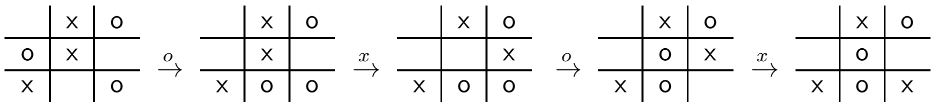
by which player O wins the game. So instead X plays

$\xrightarrow{x} \begin{array}{c|c|c} & \mathsf{X} & \mathsf{O} \\ \hline & \mathsf{O} & \mathsf{X} \\ \hline \mathsf{X} & & \mathsf{O} \end{array}$ and O responds by forcing X

$$\xrightarrow{o} \begin{array}{c|c|c} & \mathsf{X} & \mathsf{O} \\ \hline & \mathsf{O} & \mathsf{X} \\ \hline \mathsf{X} & \mathsf{O} & \end{array} \xrightarrow{x} \begin{array}{c|c|c} & \mathsf{X} & \mathsf{O} \\ \hline & \mathsf{O} & \\ \hline \mathsf{X} & \mathsf{O} & \mathsf{X} \end{array}$$

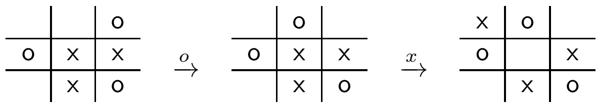
Now by Lemma 3, player X cannot win.

Suppose next that player X leaves the center but not in the first move. Now we use the games (A), (B) and (C) from Lemma 2, and note that the only missing case is that X leaves the center in the second move after the Loop-position (this follows because otherwise O has already returned to Loop). Case (A) is not in X's strategy, and case (B) becomes instead



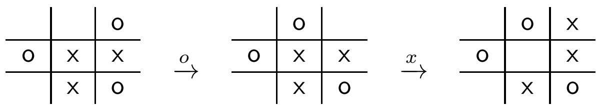
(B)

from which player O has a non-losing strategy by Lemma 3. In case (C), we get



(C)

or



(C)

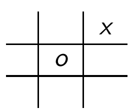
In the first case O creates a Zugzwang by moving into the center, and in the second case, O can trap X. □

Theorem 5. *Player X cannot win moving from a Loop position.*

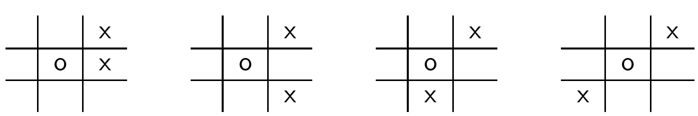
Proof. The proof follows from Lemmas 2 and 4. □

Games with two stones on the board

There are three positions with exactly two stones modulo symmetries. We begin by ruling out that player O gets to start in the center.

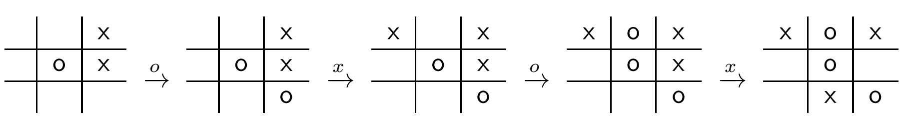
Lemma 6. *For game*  *X cannot win.*

Proof. We have the following possibilities for player X



(A) (B) (C) (D)

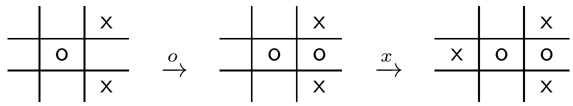
For game (A), player O can force a Loop by



(A)

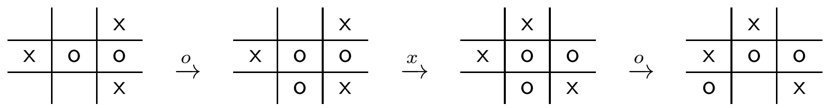
DOI 10.1515/rmm–2017–0016

For game (B), player X must respond

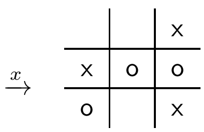


(B)

and now player O can draw the game by

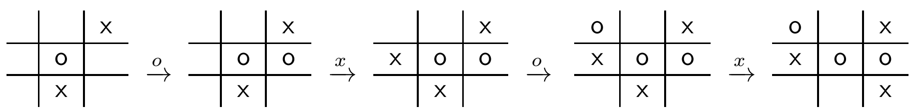


(E)



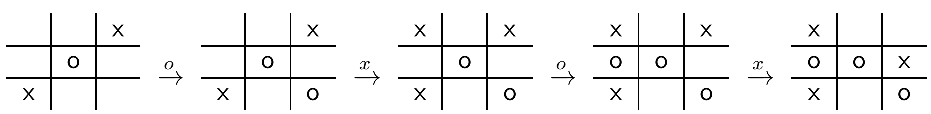
which is a Loop.

For game (C), if player O forces the sequence of positions



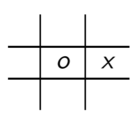
(C)

then, by Lemma 3 X cannot win. Similarly, for game (D), if O forces the sequence

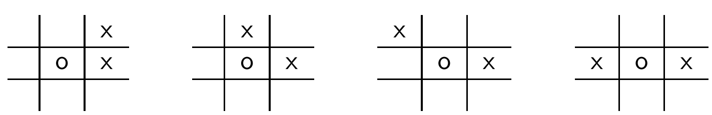


(D)

Then by Lemma 3 player X cannot win. ☐

Lemma 7. *For game*  *X cannot win.*

Proof. Here player X has the possibilities



(A) (B) (C) (D)

Games (A) and (C) are the same games as (A) and (C) in the proof of Theorem 6.

For game (B), player O can force X's moves by

$$\underset{\text{(B)}}{\begin{array}{c|c|c} & x & \\ \hline & o & x \\ \hline & & \end{array}} \xrightarrow{o} \begin{array}{c|c|c} & x & o \\ \hline & o & x \\ \hline & & \end{array} \xrightarrow{x} \begin{array}{c|c|c} & x & o \\ \hline & o & x \\ \hline x & & \end{array} \xrightarrow{o} \begin{array}{c|c|c} & x & o \\ \hline & o & x \\ \hline x & o & \end{array} \xrightarrow{x} \begin{array}{c|c|c} & x & o \\ \hline & o & \\ \hline x & o & x \end{array}$$

Then by Lemma 3, X cannot win. For game (D), player O can force X's moves by

$$\underset{\text{(D)}}{\begin{array}{c|c|c} & & \\ \hline x & o & x \\ \hline & & \end{array}} \xrightarrow{o} \begin{array}{c|c|c} & & \\ \hline x & o & x \\ \hline & o & \end{array} \xrightarrow{x} \begin{array}{c|c|c} & x & \\ \hline x & o & x \\ \hline & o & \end{array} \xrightarrow{o} \begin{array}{c|c|c} & x & o \\ \hline x & o & x \\ \hline & o & \end{array}$$

$$\xrightarrow{x} \begin{array}{c|c|c} & x & o \\ \hline & o & x \\ \hline x & o & \end{array} \xrightarrow{o} \begin{array}{c|c|c} & x & o \\ \hline o & o & x \\ \hline x & & \end{array} \xrightarrow{x} \begin{array}{c|c|c} x & & o \\ \hline o & o & x \\ \hline x & & \end{array}$$

Then, by Lemma 3, player X cannot win. ∎

The next result deals with the most delicate position. There are some ideas that are very interesting, that did not appear so far.

Lemma 8. *For game* $\begin{array}{c|c|c} & & o \\ \hline & x & \\ \hline & & \end{array}$ *X cannot win.*

Proof. We have the cases to check

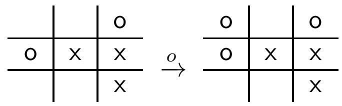
$$\underset{\text{(A)}}{\begin{array}{c|c|c} & & o \\ \hline & x & x \\ \hline & & \end{array}} \qquad \underset{\text{(B)}}{\begin{array}{c|c|c} & & o \\ \hline & x & \\ \hline & & x \end{array}} \qquad \underset{\text{(C)}}{\begin{array}{c|c|c} & & o \\ \hline & x & \\ \hline & x & \end{array}} \qquad \underset{\text{(D)}}{\begin{array}{c|c|c} & & o \\ \hline & x & \\ \hline x & & \end{array}}$$

For position (A),

$$\underset{\text{(A)}}{\begin{array}{c|c|c} & & o \\ \hline & x & x \\ \hline & & \end{array}} \xrightarrow{o} \begin{array}{c|c|c} & & o \\ \hline o & x & x \\ \hline & & \end{array} \quad \text{and X has the possibilities}$$

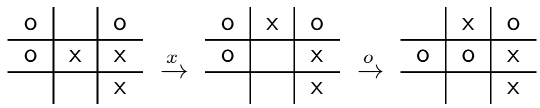
$$\underset{(1)}{\begin{array}{c|c|c} & & o \\ \hline o & x & x \\ \hline & & x \end{array}} \qquad \underset{(2)}{\begin{array}{c|c|c} & & o \\ \hline o & x & x \\ \hline & x & \end{array}} \qquad \underset{(3)}{\begin{array}{c|c|c} & & o \\ \hline o & x & x \\ \hline x & & \end{array}} \qquad \underset{(4)}{\begin{array}{c|c|c} x & & o \\ \hline o & x & x \\ \hline & & \end{array}} \qquad \underset{(5)}{\begin{array}{c|c|c} & x & o \\ \hline o & x & x \\ \hline & & \end{array}}$$

For position 1,



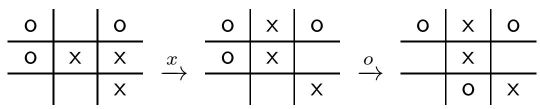
(1)

Now if



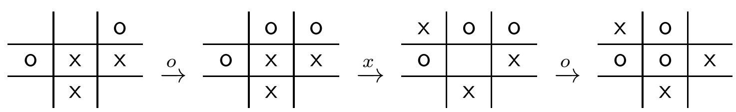
and X cannot avoid a loss,

and if



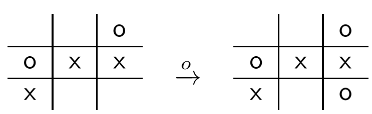
player O creates a Loop.

For position 2,



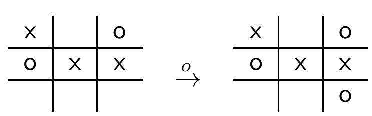
(2)

which is a zugzwang, so no matter what player X does, it loses the game. For position 3, player O creates a Loop by



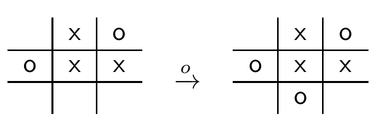
(3)

Position 4 forces player O to create a Loop by



(4)

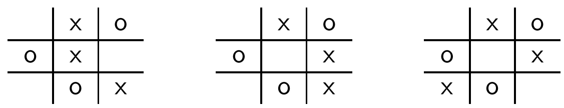
For position (5)



(5)

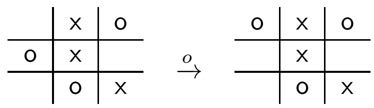
and now player X has 3 options, disregarding symmetries

DOI 10.1515/rmm–2017–0016



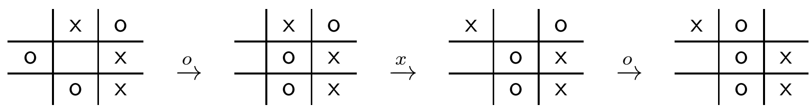
(i) (ii) (iii)

For (i), player O is forced to create a Loop by



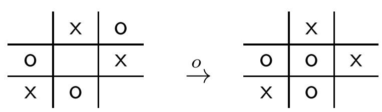
(i)

For (ii), player X is trapped by



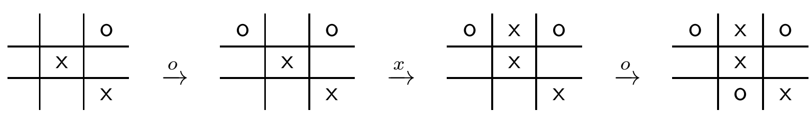
(ii)

So X does not create game (ii). Finally, game (iii) allows player O to win by a Zugzwang



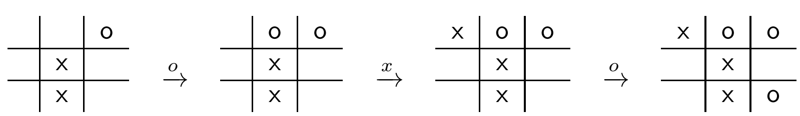
(iii)

Thus player X does not create game (iii). That finishes game (A). For game (B), all moves are forced, which creates a Loop as follows



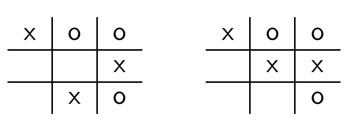
(B)

For game (C), there are the following forced moves



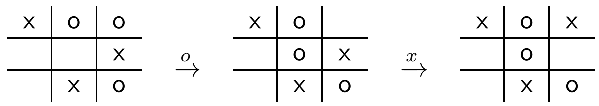
(C)

Now player X must choose one of the options



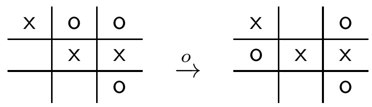
(i) (ii)

For game (i), player O forces a Loop by



(i)

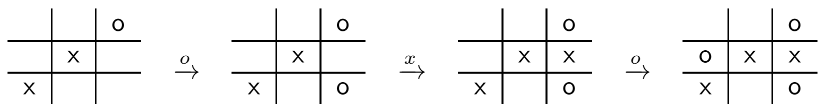
For game (ii) player O creates a Loop by



(ii)

which finishes game (C).

For game (D), player O forces a Loop by



(D)

which concludes the proof. ☐

Summing up the results

Theorem 9. *Player X cannot win.*

Proof. This follows by combining Lemmas 7, 6 and 8. ☐

Corollary 10. *Picaria is a draw.*

Proof. This is a consequence of Theorems 1 and 9. ☐

The number of positions in the second phase

To count the number of positions we apply Burnside's Lemma to count equivalent classes of positions. Besides, we only count the number of positions for the second phase, i.e., when all six pieces are placed on the board. We consider the set P the possible positions in the game and G the group acting on P. Denote by $\mathrm{Fix}(g)$ the set of positions in P that are fixed by g. Burnside's Lemma states that the number of orbits is

$$\#\mathrm{orb} = \frac{1}{|G|}\sum_{g\in G} |\mathrm{Fix}(g)|\,.$$

The elements in G are composed by the identity, denoted by e, three rotations of the board of 90, 180, and 270 degrees denoted by R_{90}, R_{180}, and R_{270}. Besides, there are four reflections, one horizontal H, one vertical V, and two diagonal D_1 and D_2. Thus

DOI 10.1515/rmm–2017–0016

$$
\begin{aligned}
\#\text{orb} &= \frac{1}{8}(|\text{Fix}(e)| + |\text{Fix}(R_{90})| + |\text{Fix}(R_{180})| + |\text{Fix}(R_{270})| \\
&\quad + |\text{Fix}(H)| + |\text{Fix}(V)| + |\text{Fix}(D_1)| + |\text{Fix}(D_2)|).
\end{aligned}
$$

For the identity e we have $\binom{9}{3,3,3} = \frac{9!}{3!3!3!}$ positions. For R_{90} and R_{180} rotations with a fixed position the board is of the respective form

$$
\begin{array}{c|c|c}
\text{b} & \text{a} & \text{d} \\ \hline
\text{a} & \text{e} & \text{c} \\ \hline
\text{b} & \text{c} & \text{d}
\end{array}
\qquad \text{and} \qquad
\begin{array}{c|c|c}
\text{d} & \text{b} & \text{c} \\ \hline
\text{a} & \text{e} & \text{a} \\ \hline
\text{c} & \text{b} & \text{d}
\end{array}
$$

Here the letters can be X, O, or empty. Notice that $|R_{90}| = |R_{270}|$. If we put all pieces on one of these boards, there would be 4 pairs, each pair with the same symbol, which is not possible. Thus, the rotations do not fix any position.

For diagonal reflections the number of fixed positions are the same for D_1 and D_2. Consider the diagonal D_1 and horizontal reflections with a fixed position of the respective forms

$$
\begin{array}{c|c|c}
\text{b} & \text{c} & \text{f} \\ \hline
\text{a} & \text{e} & \text{c} \\ \hline
\text{d} & \text{a} & \text{b}
\end{array}
\qquad \text{and} \qquad
\begin{array}{c|c|c}
\text{a} & \text{b} & \text{c} \\ \hline
\text{d} & \text{e} & \text{f} \\ \hline
\text{a} & \text{b} & \text{c}
\end{array}
$$

These are the only fixed positions by these group actions. Here the letters can be X, O, or empty. It follows that a,b, and c are precisely one each of X, O, or empty, and the same is true for d,e, and f. Thus, there are 3!3! such positions of each kind. Since there are four reflections, we have in total 3!3!4 fixed positions.

Finally, we have

$$
\begin{aligned}
\#\text{orb} &= \frac{1}{8}(|\text{Fix}(e)| + |\text{Fix}(R_{90})| + |\text{Fix}(R_{180})| + |\text{Fix}(R_{270})| \\
&\quad + |\text{Fix}(H)| + |\text{Fix}(V)| + |\text{Fix}(D_1)| + |\text{Fix}(D_2)|) \\
&= \frac{1}{8}\left(\frac{9!}{3!3!3!} + 3!3!4\right) = 228.
\end{aligned}
$$

Finally, we counted three positions that do not occur. Namely

$$
\begin{array}{c|c|c}
 & \text{o} & \text{x} \\ \hline
 & \text{o} & \text{x} \\ \hline
 & \text{o} & \text{x}
\end{array}
\qquad
\begin{array}{c|c|c}
\text{o} & & \text{x} \\ \hline
\text{o} & & \text{x} \\ \hline
\text{o} & & \text{x}
\end{array}
\qquad
\begin{array}{c|c|c}
 & \text{x} & \text{o} \\ \hline
 & \text{x} & \text{o} \\ \hline
 & \text{x} & \text{o}
\end{array}
$$

Thus we get 225 orbits. Now, the position graph for the second phase of the game contains positions where player X or O is to move. Thus, the position graph for this game contains 450 positions.

Final Remarks

Consider the following natural generalizations of Picaria. We use the same set of rules and only change the number of sides of the board as depicted.

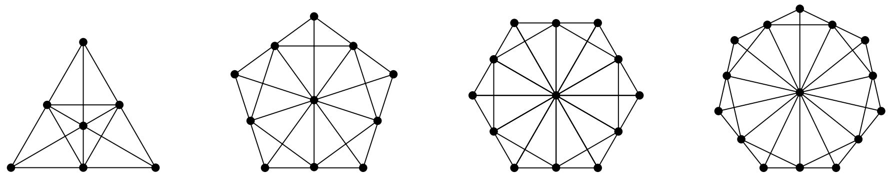


Figure 2: Relatives of Picaria, with three, five, six and seven sides respectively. Following this pattern, there are infinitely many different board games in this family. One can show that all these games are first player win in a few moves, except for Picaria, and we invite the reader to find the play-proofs of this fact. We find it compelling that the Zuni tribe played the only interesting game in this family for centuries.

Open problems

What happens if we increase the number of stones for each player, say that game parameters, $k \geqslant 3$ stones each and $s \geqslant 3$ sides, are given (otherwise the same rules). Is there any combination (k, s), other than $(3, 4)$, for which the game is a draw (provided that the total number of stones is less than the number of nodes)? Is it true that the second player never wins? If we give the second player a one stone advantage (handicap), for which combination (k, s) can he draw/win the $((k, k+1), s)$ game (that is the second player places his last placement stone after the first slide-along-edge move by the first player)? In general, how many stones advantage $l > 0$ does he require to draw/win a generalized Picaria?

References

[1] Culin, Stewart. *Games of North American Indians*, Washington DC: US gov Printing Office, 1907.

[2] Philip D. Straffin, Jr. "Position Graphs for Pong Hau K'i and Mu Torere", *Mathematics Magazine*, 68, 5, pp. 382–386, 1995.