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# The optimal rate of return for defined contribution pension systems in a stochastic framework

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**Abstract**

This paper deals with the problem of the optimal rate of return to be paid by a defined contribution pension system to its participants' savings, namely the rate that achieves the goal of the most favorable returns on their contributions jointly with the sustainability of the pension system.

We consider defined contribution pension systems provided with a funded component, and for their study we use the "theory of the logical sustainability of pension systems" already developed in several previous

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works. In this paper, we focus on pension systems in a demographically stable state, whereas the productivity of the active participants and the financial rate of return on the pension system's fund, rates that constitute the "ingredients" of the optimal rate of return on contributions, are modeled by two stochastic processes.

We show that the decisional rule defining the optimal rate of return on contributions is optimal in the sense that it is effective in terms of sustainability, and also efficient in the sense that if the system pays to its participants' contributions a rate of return that is either higher or lower than the one provided by the rule, then the pension system becomes unsustainable or overcapitalized, respectively.

**Mathematics Subject Classifications (2015).** 91B15.

**Keywords.** defined contribution pension system; logical sustainability model; stochastic rates; optimal rate of return.

# 1 Introduction

In most of the developed countries, pay-as-you-go (PAYG) pension systems, where pensions are paid from the actual contribution of the active population, face up with serious problems of sustainability due to financial, economic, and demographic uncertainties, see e.g. [1], [2], [3], [4], and [5]. Many countries tried to solve this problem introducing the Defined Contribution (DC) formula, according to which contributions are fixed and benefits depend on the value of assets and returns accumulated. In some countries, like Sweden or Italy for example, pension reforms moved towards new Notional Defined Contribution (NDC) schemes, in which contributions are virtually saved in individual accounts and, differently from conventional funded DC schemes, they are used to finance current pension benefits as in any PAYG scheme. In this regard, for an international review about NDC schemes, see e.g. [6], and refer to [7] for a more recent assessment of strengths and limitations on the application of the NDC model. In addition, for a potential implementation of NDC schemes in countries like China, Singapore, and South Korea, which are currently facing rapid population aging and rapidly increasing rates in the old-age dependency ratio, refer to [8].

Furthermore, it is worth mentioning that Sweden, which is the most well-known example of country implementing the NDC scheme, supported its pension reform also by the introduction of the so-called automatic balancing mechanism (ABM). This can be broadly defined as a measure that has to be applied as required according to the value of a sustainability indicator with the goal to maintain the system's financial equilibrium in response either to short-term economic and financial fluctuations and to long-term demographic changes. On the design of the ABM in the new Swedish PAYG pension system, see [9], whereas on the broader usefulness of ABMs for countries like Canada, Germany, Japan and Finland as well as Sweden, see [10], where the issue of a possible introduction of ABM into the Spanish state pension system is also discussed.

However, the approaches above described cannot guarantee the financial sustainability in contexts of financial, economic, and demographic instability. One of the most relevant issues concerns the choice of a rule for the rate of return to be paid on the pension liability in order to achieve the sustainability jointly with the most favourable returns on the pension liability of participants. It goes without saying that a considerable attention has been drawn to address this problem for PAYG pension systems in a steady state context; see, first of all, the pioneering works of Samuelson ([11]) and Aaron ([12]), or refer to the more recent work [13], where the complete set of factors, which determine the rate of return for a PAYG system, is identified.

Agreeing with Lee ([14]), who highlighted that most literature "... takes a comparative steady state approach and does not deal with nosteady situations and none of it addresses the specific problem of how to measure the rate of return to a non-steady state PAYG pension system..." (p. 147), it should be noted that the issue is mostly addressed by means of theoretical models, where the assumptions can significantly affect the evaluation of the pension system's state, see e.g. [15], or by proposing sophisticated forecasting models, see e.g. [16], which deeply explores the issue using eight different versions of the Swedish pension system. By means of stochastic numerical simulation, the authors in [16] found that pension systems equipped with a rate of return equal to the total growth of wages are inherently more stable than systems equipped with a rate of return equal to the average wage growth rate. Furthermore, they found that asymmetric "brake mechanisms", like the Swedish one, leads to an excessive undistributed asset accumulation. In other words the pension system inefficiently recapitalises itself over time.

In general, limited effort has been devoted to setting a rule for the rate of return on the pension liability, which could be systematically used by pension schemes in a non-steady state framework. The aim of this paper is to cover this gap.

In this regard, we basically refer to Angrisani ([17, 18]), who set up the basics of the logical sustainability model. Developed in a non-steady state context, it is based on a mathematical formalization of pension systems rather than on actuarial forecasting, and, owing to this feature, it is able to provide logical rules leading to the financial sustainability. It substantially refers to DC pension systems with a *structural funded* component, where "structural" does not mean a buffer reserve, like in the Swedish system, but refers to a funded component organically inserted in the pension scheme and whose financial returns are redistributed to contributions and benefits, either in a steady state of general stability (see [19, 20]) or in a stable state temporarily disrupted by a demographic and/or economic wave (see [21, 22]). For the described features, the logical sustainability approach designs a consistent mathematical model to be used in order to support decisions for the proper choice of the sustainable rate of return on the pension liability in a non-steady state.

Specifically, in this paper we focus on the specific rule (in the following referred to as the rule for the $\beta(t)$ stabilisation) that allows the system to avoid the uncontrolled expansion of the unfunded pension liability in relation to wages, which is one of the main causes of unsustainability. We demonstrate that this rule holds even in the stochastic framework, and provide numerical illustration of its application under the assumption that both the financial rate and the growth rate of wages are modeled by stochastic processes. By means of the numerical simulation, we show that the choice of the rate of return on the pension liability based on the proposed rule is also optimal, in the sense that if we consider two different cases, an upward or a downward change (as small as desired) that makes the rate of return systematically either higher or lower than the one provided by the rule, then the pension system becomes either unsustainable or overcapitalised.

This paper is structured as follows. In Section 2, the definitions of variables and indicators of the logical sustainability model used in this paper are extended from the continuous to the discrete framework. It is also proved that the rule for the $\beta(t)$ stabilisation holds in the discrete case, and hence holds also in the case that the financial rate and the growth rate of wages are both modeled by stochastic processes. Section 3 provides a brief description of the stochastic simulation model we use to represent the dynamics of the financial rate and the growth rate of wages. In Section 4, the optimality of the rule for the $\beta(t)$ stabilisation is numerically illustrated. Section 5 contains our main concluding comments.

# 2 The logical sustainability model and the rule for the $\beta(t)$ stabilisation in the discrete case

The main objective of this section is to prove that the rule for the stabilisation of the level of the unfunded pension liability in relation to wages, named rule for the $\beta(t)$ stabilisation (see [18]), also continues to be valid in relation to the sustainability in a stochastic framework. To achieve this goal, first we consider the logical sustainability model in the discrete case, providing the definitions of variables and indicators we use in this paper, and then we prove that the $\beta(t)$ stabilisation rule is valid in this context too. Hence, we can conclude that the rule holds even in the case that the values of both the financial rate of return on the fund and the growth rate of wages are outcomes determined by stochastic processes.

## 2.1 Basics of the logical sustainability model

In what follows, for the sake of brevity, we review only those pension system functions and indicators that are relevant for the purposes of this paper. For a deeper understanding of their meanings, as well as for a complete description of the other functions of the logical sustainability model, one can refer to [18] and [20], where the logical sustainability model is exhaustively illustrated.
We use the notations indicated in the following list:

- $t$ time unit variable (integer), i.e. the year beginning in $t-1$ and ending in $t$;
- $t_*$ year preceding the start of the time interval considered;
- $t_*+1$ first year of the time interval;
- $t_f$ last year of the time interval; it could be also equal to $+\infty$.

For any year $t \geq t_*$, the following functions are defined:

$\alpha$ $(t)$ is the contribution rate, with $\alpha(t) \geq 0$;

$W(t)$, $C(t)$, and $P(t)$ are the wages, the pension contributions, and the pension disbursements, respectively, with $W(t) > 0$, $C(t) \geq 0$, and $P(t) > 0$; naturally, it is $C(t) = \alpha(t)W(t)$.

Each one of the following functions is defined at the end of year $t$, immediately after that the annual contribution revenues have been paid in and the annual pension expenditure payments have been made.

$F(t)$ is the pension system fund, that is the aggregate value of the assets;

$L^A(t)$ is the pension liability to contributors (defined as the *latent pension liability*), with $L^A(t) \geq 0$;

$L^P(t)$ is the pension liability to retirees (defined as the *current pension liability*), with $L^P(t) > 0$;

$L^T(t)$ is the total pension liability, with $L^T(t) > 0$ and $L^T(t) = L^A(t) + L^P(t)$.

Lastly, the interest rates of return on the fund and on the pension liability in year $t$ are denoted respectively by $r(t)$ and $r_L(t)$.
Furthermore, we assume that the initial values for year $t_*$ are known for all functions above defined.
The following definitions are given.

DEFINITION 1 *A pension system is sustainable in time interval* $[t_*, t_f]$ *if and only if* $F(t) \geq 0$ *for each* $t = t_*, t_*+1, t_*+2, ...t_f$.

DEFINITION 2 *For each year* $t$ *such that* $t = t_*, t_*+1, t_*+2, \ldots$ *the unfunded pension liability is* $L^{UN}(t) = L^T(t) - F(t)$.

For each year $t$ such that $t = t_*, t_*+1, t_*+2 \ldots$, it is assumed that $L^T(t) \geq F(t)$; hence, the unfunded pension liability is subjected to $L^{UN}(t) \geq 0$.
For any year $t$ such that $t = t_*+1, t_*+2 \ldots$, the evolution equations of the fund, the pension liability, and the unfunded pension liability are respectively

$$F(t) = F(t-1)(1 + r(t)) + C(t) - P(t), \tag{1}$$

$$L^T(t) = L^T(t-1)(1+r_L(t)) + C(t) - P(t), \tag{2}$$

$$L^{UN}(t) = L^{UN}(t-1) + L^T(t-1)r_L(t) - F(t-1)r(t) \tag{3}$$

The evolution equation of wages is given by

$$W(t) = W(t-1)(1+\sigma(t)) \qquad \text{for} \quad t = t_* + 1, t_* + 2 \ldots \tag{4}$$

where $\sigma(t)$ is the wage growth rate in year $t$. Relationship (4) holds under the constraint, largely satisfied, that $1 + \sigma(t) \geq 0$ for $t = t_* + 1, t_* + 2, \ldots$ The solution of (4) is given by

$$W(t) = W(t_*) \prod_{s=t_*+1}^{t} (1 + \sigma(s)). \tag{5}$$

Assuming that the active population is constant, then $\sigma(t) = g(t)$, where $g(t)$ is the growth rate of the productivity.
Furthermore, for any year $t = t_*, t_* + 1, t_* + 2 \ldots$, we consider the definitions of the following indicators.

Function $\beta(t) = \frac{L^{UN}(t)}{W(t)}$ is the level of the unfunded pension liability in relation to wages.

Function $D_c(t) = \frac{F(t)}{L^T(t)}$, with $0 \leq D_c(t) \leq 1$, is the degree of funding of the pension liability.

## 2.2 The rule for the stabilisation of the level of the unfunded pension liability in relation to wages

In this paper, we focus only on the proof of the $\beta(t)$ stabilisation rule for the and we develop a complete extension of the logical sustainability model to the discrete framework in a forthcoming paper.

Proposition 1 (The rule for the $\beta(t)$ stabilisation) *Let us assume that* $0 \leq F(t_*) \leq L(t_*)$.
*For each* $t = t_* + 1, t_* + 2...$, *it is*

$$\beta(t) - \beta(t-1) = 0, \quad \textit{and hence} \quad \beta(t) = \beta(t_*),$$

*if and only if*

$$r_L(t) = \frac{F(t-1)}{L^T(t-1)} r(t) + \frac{L^T(t-1) - F(t-1)}{L^T(t-1)} \sigma(t) \tag{6}$$

Proof. For each $t = t_* + 1, t_* + 2...$, we calculate the difference in the level of the unfunded pension liability in relation to wages, $\beta(t)$, between $t$ and $t-1$

$$\beta(t) - \beta(t-1) = \frac{L^{UN}(t)}{W(t)} - \frac{L^{UN}(t-1)}{W(t-1)}.$$

By substituting in the previous formula the expressions of $L^{UN}(t)$ and $W(t)$, given by (3) and (4) respectively, we obtain

$$\begin{aligned}\beta(t)-\beta(t-1) &= \frac{L^{UN}(t-1)+L^{T}(t-1)r_L(t)-F(t-1)r(t)}{W(t-1)(1+\sigma(t))}+\\ &-\frac{L^{UN}(t-1)(1+\sigma(t))}{W(t-1)(1+\sigma(t))}=\\ &=\frac{L^{T}(t-1)r_L(t)-F(t-1)r(t)-L^{UN}(t-1)\sigma(t)}{W(t-1)(1+\sigma(t))}.\end{aligned} \tag{7}$$

Then $\beta(t)-\beta(t-1)=0$ if and only if

$$\begin{aligned} r_L(t) &= \frac{F(t-1)}{L^T(t-1)}r(t)+\frac{L^{UN}(t-1)}{L^T(t-1)}\sigma(t)=\\ &=\frac{F(t-1)}{L^T(t-1)}r(t)+\frac{L^T(t-1)-F(t-1)}{L^T(t-1)}\sigma(t).\end{aligned}$$

□

REMARK 1 *As we proved that the rule for the $\beta(t)$ stabilisation holds in the discrete case too, we can say that the $\beta(t)$ stabilisation rule also holds in the case that $r(t)$ and $\sigma(t)$ are outcomes determined by stochastic processes.*

# 3 The stochastic model for the financial rate and the growth rate of wages

In this section, we focus on the stochastic process we use in order to describe the behaviour of the financial rate, $r(t)$, and the wage growth rate, $\sigma(t)$. For both of them, given the probability space $(\Omega, F, P)$, we assume the rate dynamics to follow the stochastic differential equation (SDE)

$$dX(t)=\theta_{Vas}(\mu_{Vas}-X(t))dt+\sigma_{Vas}dW(t), \qquad X(0)=x_0 \tag{8}$$

where $X(t)$ represents either $r(t)$ or $\sigma(t)$, and $\theta_{Vas}$, $\mu_{Vas}$, $\sigma_{Vas}$ are positive constants and $W(t)$, with $t\geq 0$, is a Wiener process[1] under the risk-neutral probability measure (see [23]).
SDE (8) is known as a mean-reverting Ornstein-Uhlenbeck process, because

[1]Recall that a stochastic a process $W(t), t\geq 0$, is called a Wiener process if:

- $W(0)=0$
- $W(t)-W(s)\sim N(0,t-s)$
- the increments $W(t_n)-W(t_{n-1}), W(t_{n-1})-W(t_{n-2})\ldots, W(t_2)-W(t_1)$ are independent for $0\leq t_1\leq t_2\leq\cdots\leq t_n$
- the paths of the process are continuous.

when $X(t)$ is below $\mu_{Vas}$ the positive drift pulls it back up towards $\mu_{Vas}$, whereas the opposite occurs when X(t) is above $\mu_{Vas}$. Therefore, $\mu_{Vas}$ may be interpreted as the "long run mean" of the process ($E[X(t)] \to \mu_{Vas}$ as $t \to \infty$) and the coefficient $\theta_{Vas}$ is the speed of adjustment of $X(t)$ towards its long run level; letting $\theta_{Vas} \to 0^+$ it corresponds to "turn off" the mean reversion effect. Stochastic term $\sigma_{Vas} dW(t)$ captures the instantaneous volatility, i.e. constant instantaneous variance $\sigma^2_{Vas}$, which ensures that the process erratically and continuously fluctuates around the level $\mu_{Vas}$.

Figure 1 shows three sample paths of equation (8) for $\sigma_{Vas} = 0.05$, $\mu_{Vas} = 1$, and for increasing values of $\theta_{Vas}$. It can be seen that the smaller $\theta_{Vas}$, the more $X(t)$ drifts away from $\mu_{Vas}$. For $\theta_{Vas} = 2$, $X(t)$ displays only small and short-lived deviations from $\mu_{Vas}$.
The main features of the process are:

- the rate $X(t)$, for each time $t$, can be negative with positive probability;
- a closed form solution for $X(t)$ is available, see [23]; it is

$$X(t) = X(s)e^{-\theta_{Vas}(t-s)} + \mu_{Vas}(1 - e^{-\theta_{Vas}(t-s)}) + \sigma_{Vas} \int_s^t e^{-\theta_{Vas}(t-u)}\, dW(u)$$

where $s \leq t$.

In particular, as in [24], $X(t)$ is a Gaussian process with mean and variance given respectively by

$$E[X(t)|X(s)] = \mu_{Vas} + (X(s) - \mu_{Vas})e^{-\theta_{Vas}(t-s)}$$

$$Var[X(t)|X(s)] = \frac{\sigma^2_{Vas}}{2\theta_{Vas}}(1 - e^{-2\theta_{Vas}(t-s)}).$$

For simulation purpose, we need to discretize the stochastic differential equation (8), which turns out to be the limiting case as $t_i - t_{i-1} \to 0$ of the discrete first order auto-regressive process (see [24])

$$\begin{aligned} X(t_i) = {} & X(t_{i-1})e^{-\theta_{Vas}(t_i - t_{i-1})} + \\ & + \mu_{Vas}(1 - e^{-\theta_{Vas}(t_i - t_{i-1})}) + \sigma_{Vas}\sqrt{\frac{(1 - e^{-2\theta_{Vas}(t_i - t_{i-1})})}{2\theta_{Vas}}} N_{0,1}. \end{aligned} \tag{9}$$

# 4 The optimality of the $\beta(t)$ stabilisation rule in a stochastic framework

In this section, we numerically show the optimality of the $\beta(t)$ stabilisation rule, in the sense that it is effective and efficient in terms of sustainability under the assumption that rates $r(t)$ and $\sigma(t)$ are stochastic.

Specifically, we show that paying the rate of return on the pension liability according to this rule is an efficient choice in the meaning that if the rate of

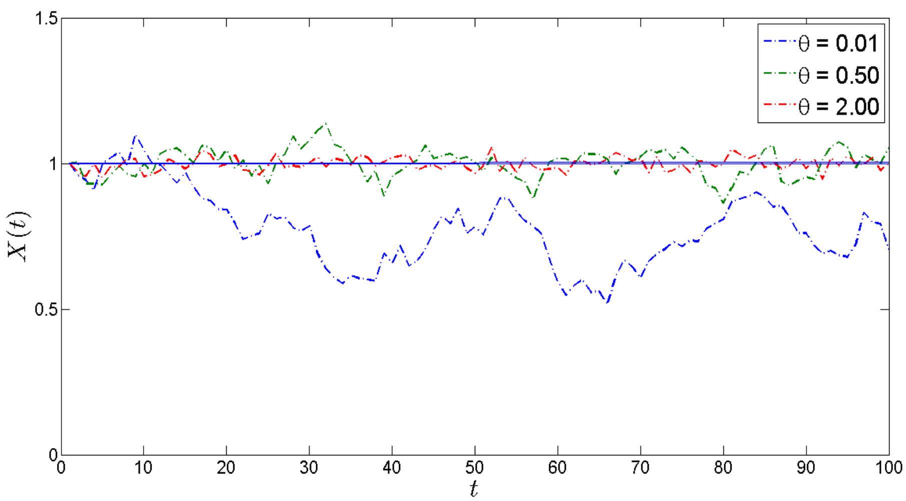


Figure 1: Three sample paths of the mean-reverting Ornstein-Uhlenbeck process.

return on the pension liability is systematically higher (even if only by a quantity as small as desired) than that set by the rule, then the pension system does not remain sustainable over time. Differently, if the rate of return on the pension liability is systematically lower (even if only by a quantity as small as desired) than that set by the rule, then the pension system is overcapitalised, namely it accumulates more resources than it actually needs, without there being a proper redistribution of the system's returns to the participants' pension credit.

Despite being fully aware of the relevance of the demographic aspects, in our numerical simulation we consider a stable demographic structure solely for sake of simplicity and also because this is not the goal of our work. On the other hand, it is worth highlighting that this simplifying assumption does not affect the validity, analytically proved, of the rule for the $\beta(t)$ stabilisation.

## 4.1 The model and its assumptions

In our numerical illustrations, in order to simulate the dynamics of a DC pension system in a discrete time context, we consider the actuarially-consistent model already developed in an earlier work, see [20], whose assumptions are as follows:

- the discrete time unit is one year;

- the pension system is considered starting from the year of first entries in the pension system;

- the time interval is $[0, +\infty)$; we observe the first 300 years;

- the number of new entries (in our example, only male gender) in the pension system is constant over time and is put equal to 1,000 units per year;
- each new member joins the scheme at age 25 and works until age 65;
- at age 65 the member retires and receives a pension until he/she is alive; there is a maximum age, $\omega$, up to which it is possible to survive; no survivor pension is considered;
- mortality rates are assumed to be constant over time; these values are those of the Life Table for Italy, Males, Last Modified March 2011, downloaded from the HMD ([25]); according to this Life Table, $\omega$ is equal to 110 years;
- at the initial year, the "wage step", namely the ratio of the average wage at age $x$ to the average wage at age $x-1$, is constant and equal to 2%. The productivity rate, $g$, which is evaluated on a yearly basis and is used to annually increase the average wage for each age group $x$, is constant in relation to age and time; therefore, the wage-pattern is stable over time;
- contribution amounts paid in by workers who die during their active lives are redistributed to survivors of the same age group;
- for each member, his/her pension is calculated dividing the individual total pension credit by the annuity divisor at the retirement age, i.e. 65 years (in the annuity divisor the technical rate is set equal to zero, hence, the annuity divisor coincides with life expectancy at retirement age); as a result of the assumption on mortality rates, the annuity divisor at retirement age remains unchanged over time; pensions are paid at the end of the year; pensions are revalued each year according to the rate of return on the pension liability, $r_L$ (it is evaluated on a yearly basis);
- rate $r_L$ is equally returned to pension liability both for workers and for pensioners;
- the fund is revalued each year according to the rate of return, $r$ (it is evaluated on a yearly basis).

As a result of the previous assumptions, the pension system achieves a demographic stabilisation. Specifically, as a consequence of the assumptions about a constant number of entries, constant mortality rates, and constant age of both entry and retirement, the number of workers becomes stable after forty years and the number of pensioners also becomes stable after a pension system lifecycle. Hence, the ratio of workers to pensioners is also stable and it is equal to about 2.4. After the first forty years, the rate of growth of the contribution base, $\sigma(t)$, coincides with the productivity growth rate, $g(t)$, since the number of workers is constant over time.

| | $r(t)$ | $\sigma(t)$ |
|---|---|---|
| $\hat{\theta}_{Vas}$ | 2.6246 | 0.7342 |
| $\hat{\mu}_{Vas}$ | 0.0322 | 0.0171 |
| $\hat{\sigma}_{Vas}$ | 0.1626 | 0.0289 |

Table 1: Values of the stochastic process parameters for $r(t)$ and $\sigma(t)$.

## 4.2 Economical and financial assumptions

In order to model the dynamics of both the financial rate and the growth rate of productivity, we use the stochastic process provided by equation (9).

For sake of reality, we estimated the values of the process parameters, respectively $\hat{\theta}_{Vas}$, $\hat{\mu}_{Vas}$ and $\hat{\sigma}_{Vas}$ both for $r(t)$ and $\sigma(t)$, by fitting the data of the real return on the Buffer Fund and the real growth in earnings for Sweden for years 1960-2012, as in the Orange Report for 2013, see [26], p. 69. On the other hand, this choice derives from the fact that, to our knowledge, the Swedish pension system is the only one providing these data.

The estimated values of the stochastic process parameters for $r(t)$ and $\sigma(t)$, respectively, are reported in Table 1. In Figures 2 and 3, the historical trend, as in the Orange Report for 2013, and three trajectories, based on equation (9), are graphed for $r(t)$ and $\sigma(t)$, respectively.

As expected, the long term average values of the two processes for $r(t)$ and $\sigma(t)$, which are equal to 3.22% and 1.71% respectively, are very closed to the *constant* values used in the base projection scenario of the Swedish pension system, which respectively are 3.25% and 1.8%, see [26], p. 65.

On the basis of the actuarial model used in [20] to simulate the pension system evolution, 20,000 possible development scenarios, each of which is 300 years length, are simulated. Every development scenario is described, year by year, using the whole of the variables representing the system itself. Specifically, in every development scenario, despite the fact that the long-term average value for $r(t)$, the financial rate of return on the fund, is equal to 3.22%, see Table 1, the rate of return on the pension liability is set to 4% per year, namely the rate paid on the pension liability is higher than the average financial rate. This assumption generally determines a pension liability higher than the coverage provided by the fund and, hence, determines a degree of covering that progressively reduces. The rate of return on the pension liability, initially set to 4% per year, remains unchanged as long as $D_c(t) \geq 50\%$, whereas it is constrained to follow the $\beta(t)$ stabilisation rule starting from the first year when $D_c(t) < 50\%$. As a result of the numerical simulation, it is found that $D_c(t)$ falls below the 50% level in 18,903 out of the 20,000 scenarios considered, namely in more than 90 percent (94.52%) of the total scenarios. Hence, as a consequence of the application of the rule, $\beta(t)$ is found to be blocked up to year 300 in all these scenarios. Referring only to the 18,903 scenarios with $\beta(t)$ blocked, it is found that $D_c(t)$ has been non-negative, namely the fund has been non-negative, up to year 300 in 17,876 scenarios, namely in about 95% (94.57%) of the 18,903

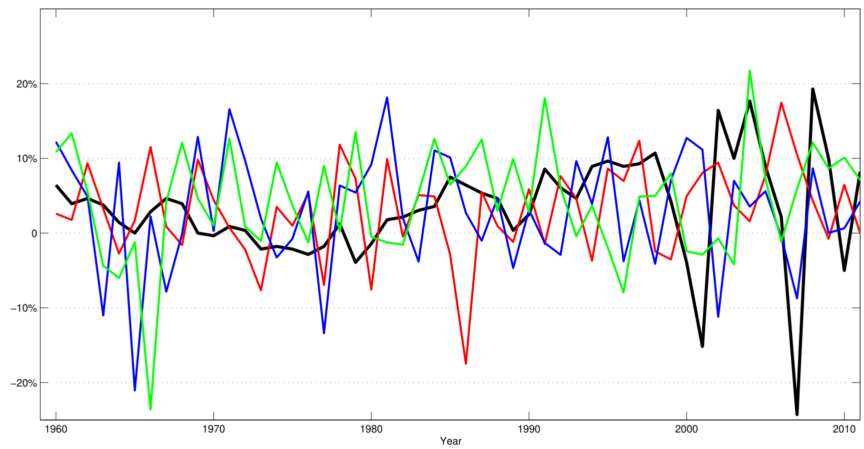


Figure 2: Graph of $r(t)$: in black, the historical trend as in the Orange Report for year 2013; in other colors, three trajectories simulated by our model.

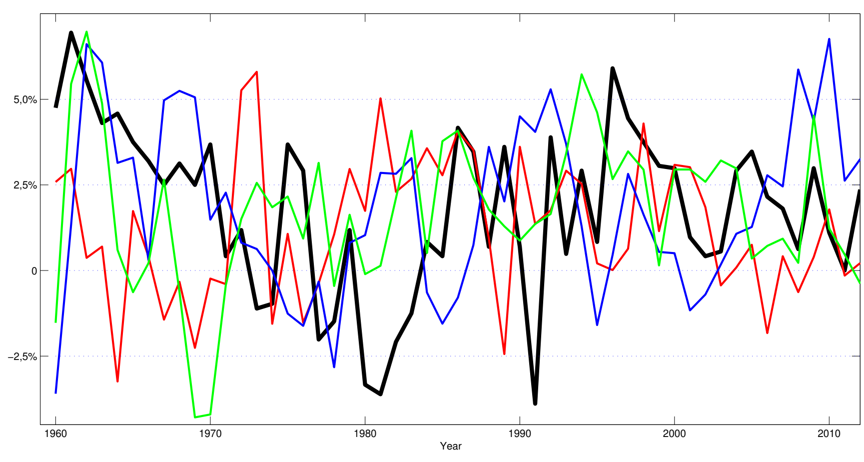


Figure 3: Graph of $\sigma(t)$: in black, the historical trend as in the Orange Report for year 2013; in colors, three trajectories simulated by our model.

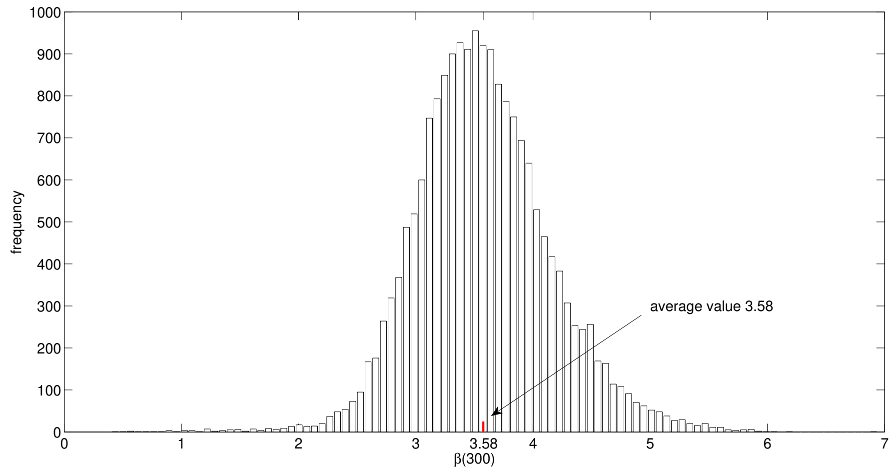


Figure 4: $\beta(t)$ distribution at $t = 300$ for all the 18,903 scenarios considered.

scenarios considered. In the 1,027 remaining scenarios, $D_c(t)$, and namely the fund, is found to be negative for at least one year. For the extension of the logical sustainability model in the case of the negative fund, see [21]. However, it should be noted that rule 6 also holds in order to stabilise $\beta(t)$ in the case of the negative fund, as can be easily verified.
In Figures 4 and 5 the $\beta(t)$ distribution and the $D_c(t)$ distribution at $t = 300$ are shown for all the 18,903 scenarios, respectively.

## 4.3 The optimality of the $\beta(t)$ stabilisation rule

As regards the empirical test on the optimality of the rule for the $\beta(t)$ stabilisation, we consider only the 17,876 scenarios where the pension system has both $\beta(t)$ blocked and the fund non-negative up to year 300. In each one of these scenarios, starting from the year in which $D_c(t)$ falls below the 50% level, with regards to the rate of return on the pension liability, the system applies rule (6) modified solely by an additional term, set equal to 0.3% in absolute value. We consider two different cases:

(a) the system applies the rule

$$r_L(t) = \frac{F(t-1)}{L^T(t-1)} r(t) + \frac{L^T(t-1) - F(t-1)}{L^T(t-1)} \sigma(t) + 0.003 \tag{10}$$

in this case, the system applies rule (6) slightly increased by 0.3%; in about all the scenarios considered the fund becomes negative;

(b) the system applies the rule

$$r_L(t) = \frac{F(t-1)}{L^T(t-1)} r(t) + \frac{L^T(t-1) - F(t-1)}{L^T(t-1)} \sigma(t) - 0.003 \tag{11}$$

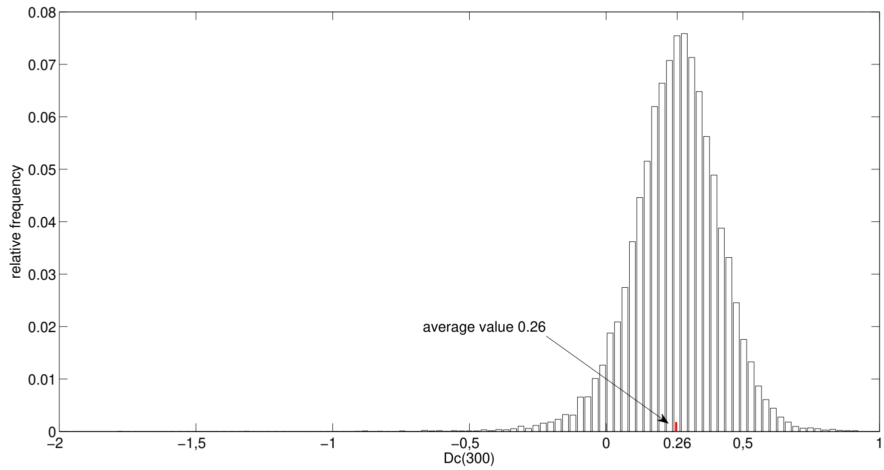


Figure 5: $D_c(t)$ distribution at $t = 300$ for all the 18,903 scenarios considered.

in this case, the system applies rule (6) slightly decreased by 0.3%; in all the scenarios considered, the fund remains non-negative and an overcapitalisation of the system itself is occurred.

In both cases, in relation to the 17,876 scenarios modified, we analyse the sustainability of the pension system by testing either the fund non-negativity or, equivalently, the non-negativity of the degree of funding of the pension liability, $D_c(t)$, considering the development of these 17,876 scenarios modified up to year 300.

### 4.3.1 Case (a): the pension system is unsustainable

It is found that the number of the modified scenarios with the non-negative fund up to year 300 dramatically decreases over time moving from 17,876 to 310. Hence, given that the system has systematically recognised on the pension liability a rate of return that has been slightly higher than that provided by rule (6), then this has entailed the fund being non-negative up to year 300 only in 1.73% of the scenarios, versus the 17,876 scenarios where the fund has been non-negative up to year 300 by the effect of the rule tout court, Indeed, the application of a higher rate of return (even only a slightly higher one) than that provided by (6), determines a systematic expansion of $\beta(t)$, which makes the pension system not sustainable under a fixed contribution rate.

The three panels of Figure 6 and Figure 7 show the $\beta(t)$ distribution and the $D_c(t)$ distribution, respectively, at years 100, 200, and 300, for the scenarios having the fund non-negative up to the corresponding year. It is worth noting that as time increases the number of the scenarios having the fund non-negative decreases whereas the average value of the $\beta(t)$ distribution increases.

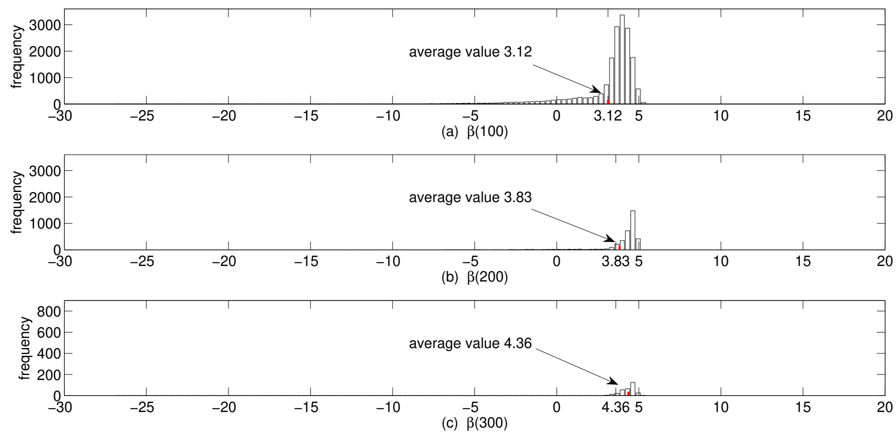


Figure 6: $\beta(t)$ distribution for the scenarios with the fund non-negative up to the corresponding year.

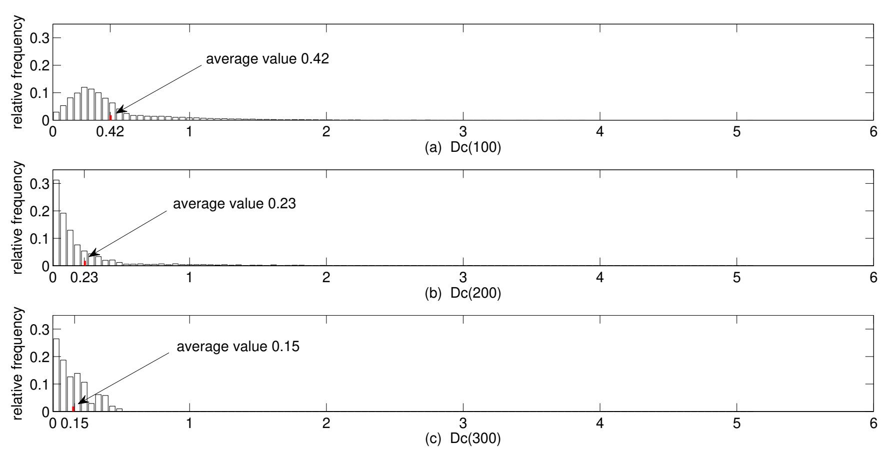


Figure 7: $D_c(t)$ distribution for the scenarios with the fund non-negative up to the corresponding year.

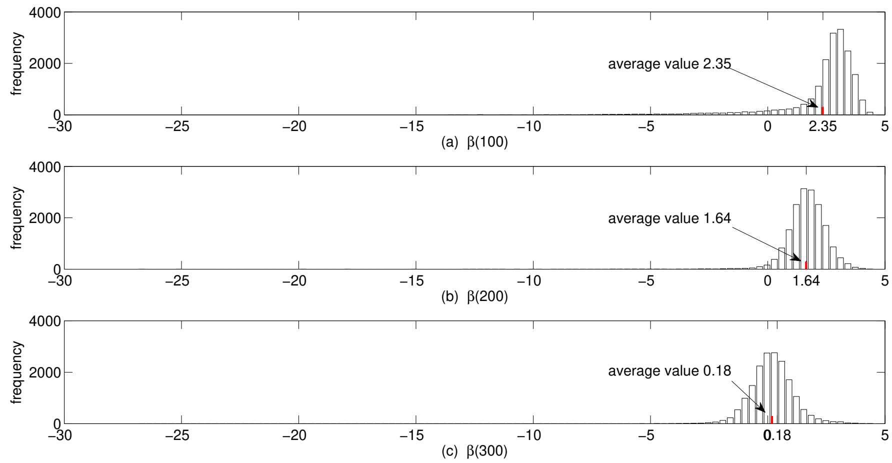


Figure 8: $\beta(t)$ distribution for all the scenarios considered.

### 4.3.2 Case (b): the pension system overcapitalises

Always referring to the set of the 17,876 scenarios with both $\beta(t)$ blocked and a non-negative fund up to year 300, we check the sustainability of the pension system in the corresponding scenarios modified looking at the fund, or equivalently at the degree of funding of the pension liability, $D_c(t)$, up to year 300. In particular, it should be noticed that the non-application of the $\beta(t)$ stabilisation rule involves the $\beta(t)$ not remaining stable over time, and, as the rate of return paid on the pension liability is systematically lower than that fixed by the rule, we can notice a progressive reduction in the $\beta(t)$ value, see Figure 8. In this case, it is found that in all the modified scenarios the fund has been systematically non-negative up to year $100, 200$, and 300, and that the average value of the $D_c(t)$ distribution has been increasing over time, see Figure 9, where the $D_c(t)$ distribution is shown for years $100, 200$, and 300. This result proves that the pension system recognises an inefficient rate of return as it redistributes a return less than what it could have actually returned thereby causing an unnecessary overcapitalisation of the system itself.

# 5 Conclusions

The choice of the rate of return on the pension liability is a crucial issue very difficult to carry out because of the intrinsic stochastic and dynamic nature of the problem. Although in literature a significant effort has been devoted to address this issue by proposing sophisticated projections models, the existing approaches cannot be logically used in practice because they do not provide a rule able to ensure the financial sustainability in a logical mathematical key,

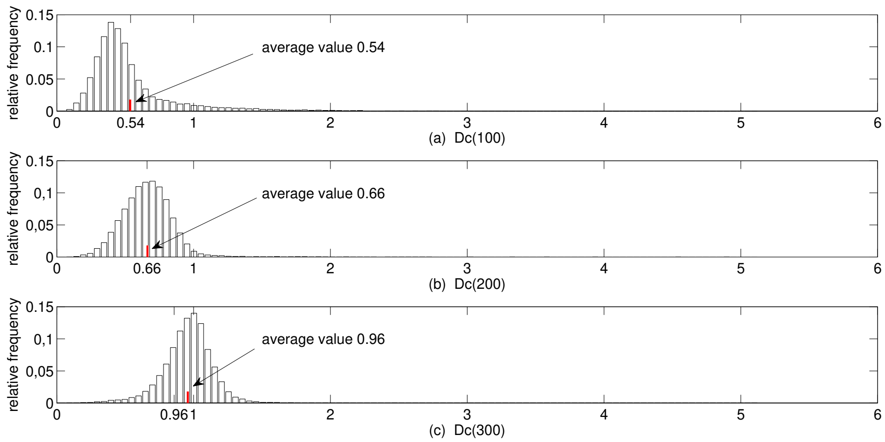


Figure 9: $D_c(t)$ distribution for all the scenarios considered.

whichever the uncertainty is.

In pursuing this goal, this paper focuses on a rule of choice for the rate of return on the pension liability, already introduced in the deterministic framework of the logical sustainability model for pension systems, see [17, 18], where it is proved that the application of this rule allows the system to control the level of the unfunded pension liability. In this paper it is proved that this rule maintains its validity in relation to sustainability even in the stochastic framework, and a numerical exemplification of the application of this rule is provided in the case that both the financial rate and the wage growth rate are modeled by stochastic processes. The numerical simulation illustrates also the performance in terms of optimality of this decisional rule, whose application allows the pension system to avoid either an uncontrolled expansion of the unfunded pension liability or an accumulation of undistributed assets.

Finally, it is worth noting that the proposed rule on the rate of return on the pension liability systematically follows the actual dynamics of the financial and wage growth rates with no need to activate balance mechanisms a posteriori to restore the financial equilibrium. For this reason, the proposed rule, jointly with a proper choice of the contribution rate, can be considered as a main tool for a more general support to the pension system management in order to the sustainability.

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