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NEW ASPECTS CONCERNING DYNAMICS OF RIGID SOLID BODY IN PLANE-PARALLEL MOTION SUBJECTED TO REAL CONSTRAINTS. PART ONE

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ABSTRACT
The present paper approaches in an original manner the dynamic analysis of a wheel which climbs on an inclined plane under the action of a horizontal force. The wheel rolls and slides in the same time. The two movements, rolling and sliding are considered to be independent of each other. Therefore we are dealing with a solid rigid body with two degrees of freedom. The difficulty of approaching the problem lies in the fact that in the differential equations describing the motion of the solid rigid body are also present the constraint forces and these are unknown. For this reason they must be eliminated from the differential equations of motion. The paper presents as well an original method of the constraint forces elimination.

KEYWORDS: Constraint forces, pulled wheel, dynamic study, rolling, sliding

1. Introduction

We will consider a wheel climbing on an inclined plane. The wheel is acted upon by an active horizontal force which causes the wheel movement and a viscous resisting force that opposes to the movement of the wheel. The sliding friction force, the rolling friction moment, the viscous resisting moment and the force of gravity are also acting upon the wheel. For the problem to be more clearly understood, one should take a look at Figure no.1. Unlike the case of rolling without sliding when the sliding friction force was unknown, in this case, the value of the sliding friction force is known and equal to the product between the μ sliding friction coefficient and the magnitude of the normal reaction force $\overline{N}$. This is because the sliding occurs when the sliding friction force magnitude has reached its maximum value. The two movements, translation and rotation, are considered to be independent one of each other. As a consequence the wheel may be regarded as a solid rigid body in plane motion with two degrees of freedom. The dynamics of the pulled wheel in the case of rolling without sliding has been studied before in the literature. Thus, the mathematical model describing the dynamic behavior of the pulled wheel in the case of pure rolling (rolling without sliding) was presented by Polidor (2006), Rădoi & Deciu (1993), Voinea, Voiculescu & Ceauşu (1983), Staicu (1998), Vâlcovici, Bălan & Voinea (1968).

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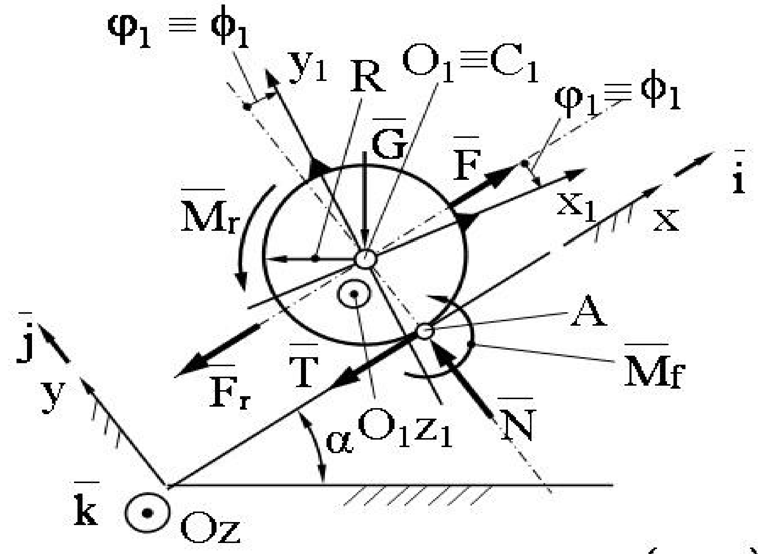
$$\overline{M}_f = -s \cdot N \cdot \mathrm{sgn}(\omega_{z_1}) \cdot \overline{k} \qquad \overline{T} = T \cdot \overline{i}$$



$$T = -\mu \cdot N \cdot \mathrm{sgn}(v_{O_1x})$$

Figure no. 1: *Pulled wheel. Rolling in the presence of sliding*

We aim ourselves to perform the dynamic study of the wheel, namely to determine its movement under the action of forces acting upon it.

2. Establishing the Differential Equations of Motion

The differential equations that describe the motion of the pulled wheel may be written in projections on the axes of the fixed reference frame, in the presence of constraints forces as follows:

$$m_1 \cdot \dot{v}_{O_1x} = F - G \cdot \sin(\alpha) - c_{1t} \cdot v_{O_1x} - \mu \cdot N \cdot \mathrm{sgn}(v_{O_1x}) \tag{1}$$

$$m_1 \cdot \dot{v}_{O_1y} = -G \cdot \cos(\alpha) - c_{1t} \cdot v_{O_1y} + N \tag{2}$$

$$J_{z_1} \cdot \dot{\omega}_{z_1} = -c_{1r} \cdot \omega_{z_1} - \mu \cdot N \cdot \mathrm{sgn}(v_{O_1x}) \cdot R - s \cdot N \cdot \mathrm{sgn}(\omega_{z_1}) \tag{3}$$

$$\dot{x}_{O_1} = v_{O_1x} \tag{4}$$

$$\dot{y}_{O_1} = v_{O_1y} \tag{5}$$

$$\dot{\varphi}_1 = \omega_{z_1} \tag{6}$$

where m_1 is the mass of the wheel, G the gravity force, F applied force parallel with the inclined plane, α inclined plane angle, c_{1t} viscous damping coefficient corresponding to the translational movement, c_{1r} viscous damping coefficient corresponding to the movement of rotation, s rolling friction coefficient, μ sliding friction coefficient, v_{O_1x} projection on Ox axis of O_1 point velocity, v_{O_1y} projection on Oy axis of O_1 point velocity, J_{z_1} inertia moment of the wheel relatively to O_1z_1 axis, R radius of the wheel, x_{O_1} and y_{O_1}

point O_1 coordinates relatively to the fixed reference frame, ω_{z_1} angular velocity of the wheel, φ_1 angle of self-rotation of the wheel. Figure no. 1 gives extra clear explanations. The relations (1-3) may be written very concisely in matrix form as follows:

$$M_{O_1} \cdot \dot{v} = Q_1 + Q_1^a + Q_1^c \tag{7}$$

The relations (4-6) may be written in matrix form as follows:

$$\dot{x} = v \tag{8}$$

$$M_{O_1} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_1 & 0 \\ 0 & 0 & J_{z_1} \end{bmatrix} \tag{9}$$

$$\dot{v} = \left[\dot{v}_{O_1x} \;\vdots\; \dot{v}_{O_1y} \;\vdots\; \dot{\omega}_{z_1}\right]^T \tag{10}$$

$$Q_1 = \left[F - G \cdot \sin(\alpha) \;\vdots\; -G \cdot \cos(\alpha) \;\vdots\; 0\right]^T \tag{11}$$

$$Q_1^a = -C \cdot v \tag{12}$$

$$C = \begin{bmatrix} c_{1t} & 0 & 0 \\ 0 & c_{1t} & 0 \\ 0 & 0 & c_{1r} \end{bmatrix} \tag{13}$$

$$v = \left[v_{O_1x} \;\vdots\; v_{O_1y} \;\vdots\; \omega_{z_1}\right]^T \tag{14}$$

$$Q_1^c = L_{\lambda f} \cdot \lambda \tag{15}$$

$$L_{\lambda f} = \left[-\mu \cdot \mathrm{sgn}(v_{O_1x}) \;\vdots\; 1 \;\vdots\; -\mu \cdot \mathrm{sgn}(v_{O_1x}) \cdot R - s \cdot \mathrm{sgn}(\omega_{z_1})\right]^T \tag{16}$$

$$\lambda = N \tag{17}$$

$$\dot{x} = \left[\dot{x}_{O_1} \;\vdots\; \dot{y}_{O_1} \;\vdots\; \dot{\varphi}_1\right]^T \tag{18}$$

where the exponent" T "denotes the transposition matrix operation. The matrix relation (7) may be written in the following equivalent form:

$$M_{O_1} \cdot \dot{v} = Q_1 + Q_1^a + L_{\lambda f} \cdot \lambda \tag{19}$$

Further on, we want to determine the motion of the wheel under the action of given forces and constraints forces. In order to do this we have to observe first that the constraint force N is unknown which makes the integration of the differential equations system to be impossible. For this reason the constraint force N must be removed from the differential equations system. In the next section a method of constraint force elimination will be presented.

3. Method of Constraint Forces Elimination

We will multiply the matrix relation (19) to the left with the matrix $M_{O_1}^{-1}$ and the following result will be obtained:

$$\dot{v} = M_{O_1}^{-1} \cdot Q_1 + M_{O_1}^{-1} \cdot Q_1^a + M_{O_1}^{-1} \cdot L_{\lambda f} \cdot \lambda \tag{20}$$

The matrix relation (20) will be multiplied to the left with the matrix L_λ^T and we will obtain the following relationship:

$$L_\lambda^T \cdot \dot{v} = L_\lambda^T \cdot M_{O_1}^{-1} \cdot Q_1 + L_\lambda^T \cdot M_{O_1}^{-1} \cdot Q_1^a + L_\lambda^T \cdot M_{O_1}^{-1} \cdot L_{\lambda f} \cdot \lambda \tag{21}$$

$$L_\lambda = \left[0 \;\vdots\; 1 \;\vdots\; 0\right]^T \tag{22}$$

But the left member of the relation (21) is equal to zero:

$$L_\lambda^T \cdot \dot{v} = 0 \tag{23}$$

The relation (21) becomes:

$$L_\lambda^T \cdot M_{O_1}^{-1} \cdot Q_1 + L_\lambda^T \cdot M_{O_1}^{-1} \cdot Q_1^a + L_\lambda^T \cdot M_{O_1}^{-1} \cdot L_{\lambda f} \cdot \lambda = 0 \tag{24}$$

The following notations will be introduced in the relation (24):

$$L_{\lambda}^{T} \cdot M_{O_1}^{-1} \cdot L_{\lambda f} = A \quad (25)$$

$$L_{\lambda}^{T} \cdot M_{O_1}^{-1} \cdot Q_1^{a} = \widetilde{Q}_1^{a} \quad (26)$$

$$L_{\lambda}^{T} \cdot M_{O_1}^{-1} \cdot Q_1 = \widetilde{Q}_1 \quad (27)$$

The matrix relation (24) becomes:

$$\widetilde{Q}_1 + \widetilde{Q}_1^{a} + A \cdot \lambda = 0 \quad (28)$$

From the relation (28) the unknown" λ "will be determined:

$$\lambda = -A^{-1} \cdot (\widetilde{Q}_1 + \widetilde{Q}_1^{a}) \quad (29)$$

The system of differential equations describing the motion of the wheel in the presence of constraints may be written as follows:

$$M_{O_1} \cdot \dot{v} = Q_1 + Q_1^{a} + L_{\lambda f} \cdot \left[-A^{-1} \cdot (\widetilde{Q}_1 + \widetilde{Q}_1^{a}) \right] \quad (30)$$

The matrix relations (8) and (30) form together a system of six differential equations of the first order having six unknowns. These unknowns will be determined using numerical integration methods.

4. Numerical Application

In this section the system of differential equations established in the previous chapter will be integrated using numerical integration methods and some results will be obtained. MATLAB software was used to develop a computing program. In order to run it the following input data are required:

$$m_1 = 1\,[kg] \quad (31)$$

$$J_{z_1} = 1\,[kg \cdot m^2] \quad (32)$$

$$c_t = 1\,[kg/s] \quad (33)$$

$$c_r = 1\,[kg \cdot m/s] \quad (34)$$

$$g = 10\,[m/s^2] \quad (35)$$

$$s = 0.2\,[meters] \quad (36)$$

$$\mu = 0.4 \quad (37)$$

$$F = 25\,[N] \quad (38)$$

$$\alpha = \pi/6\,[radians] \quad (39)$$

$$R = 1\,[meters] \quad (40)$$

The following initial conditions are also required:

$$v_{O_1 x}^{0} = 0\,[m/s] \quad (41)$$

$$v_{O_1 y}^{0} = 0\,[m/s] \quad (42)$$

$$\omega_{1z}^{0} = 0\,[radians/s] \quad (43)$$

$$x_{O_1}^{0} = 0\,[meters] \quad (44)$$

$$y_{O_1}^{0} = R = 1\,[meters] \quad (45)$$

$$\varphi_1^{0} = 0\,[radians] \quad (46)$$

The movement of the wheel has been studied for ten seconds.

After running the program, the results presented in the figures below will be obtained (Figure no. 2-Figure no. 4).

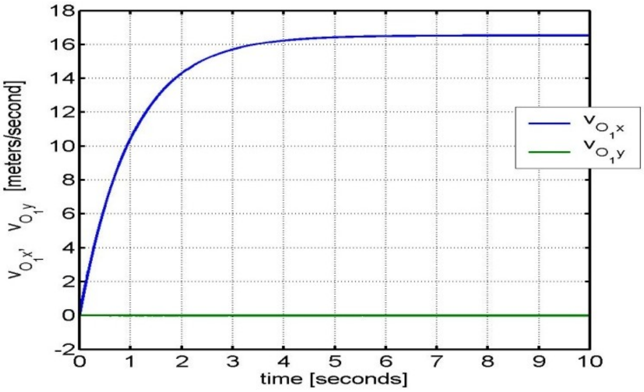


Figure no. 2: *Variation of the speed components of the O_1 point versus time*

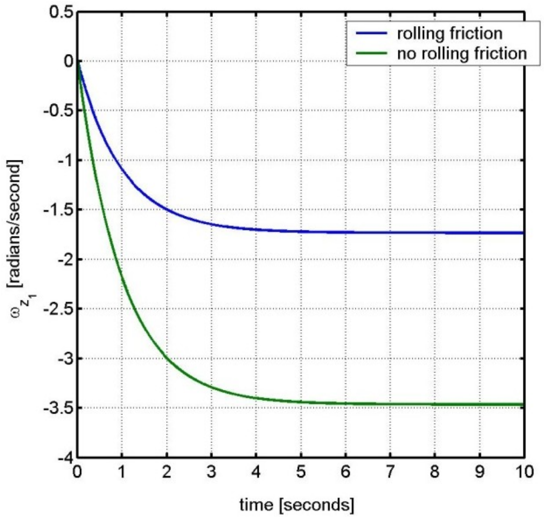


Figure no. 3: *Variation of the angular speed of the wheel versus time*

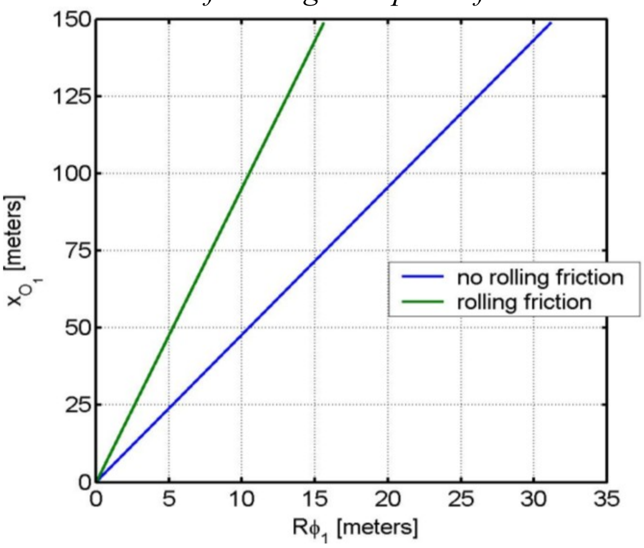


Figure no. 4: *Variation of the* x_{O_1} *abscissa of the* O_1 *point of the wheel versus* $R \cdot \varphi_1$ *product*

The movement has been studied for two distinct cases, namely: in the first case rolling friction was considered, and in the second case rolling friction was neglected. As for the speed of O_1 point, see Figure no.2, the results in the two cases are the same and therefore only one case was presented namely when rolling friction was taken into account.

5. Conclusions

The research established the mathematical model of the dynamic analysis of the pulled wheel which climbs on an inclined plane.

If the sliding friction coefficient "μ" is equal to zero, the angular velocity of the wheel is equal to zero for the entire duration of the movement. Therefore, in this case, the movement of the wheel is a pure sliding.

An original method of eliminating constraint forces from the differential equations system describing the motion of the solid rigid body was also presented.

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