Games and Puzzles

Star tangrams

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Abstract: *The Tangram is a puzzle in which seven tiles are arranged to make various shapes. Four families of tangram shapes have been mathematically defined and their members enumerated. This paper defines a fifth family, enumerates its members, explains its taxonomic relationship with the previously-defined families, and provides some interesting examples.*

Keywords: Tangrams, combinatorics, spatial puzzles.

Introduction

In the puzzle known as the Tangram, players rearrange the seven tiles shown in Figure 1 to make various shapes.

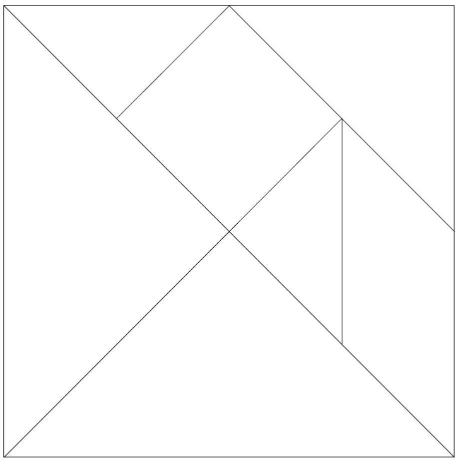

Figure 1: Tangram.

Some shapes are more or less representational; others are simply geometric forms. Shapes are often posed as "problems" — that is, as mere silhouettes or outlines to be matched by the player's arrangement of the tiles. Problems commonly require that all seven tiles be used and that no subset of tiles be spatially separated from the others. Though the Tangram can be played competitively to see who can solve a given problem first, it is often played alone simply for aesthetic pleasure. The Tangram's modern form, which has been traced back to the early 19th century, evidently derives from designs for elaborate Chinese banquet-table suites and may have roots as deep as Liu Hui's 4th-century proof of the Pythagorean theorem. Devotees of the Tangram have included Lewis Carroll, Edgar Allen Poe, Napoleon Bonaparte, and John Quincy Adams [2].

Four families of tangram shapes (or problems) have been mathematically defined and the members of each have been enumerated [1, 3]. The purposes of this paper are to define a fifth type, to count its members, to explain its taxonomic relationship with the previously-defined types, and to provide some interesting examples.

Counting the Stars

A star tangram is simply one in which all seven tiles meet at a point (see Figure 2). In constructing such a tangram, it is natural to think of the tiles as converging at the central point; in appreciating the result as a star, however, it is more natural to think of them as radiating out from it. Because one of the tiles is a square, it will take up 90 degrees of the center; the remaining 270 degrees must accommodate the remaining six tiles. Since no tile has an interior angle less than 45 degrees, we have no way to compensate for an angle greater than 45 degrees. So each of the remaining six tiles must contribute a 45-degree angle to the central array. Note that each tile affords two orientations for touching the central point in this manner. For the five triangles, the second orientation can be found by either rotating or flipping the tile; for the parallelogram, flipping is necessary.

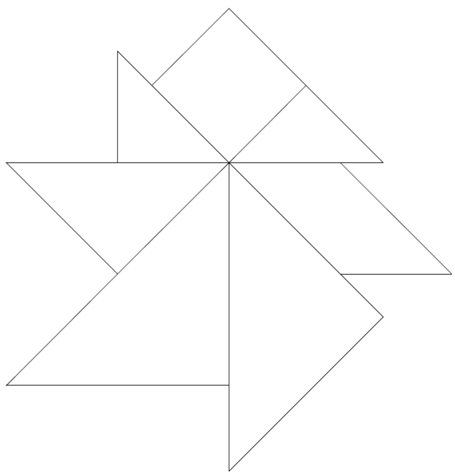

Figure 2: Star tangram.

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To determine how many star tangrams are possible (the "star-tangram number") it is convenient to begin by considering only the arrangement of the tiles without regard to their orientations. Following [1], we label each of the two large triangles L; the medium one M; each of the two small ones S; the square Q; and the parallelogram Z. Adopting Q as our reference tile, we can think of the remaining tiles as candidates for the six positions shown in Figure 3.

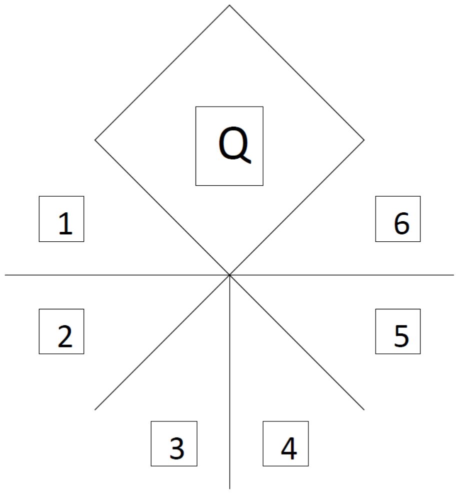

Figure 3: Six positions.

The number of possible arrangements may be thought of as the number of six-letter "words" that can be formed using L, L, S, S, M, and Z. Begin (arbitrarily) with the L's. They may occupy any two of the six positions. Their placement will leave only four positions for the two S's. *Their* placement will leave M a choice between two positions and Z must settle for the spot M leaves empty. The total number of possible arrangements, then, is:

$$\binom{6}{2}\binom{4}{2}\binom{2}{1}\binom{1}{1} = 180.$$

To denote the orientations of the tiles, it is helpful to consider Figure 2 in light of Figure 3. Note that in Figure 2 , as we move counterclockwise through the six positions, the leading abutting edge of each tile is always shorter than the trailing abutting edge. Taking capitalization to denote this kind of orientation and lower case to denote the reverse, we can specify Figure 2 's star, with regard both to arrangement of the tiles and to their orientations, by the designation SMLLZS. Reversing the orientation of the second tile would give us a new star designated SmLLZS.

How many such designations are possible? Because each of the six tiles can be placed, independently of the others, in either of two orientations, we must multiply the number of possible arrangements (180) by 2^6, this product being 11,520. It is conventional, however, to consider a tangram and its mirror image to be a single tangram. From this point of view, each of our designations has a "looking-glass self" that we should not have counted. This is easily corrected: we simply divide by two, arriving at 5,760. (The correction would be a bit more complicated were it not that star tangrams, due to the odd couple M and Z, cannot be bilaterally symmetric.) It is also conventional to treat tangrams as the same when one is the result of rotating the other. This demands no correction from us, however, because our "word count" technique already ignores rotations. The designation SMLLZS says nothing about where, say, the square Q is sitting — even though, in Figure 2, we happened to place it at the top.

The Same Star?

But there is still a comparability problem. Previously-published numbers also consider any two tangrams having the same outline (or silhouette) to be a single tangram, regardless of their internal construction. Is it possible for a star's outline to remain the same despite a change in its internal construction? It is indeed, due to the square tile Q's interchangeability with the two small triangles (S and S) when the latter abut fully along their hypotenuses. (There are a few other promising-looking interchangeabilities, but they fail in one way or another to preserve star-ness.) This means we have, in effect, included some pairs of "twin" outlines, counting two where we should (for present purposes) have counted only one. To figure out how often this has occurred, it is helpful to return to Figure 3 and begin by considering how many places our two-piece square (SS) might occupy. Clearly, it can occupy positions 1 and 2, 2 and 3, 3 and 4, 4 and 5, or 5 and 6. Once it is in any one of these five pairs of adjacent positions, we will have four remaining positions for L and L, then two for M, and, finally, only one for Z. The number of arrangements, then, is:

$$\binom{5}{1}\binom{4}{2}\binom{2}{1}\binom{1}{1} = 60.$$

Now, the orientations of SS are fixed, but those of the other four tiles are free to vary; accordingly, we multiply by 2^4 to get 960. As before, we must divide by two to correct for mirror images. The 480 remaining tangrams, however, comprise 240 pairs of outline-twins differing only in that one has tiles Q and SS where the other has tiles SS and Q. (Figure 10, below, shows one member of such a pair.) Counting each pair as a single tangram, then, we must deduct 240 from 5,760. So the star-tangram number turns out to be 5,520.

A Catalog of Stars

Can we be sure that this figure is correct — that we have not, for example, overlooked some other way in which a star tangram can be produced more than once? There is a way to find out. In Figure 3, the Q piece is already inserted. Let A denote the collection of star tangrams obtained by inserting the other pieces into Figure 3 in all possible ways. As we have already seen, there are 180 ways of allocating the pieces to the six sectors and 64 ways of orienting those pieces within the sectors. Hence A will contain $180 \times 64 = 11,520$ tangrams.

One thing that is certain is that A will contain every possible star tangram at least once. To find how many of these are different we program a computer to produce all the tangrams in A and output them as a file of 11,520 records, each record being a string of symbols giving the directions and lengths of the segments of the tangram's outline as we go round it in clockwise order. (This is somewhat similar to what was done in [1] to produce snug tangrams, but a bit more complicated since there are more possibilities for the lengths of the segments.)

We pass this file to another program which converts each record to a standard form ("standard" in the sense that tangrams that are equivalent under rotation or reflection will have identical standard forms, and conversely). In the final step, the program eliminates duplicate records, leaving just one record for each distinct tangram. All this was done, and the final list turned out to contain exactly 5,520 tangrams! This not only confirms the theoretical result but gives us an added bonus. From each record in the final output we can produce, electronically, a picture of the corresponding tangram. This has been done and there is now extant a "Picture Gallery" of all the star tangrams. (The first 480 entries of this gallery appear below in the Appendix.)

Other Families

How do these results relate to the four tangram numbers previously determined? Earliest — and by far smallest — is the convex-tangram number, famously proven in 1942 to be only 13 [3]. Convex tangrams, one of which is shown in Figure 1, have outlines free of indentations. None of the 13 is a star.

The convex tangrams compose a very small subset indeed of what more recently have been termed snug tangrams, the snug-tangram number having been determined, with computer assistance, to be 4,842,205 [1]. A snug tangram is defined as follows. Suppose the square Q has sides of length 1. Then the hypotenuse of a small triangle S has length $\sqrt{2}$. Indeed, each side of each tile consists either of a single segment of length 1, a single segment of length $\sqrt{2}$, two segments of length 1, or two segments of length $\sqrt{2}$. Snug tangrams obey the rule that when segments are in contact, they are in contact along their whole lengths with endpoints touching endpoints (a condition not all stars satisfy). Snug tangrams must also be free of internal voids (a condition all stars do satisfy).

The 45-degree angles in tiles L, M, S, and Z are all formed by an edge whose length is an integer (1 or 2) and an edge whose length is a multiple of $\sqrt{2}$ ($\sqrt{2}$ or $2\sqrt{2}$). This means that snugness will significantly constrain the placement of tiles in Figure 3. The tile that fills position 1 must have a trailing edge of integer length (since Q has sides of length 1) and a leading edge whose length is a multiple of $\sqrt{2}$. But then the next tile must have a trailing edge whose length is a multiple of $\sqrt{2}$ and a leading edge of integer length. And on it goes, from one position to the next, with the orientation of each new tile determined by that of the preceding one. The result is that very few stars are snug — only 81, or about 1.5 percent. (We leave the proof to the readers. For two examples, see Figures 6 and 8.)

A "bay" is an internal void that meets the perimeter of its host tangram at a single point. A bay tangram has a bay but, otherwise, satisfies the definition of snugness (as in Figure 4). There are 731,951 bay tangrams [1]. A “hole” is an internal void that meets the perimeter of its host tangram at no point. A hole tangram has a hole but, otherwise, satisfies the definition of snugness (as in Figure 5). Their number is 9,360 [1]. Stars accommodate neither bays nor holes.

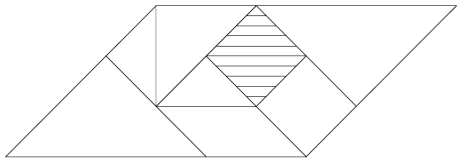

Figure 4: Bay.

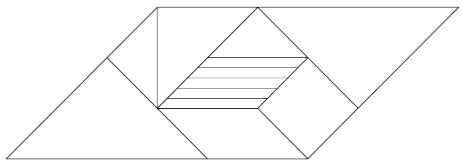

Figure 5: Hole.

Though there are over four hundred star tangrams for every one convex tangram, stars are rare compared with the other three types. For every star tangram, there are 877 snug tangrams and more than 130 bay tangrams (which seems quite remarkable); even the hole tangrams outnumber stars by a factor of nearly 1.7

Some Examples

Though star tangrams tend to have complex outlines and therefore tend to be relatively easy as problems, there are exceptions (e.g., Figures 6, 7, 8); and certainly the novel shapes they can take include many that are interesting, attractive, and suggestive (e.g., Figures 9, 10, 11, 12).

The examples in the Appendix offer a hint of the rich diversity of star-tangram shapes. They also present some amusing challenges for puzzlers — including various forms of wildlife and even Guinan from *Star Trek*. Can you find them?

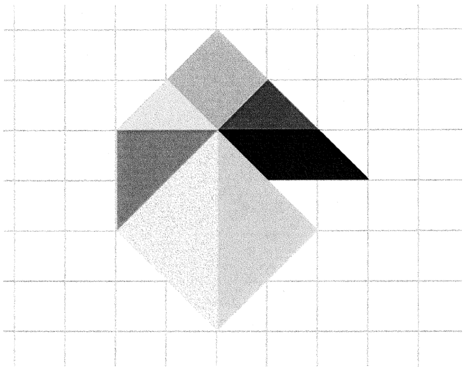

Figure 6: Example 1.

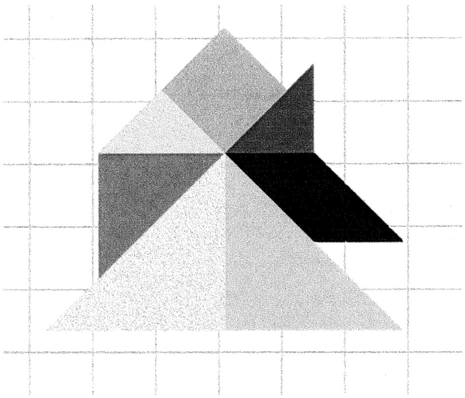

Figure 7: Example 2.

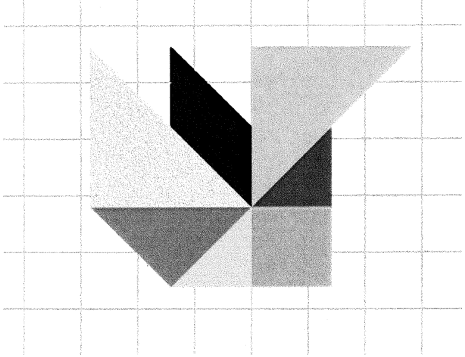

Figure 8: Example 3.

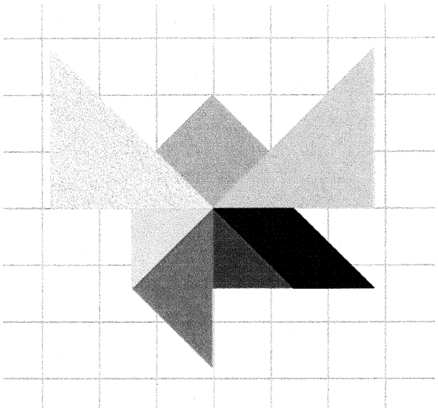

Figure 9: Example 4.

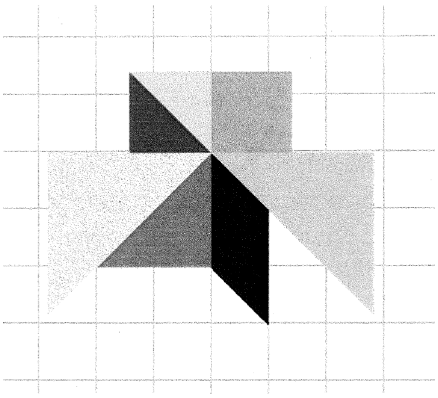

Figure 10: Example 5.

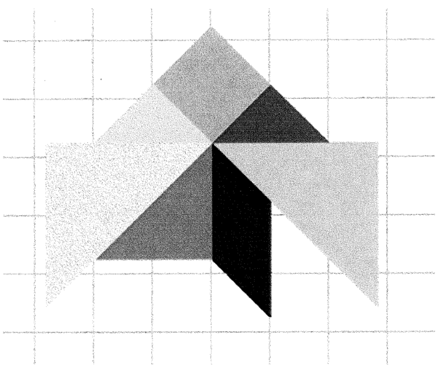

Figure 11: Example 6.

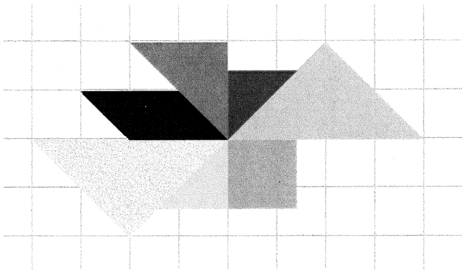

Figure 12: Example 7.

Acknowledgements

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References

[1] R. C. Read. "The Snug Tangram and Some Other Contributions to the Corpus of Mathematical Trivia", *Bulletin of the ICA*, 40, 2004.

[2] J. Slocum, J. Botermans, D. Gebhardt, M. Ma, X. Ma, H. Raizer, D. Sonneveld, and C. Splunteren. *The Tangram Book*, New York: Sterling, 2004.

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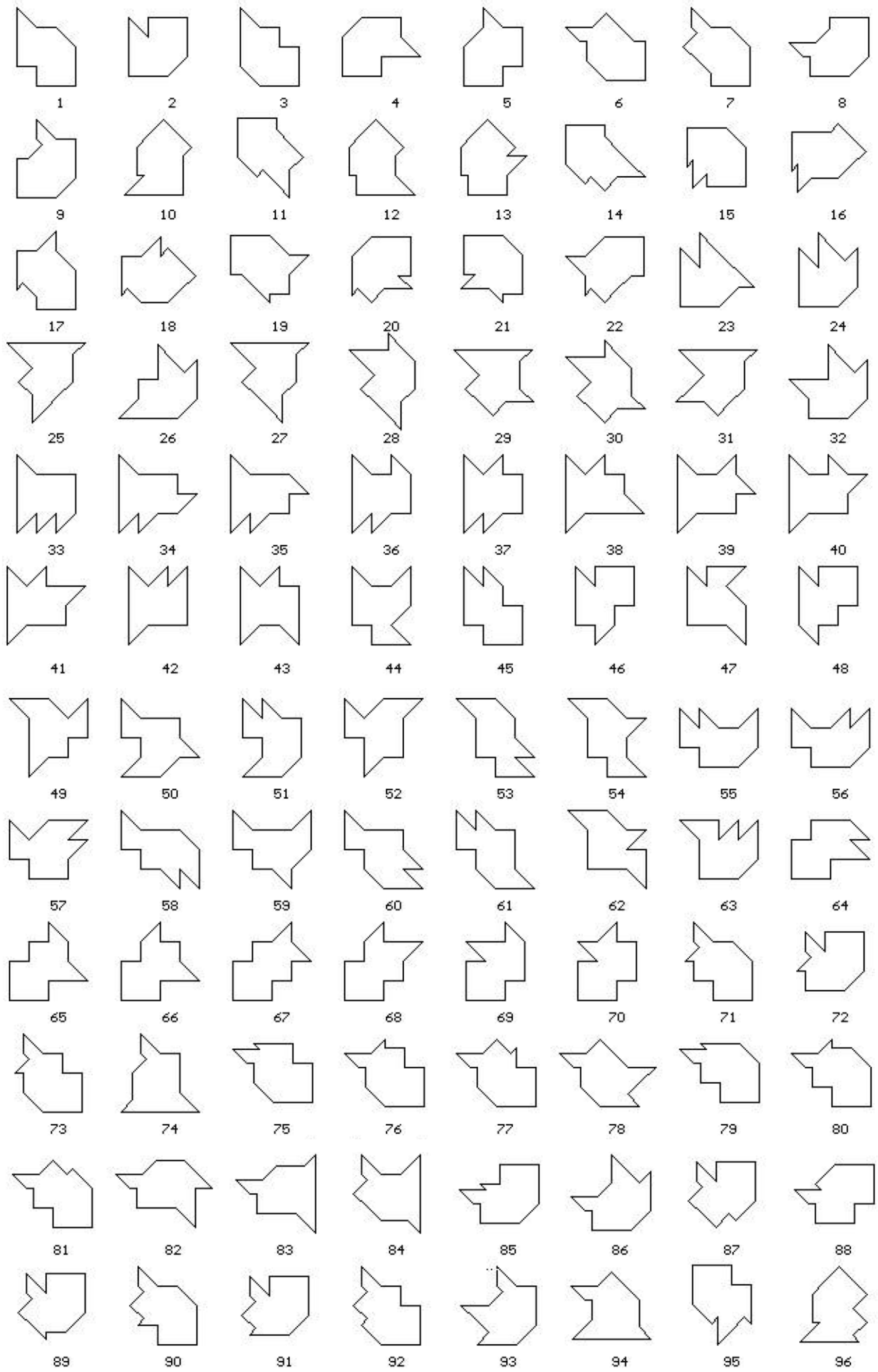
Appendix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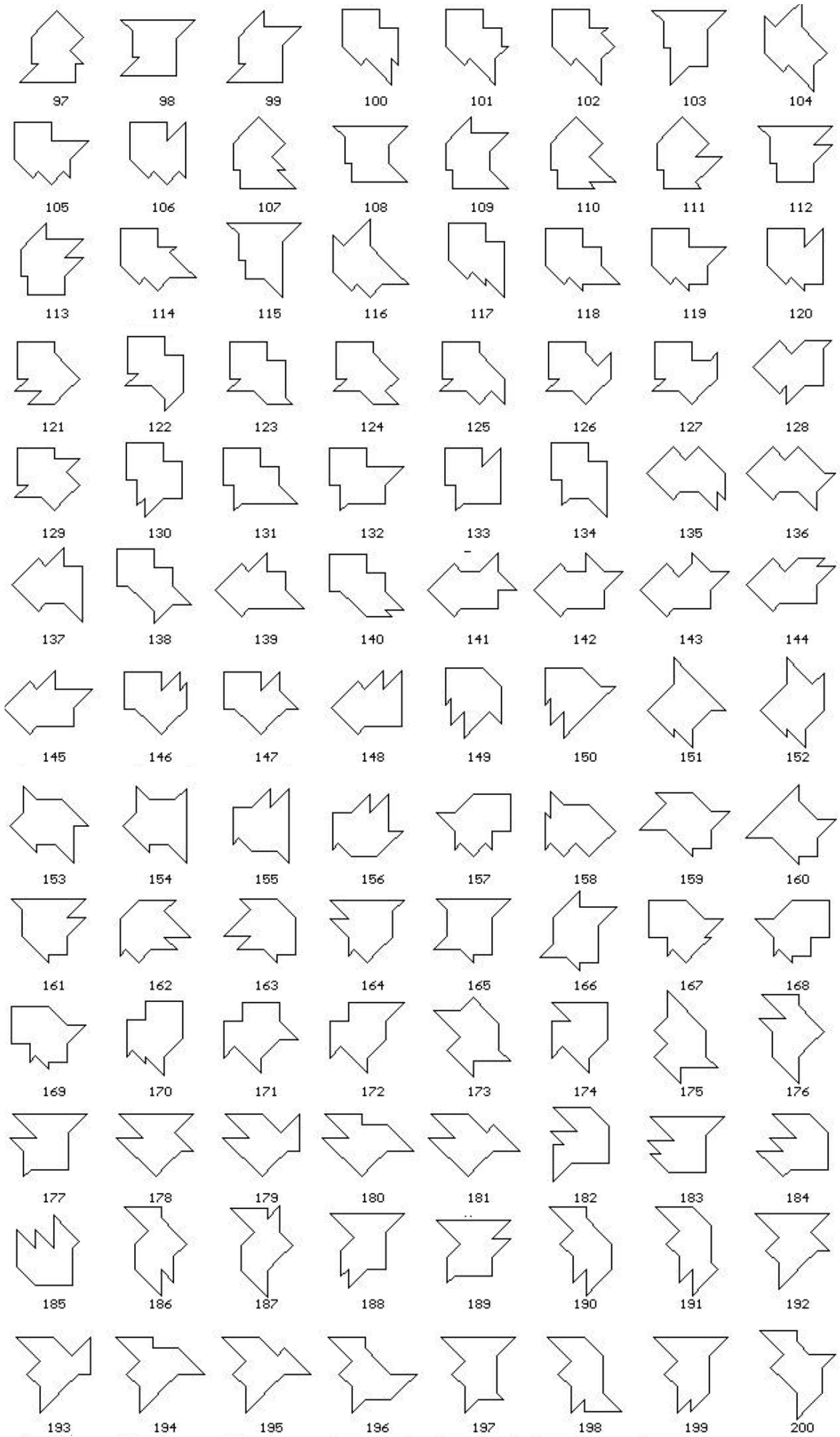

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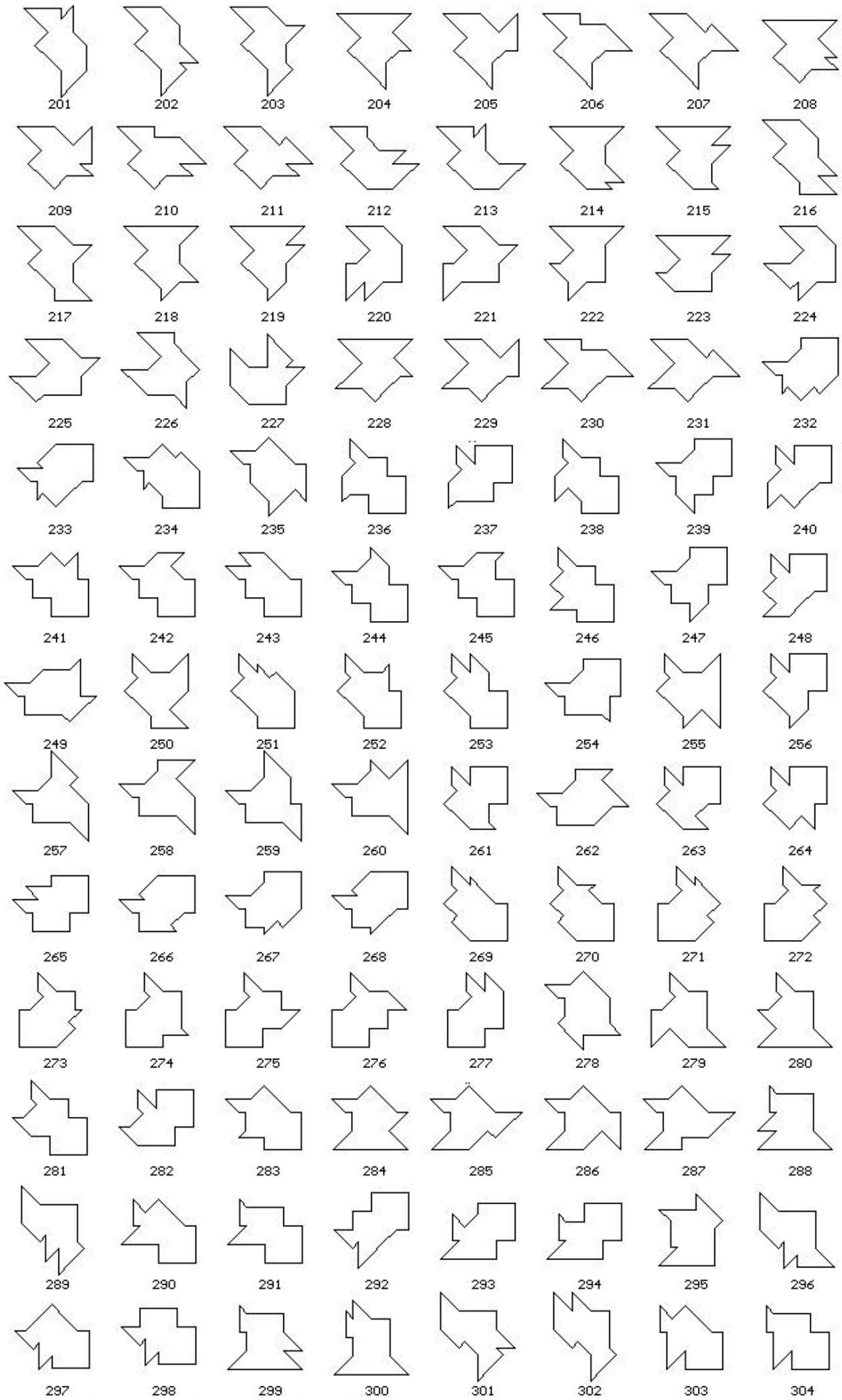
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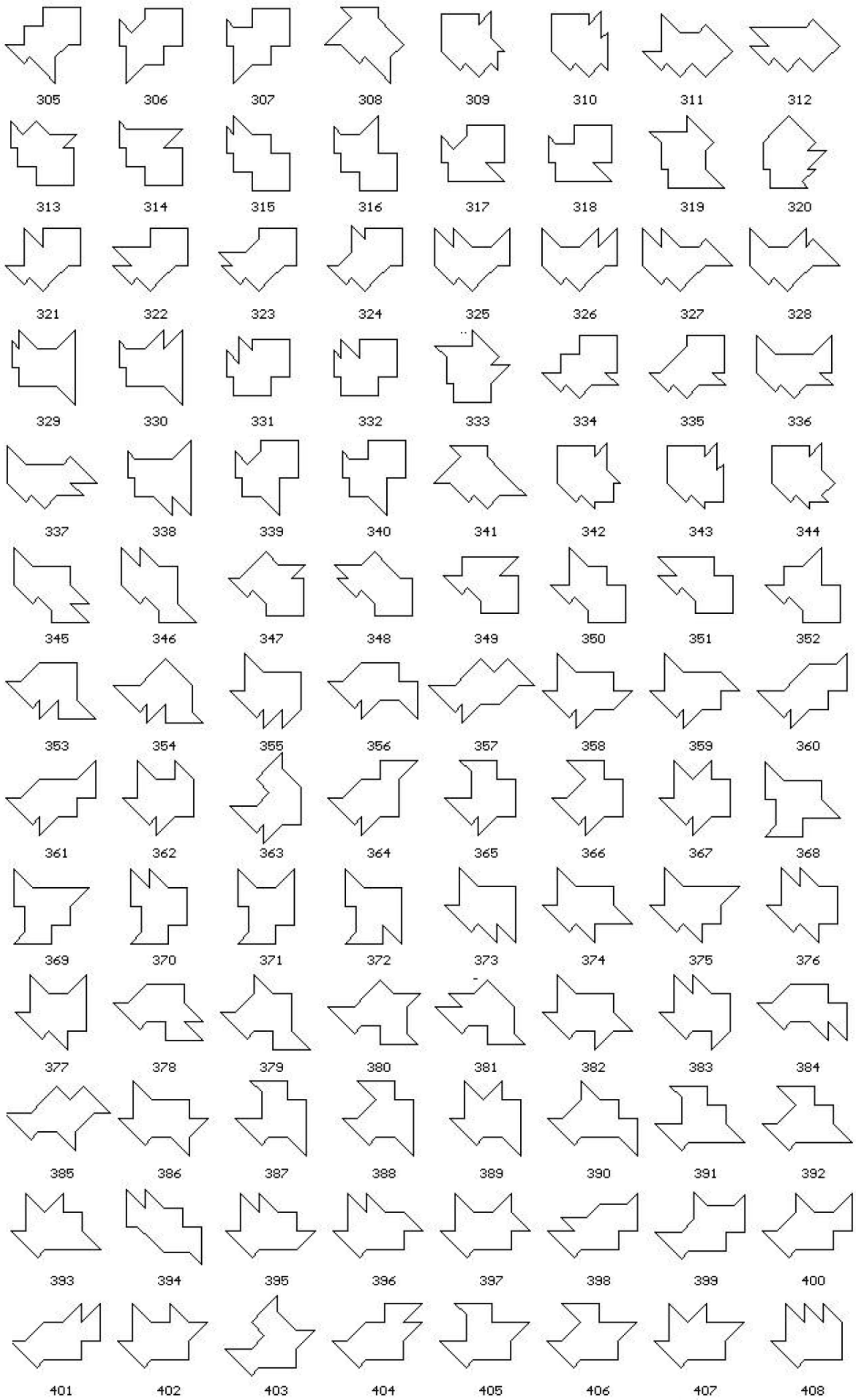
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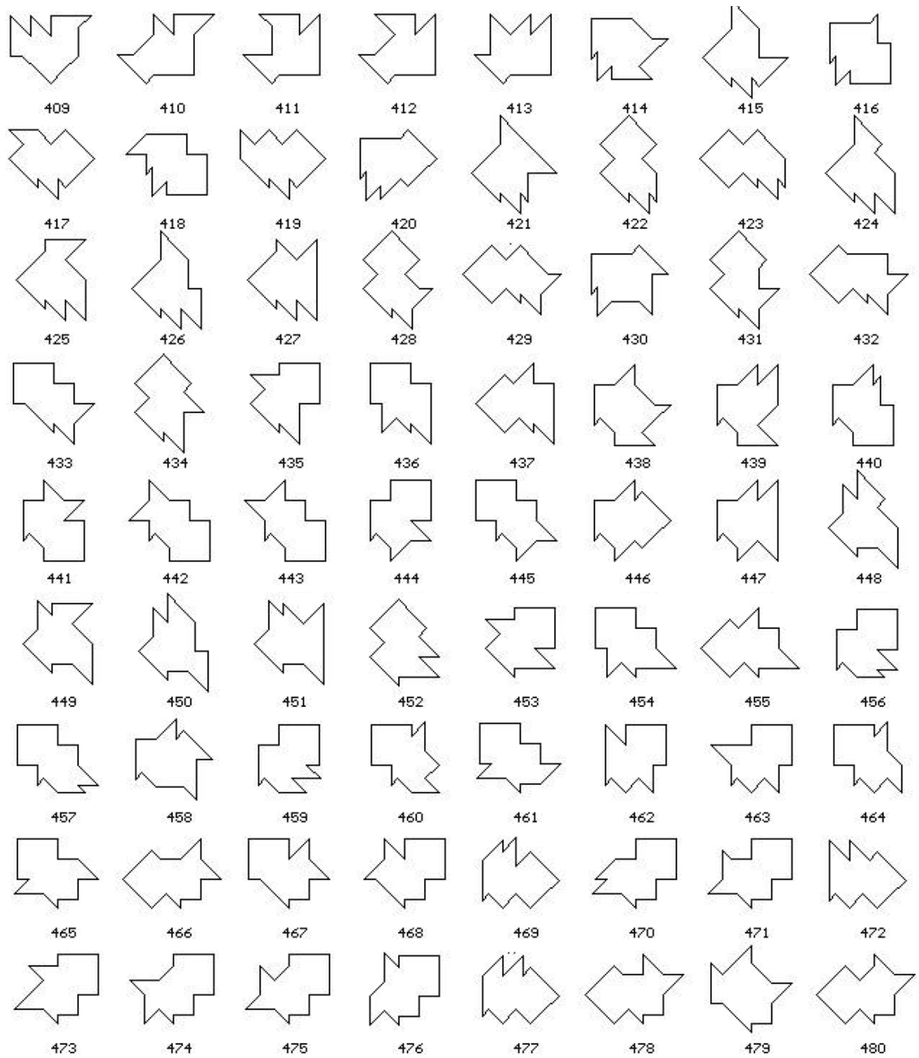

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