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A generalization of André-Jeannin's symmetric identity

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Abstract. In 1997, Richard André-Jeannin obtained a symmetric identity involving the reciprocal of the Horadam numbers W_n, defined by a three-term recurrence $W_{n+2} = PW_{n+1} - QW_n$ with constant coefficients. In this paper, we extend this identity to sequences $\{a_n\}_{n\in\mathbb{N}}$ satisfying a three-term recurrence $a_{n+2} = p_{n+1}a_{n+1} + q_{n+1}a_n$ with arbitrary coefficients. Then, we specialize such an identity to several q-polynomials of combinatorial interest, such as the q-Fibonacci, q-Lucas, q-Pell, q-Jacobsthal, q-Chebyshev and q-Morgan-Voyce polynomials.

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1 Introduction

Let $W_n = W_n(a, b; P, Q)$ be the *Horadam numbers* [8, 9], defined by the linear recurrence

$$W_{n+2} = PW_{n+1} - QW_n$$

with the initial conditions $W_0 = a$ and $W_1 = b$, where a, b, P and Q are constants (or symbols) with $PQ \neq 0$. Several classical combinatorial sequences are of this kind. This is true, for instance, for the *Fibonacci*, *Lucas*, *Pell* and *Jacobsthal numbers*, the *Chebyshev polynomials* and the *Morgan-Voyce polynomials*.

In [2], Richard André-Jeannin proved, for all $m, n \in \mathbb{N}$, the symmetric identity

$$U_n \sum_{k=1}^{m} \frac{Q^k}{W_k W_{n+k}} = U_m \sum_{k=1}^{n} \frac{Q^k}{W_k W_{m+k}} \tag{1}$$

where $U_n = W_n(0, 1; P, Q)$. For instance, for the *Fibonacci numbers* $F_n = W_n(0, 1; 1, -1)$ and for the *Lucas numbers* $L_n = W_n(2, 1; 1, -1)$, we have $U_n = W_n(0, 1; 1, -1) = F_n$. Hence, in this case, we have the identities [6]

$$F_n \sum_{k=1}^{m} \frac{(-1)^k}{F_k F_{n+k}} = F_m \sum_{k=1}^{n} \frac{(-1)^k}{F_k F_{m+k}}$$
$$F_n \sum_{k=1}^{m} \frac{(-1)^k}{L_k L_{n+k}} = F_m \sum_{k=1}^{n} \frac{(-1)^k}{L_k L_{m+k}} .$$

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Similarly, for the *Chebyshev polynomials* of the first and second kind $T_n(x) = W_n(1, x; 2x, 1)$ and $U_n(x) = W_n(1, 2x; 2x, 1)$ we have $U_n = W_n(0, 1; 2x, 1) = U_{n-1}(x)$ and

$$U_{n-1}(x) \sum_{k=1}^{m} \frac{1}{T_k(x)T_{n+k}(x)} = U_{m-1}(x) \sum_{k=1}^{n} \frac{1}{T_k(x)T_{m+k}(x)}$$

$$U_{n-1}(x) \sum_{k=1}^{m} \frac{1}{U_k(x)U_{n+k}(x)} = U_{m-1}(x) \sum_{k=1}^{n} \frac{1}{U_k(x)U_{m+k}(x)} .$$

Finally, for the *Morgan-Voyce polynomials* [13] [19, 20]

$$M_n(x) = W_n(1, x+2; x+2, 1) = \sum_{k=0}^{n} \binom{n+k+1}{n-k} x^k$$

$$N_n(x) = W_n(1, x+1; x+2, 1) = \sum_{k=0}^{n} \binom{n+k}{n-k} x^k$$

we have $U_n = W_n(0, 1; x+2, 1) = M_{n-1}(x)$ and

$$M_{n-1}(x) \sum_{k=1}^{m} \frac{1}{M_k(x)M_{n+k}(x)} = M_{m-1}(x) \sum_{k=1}^{n} \frac{1}{M_k(x)M_{m+k}(x)}$$

$$M_{n-1}(x) \sum_{k=1}^{m} \frac{1}{N_k(x)N_{n+k}(x)} = M_{m-1}(x) \sum_{k=1}^{n} \frac{1}{N_k(x)N_{m+k}(x)} .$$

In this paper, we extend André-Jeannin's identity (1) to sequences $\{a_n\}_{n\in\mathbb{N}}$ satisfying a three-term recurrence $a_{n+2} = p_{n+1}a_{n+1} + q_{n+1}a_n$ with arbitrary coefficients. Then, we specialize such an identity to the particular case in which the coefficients of the recurrence are given by $p_n = X(q^n x)$ and $q_n = Y(q^n x)$. Finally, we exemplify this identity for several q-polynomials of combinatorial interest, such as the q-Fibonacci, q-Lucas, q-Pell, q-Jacobsthal, q-Chebyshev and q-Morgan-Voyce polynomials.

2 The main result

André-Jeannin's identity (1) is a simple consequence of the next Lemma (whose proof is reported for completeness).

Lemma 2.1 *Given a sequence $\{a_n\}_{n\in\mathbb{N}}$, let $\{A_{n,k}\}_{n,k\in\mathbb{N}}$ be the sequence where $A_{n,k} = a_k - a_{n+k}$. Then, for every $m, n \in \mathbb{N}$, we have the identity*

$$\sum_{k=1}^{m} A_{n,k} = \sum_{k=1}^{n} A_{m,k} .$$

Proof. If $m \geq n$, then we have

$$\sum_{k=1}^{m} A_{n,k} = \sum_{k=1}^{m} (a_k - a_{n+k}) = (a_1 + \cdots + a_m) - (a_{n+1} + \cdots + a_{n+m})$$

$$= (a_1 + \cdots + a_n + a_{n+1} + \cdots + a_m) - (a_{n+1} + \cdots + a_m + a_{m+1} + \cdots + a_{m+n})$$
$$= (a_1 + \cdots + a_n) - (a_{m+1} + \cdots + a_{m+n}) = \sum_{k=1}^{n} (a_k - a_{m+k}) = \sum_{k=1}^{n} A_{m,k} .$$

A similar argument holds for $n \geq m$. This completes the proof. □

We also need the following result.

THEOREM 2.2 *Let* $\{a_n\}_{n\in\mathbb{N}}$ *be a sequence satisfying a three-term recurrence*

$$a_{n+2} = p_{n+1} a_{n+1} + q_{n+1} a_n \tag{2}$$

with $a_n \neq 0$ *for all* $n \geq 1$. *Then there exists a sequence* $\{A_n\}_{n\in\mathbb{N}}$ *with the following property: for every* $k \in \mathbb{N}$, *the sequence* $\{B_n^{(k)}\}_{n\in\mathbb{N}}$, *where*

$$B_n^{(k)} = A_k a_{n+k} - A_{n+k} a_k ,$$

satisfies the three-term recurrence

$$B_{n+2}^{(k)} = p_{n+k+1} B_{n+1}^{(k)} + q_{n+k+1} B_n^{(k)} \tag{3}$$

with the initial values $B_0^{(k)} = 0$ *and* $B_1^{(k)} = (-1)^k q_k^*$, *where* $q_k^* = q_k q_{k-1} \cdots q_2 q_1$.

Proof. Let us suppose that the sequence $\{A_n\}_{n\in\mathbb{N}}$ exists. Then, by recurrence (2), we have

$$\begin{aligned}
B_{n+2}^{(k)} &= A_k a_{n+k+2} - A_{n+k+2} a_k \\
&= A_k (p_{n+k+1} a_{n+k+1} + q_{n+k+1} a_{n+k}) - A_{n+k+2} a_k \\
&= p_{n+k+1} A_k a_{n+k+1} + q_{n+k+1} A_k a_{n+k} - A_{n+k+2} a_k \\
&= p_{n+k+1} (A_k a_{n+k+1} - A_{n+k+1} a_k) + p_{n+k+1} A_{n+k+1} a_k + \\
&\qquad + q_{n+k+1} (A_k a_{n+k} - A_{n+k} a_k) + q_{n+k+1} A_{n+k} a_k - A_{n+k+2} a_k \\
&= p_{n+k+1} B_{n+1}^{(k)} + q_{n+k+1} B_n^{(k)} - (A_{n+k+2} - p_{n+k+1} A_{n+k+1} - q_{n+k+1} A_{n+k}) a_k .
\end{aligned}$$

Now, if we assume that the sequence $\{A_n\}_{n\in\mathbb{N}}$ satisfies the recurrence

$$A_{n+2} = p_{n+1} A_{n+1} + q_{n+1} A_n \tag{4}$$

then, by the above remarks, we obtain identity (3). Moreover, for every $k \in \mathbb{N}$, we have

$$\begin{aligned}
B_0^{(k)} &= A_k a_k - A_k a_k = 0 \\
B_1^{(k)} &= A_k a_{k+1} - A_{k+1} a_k = \begin{vmatrix} a_{k+1} & a_k \\ A_{k+1} & A_k \end{vmatrix} .
\end{aligned}$$

Assuming $k \geq 1$ and using recurrence (2), we have

$$B_1^{(k)} = \begin{vmatrix} p_k a_k + q_k a_{k-1} & a_k \\ p_k A_k + q_k A_{k-1} & A_k \end{vmatrix} = \begin{vmatrix} q_k a_{k-1} & a_k \\ q_k A_{k-1} & A_k \end{vmatrix} = -q_k \begin{vmatrix} a_k & a_{k-1} \\ A_k & A_{k-1} \end{vmatrix} = -q_k B_1^{(k-1)} .$$

Consequently, we have

$$B_1^{(k)} = (-1)^k q_k q_{k-1} \cdots q_2 q_1 B_1^{(0)} = (-1)^k q_k^* \left(A_0 a_1 - A_1 a_0\right).$$

Now, we choose A_0 and A_1 so that $A_0 a_1 - A_1 a_0 = 1$. Specifically, since $a_1 \neq 0$, we choose $A_0 = (A_1 a_0 + 1)/a_1$. In conclusion, there exists at least a sequence $\{A_n\}_{n\in\mathbb{N}}$ satisfying recurrence (4) and having the requested property. □

Now, we can prove next

THEOREM 2.3 *Let* $\{a_n\}_{n\in\mathbb{N}}$ *be a sequence satisfying a three-term recurrence*

$$a_{n+2} = p_{n+1} a_{n+1} + q_{n+1} a_n \tag{5}$$

with $a_n \neq 0$ *for all* $n \geq 1$. *Then, for every* $m, n \in \mathbb{N}$, *we have the identity*

$$\sum_{k=1}^{m} (-1)^k q_k^* \frac{b_n^{(k)}}{a_k a_{n+k}} = \sum_{k=1}^{n} (-1)^k q_k^* \frac{b_m^{(k)}}{a_k a_{m+k}} \tag{6}$$

where $q_k^* = q_k q_{k-1} \cdots q_2 q_1$, *and where the coefficients* $b_n^{(k)}$ *are defined by the recurrence*

$$b_{n+2}^{(k)} = p_{n+k+1} b_{n+1}^{(k)} + q_{n+k+1} b_n^{(k)} \tag{7}$$

with the initial values $b_0^{(k)} = 0$ *and* $b_1^{(k)} = 1$.

Proof. Consider the sequence $\{B_n^{(k)}\}_{n\in\mathbb{N}}$ defined in Theorem 2.2. Since $B_n^{(k)} = A_k a_{n+k} - A_{n+k} a_k$ and $a_n \neq 0$ for all $n \geq 1$, we have

$$\frac{B_n^{(k)}}{a_k a_{n+k}} = \frac{A_k}{a_k} - \frac{A_{n+k}}{a_{n+k}}.$$

So, by Lemma 2.1, we have the identity

$$\sum_{k=1}^{m} \frac{B_n^{(k)}}{a_k a_{n+k}} = \sum_{k=1}^{n} \frac{B_m^{(k)}}{a_k a_{m+k}}.$$

Finally, since $B_n^{(k)} = (-1)^k q_k^* b_n^{(k)}$, we have identity (6). □

Notice that the coefficients $b_n^{(k)}$ can be obtained by two linearly independent solutions of recurrence (5). Indeed, we have

PROPOSITION 2.4 *If* $\{x_n\}_{n\in\mathbb{N}}$ *and* $\{y_n\}_{n\in\mathbb{N}}$ *are two linearly independent solutions of recurrence (5), then the coefficients* $b_n^{(k)}$ *can be expressed as*

$$b_n^{(k)} = \frac{x_k y_{n+k} - x_{n+k} y_k}{x_k y_{k+1} - x_{k+1} y_k}. \tag{8}$$

Proof. The sequence $\{b_n^{(k)}\}_{n\in\mathbb{N}}$ satisfies recurrence (7). So, it belongs to the vector space generated by the two sequences $\{x_{n+k}\}_{n\in\mathbb{N}}$ and $\{y_{n+k}\}_{n\in\mathbb{N}}$. This means that there exist two scalars $\lambda, \mu \in \mathbb{R}$ such that

$$b_n^{(k)} = \lambda x_{n+k} + \mu y_{n+k} \qquad \forall n \in \mathbb{N}.$$

By imposing the initial conditions $b_0^{(k)} = 0$ and $b_1^{(k)} = 1$, we obtain the system

$$\begin{cases} x_k\lambda + y_k\mu = 0 \\ x_{k+1}\lambda + y_{k+1}\mu = 1 \end{cases}$$

whose unique solution (by Cramer's theorem) is given by

$$\lambda = \frac{1}{\Delta_k}\begin{vmatrix} 0 & y_k \\ 1 & y_{k+1} \end{vmatrix} = -\frac{y_k}{\Delta_k} \qquad \text{and} \qquad \mu = \frac{1}{\Delta_k}\begin{vmatrix} x_k & 0 \\ x_{k+1} & 1 \end{vmatrix} = \frac{x_k}{\Delta_k}$$

where

$$\Delta_k = \begin{vmatrix} x_k & y_k \\ x_{k+1} & y_{k+1} \end{vmatrix} = x_k y_{k+1} - x_{k+1} y_k .$$

Notice that $\Delta_k \neq 0$ for all $k \in \mathbb{N}$, since we are considering two linearly independent solutions of recurrence (5). In conclusion, we have obtained identity (8). $\square$

3 A first specialization

Let $X(x)$ and $Y(x)$ be two expressions such that $X(x), Y(x) \neq 0$. Let $\{\mathcal{W}_n(q,x)\}_{n\in\mathbb{N}}$ be the sequence defined by the recurrence

$$\mathcal{W}_{n+2}(q,x) = X(q^{n+1}x)\mathcal{W}_{n+1}(q,x) + Y(q^{n+1}x)\mathcal{W}_n(q,x) \tag{9}$$

with the initial values $\mathcal{W}_0(q,x) = 1$ and $\mathcal{W}_1(q,x) = X(x)$. Furthermore, let $\{\mathcal{W}_n^{(a,b)}(q,x)\}_{n\in\mathbb{N}}$ be the sequence defined by recurrence (9) and by the initial values $\mathcal{W}_0^{(a,b)}(q,x) = a$ and $\mathcal{W}_1^{(a,b)}(q,x) = b$ (with $b \neq 0$).

Theorem 3.1 *We have* $\mathcal{W}_n^{(0,1)}(q,x) = \mathcal{W}_{n-1}(q,qx)$, *for all* $n \in \mathbb{N}$.

Proof. Set $U_n(q,x) = \mathcal{W}_{n-1}(q,qx)$. Replacing n by $n-1$ and x by qx in recurrence (9), we have

$$\mathcal{W}_{n+1}(q,qx) = X(q^{n+1}x)\mathcal{W}_n(q,qx) + Y(q^{n+1}x)\mathcal{W}_{n-1}(q,qx)$$

that is

$$U_{n+2}(q,x) = X(q^{n+1}x)U_{n+1}(q,x) + Y(q^{n+1}x)U_n(q,qx).$$

So, the terms $U_n(q,x)$ satisfy recurrence (9). Moreover $U_1(q,x) = \mathcal{W}_0(q,qx) = 1$. Finally, for $n = -1$ in (9), we have $\mathcal{W}_1(q,x) = X(x)\mathcal{W}_0(q,x) + Y(x)\mathcal{W}_{-1}(q,x)$, that is $X(x) = X(x) + Y(x)\mathcal{W}_{-1}(q,x)$, from which we have $U_0(q,x) = \mathcal{W}_{-1}(q,qx) = 0$. $\square$

THEOREM 3.2 *For every* $m, n \in \mathbb{N}$, $m, n \geq 1$, *we have the identity*

$$\sum_{k=1}^{m}(-1)^k Q_k(q,x)\,\frac{\mathcal{W}_{n-1}(q,q^{k+1}x)}{\mathcal{W}_k^{(a,b)}(q,x)\mathcal{W}_{n+k}^{(a,b)}(q,x)} = \sum_{k=1}^{n}(-1)^k Q_k(q,x)\,\frac{\mathcal{W}_{m-1}(q,q^{k+1}x)}{\mathcal{W}_k^{(a,b)}(q,x)\mathcal{W}_{m+k}^{(a,b)}(q,x)} \tag{10}$$

where $Q_k(q,x) = Y(qx)Y(q^2x)\cdots Y(q^{k-1}x)Y(q^kx)$. *In particular, we have the identity*

$$\sum_{k=1}^{m}(-1)^k Q_k(q,x)\,\frac{\mathcal{W}_{n-1}(q,q^{k+1}x)}{\mathcal{W}_k(q,x)\mathcal{W}_{n+k}(q,x)} = \sum_{k=1}^{n}(-1)^k Q_k(q,x)\,\frac{\mathcal{W}_{m-1}(q,q^{k+1}x)}{\mathcal{W}_k(q,x)\mathcal{W}_{m+k}(q,x)} \tag{11}$$

Proof. The terms $\mathcal{W}_n^{(a,b)}(q,x)$ satisfy recurrence (5) with $p_n = X(q^nx)$ and $q_n = Y(q^nx)$. So $q_k^* = Y(q^kx)Y(q^{k-1}x)\cdots Y(q^2x)Y(qx) = Q_k(q,x)$ and the coefficients $b_n^{(k)} = b_n^{(k)}(q,x)$ appearing in the statement of Theorem 2.3 are defined by the recurrence

$$b_{n+2}^{(k)}(q,x) = X(q^{n+k+1}x)b_{n+1}^{(k)}(q,x) + Y(q^{n+k+1}x)b_n^{(k)}(q,x)$$

with the initial values $b_0^{(k)}(q,x) = 0$ and $b_1^{(k)}(q,x) = 1$. Hence, by Theorem 3.1, we have

$$b_n^{(k)}(q,x) = U_n(q,q^kx) = \mathcal{W}_{n-1}(q,q^{k+1}x)\,.$$

In conclusion, identity (6) becomes identity (11). □

The results obtained in Theorem 3.2 can be extended to the bisection sequences $\{\mathcal{W}_{2n}^{(a,b)}(q,x)\}_{n\in\mathbb{N}}$ and $\{\mathcal{W}_{2n+1}^{(a,b)}(q,x)\}_{n\in\mathbb{N}}$. If $E_n^{(a,b)}(q,x) = \mathcal{W}_{2n}^{(a,b)}(q,x)$ and $O_n^{(a,b)}(q,x) = \mathcal{W}_{2n+1}^{(a,b)}(q,x)$, then we have

THEOREM 3.3 *The terms* $E_n^{(a,b)}(q,x)$ *and* $O_n^{(a,b)}(q,x)$ *satisfy the three-term recurrences*

$$E_{n+2}(q,x) = R_{n+1}(q,x)E_{n+1}(q,x) + S_{n+1}(q,x)E_n(q,x) \tag{12}$$

$$O_{n+2}(q,x) = R_{n+1}^+(q,x)O_{n+1}(q,x) + S_{n+1}^+(q,x)O_n(q,x) \tag{13}$$

where

$$R_{n+1}(q,x) = Y(q^{2n+3}x) + X(q^{2n+2}x)X(q^{2n+3}x) + \frac{X(q^{2n+3}x)}{X(q^{2n+1}x)}\,Y(q^{2n+2}x) \tag{14}$$

$$S_{n+1}(q,x) = \frac{X(q^{2n+3}x)}{X(q^{2n+1}x)}\,Y(q^{2n+1}x)Y(q^{2n+2}x) \tag{15}$$

and $R_{n+1}^+(q,x) = R_{n+1}(q,qx)$ *and* $S_{n+1}^+(q,x) = S_{n+1}(q,qx)$.

Proof. By recurrence (9), we have the system

$$\begin{cases} E_{n+1}(q,x) = Y(q^{2n+1}x)E_n(q,x) + X(q^{2n+1}x)O_n(q,x) \\ O_{n+1}(q,x) = X(q^{2n+2}x)E_{n+1}(q,x) + Y(q^{2n+2}x)O_n(q,x) \end{cases}$$

from which it is straightforward to obtain recurrences (12) and (13). □

Moreover, if $E_n(q,x) = \mathcal{W}_{2n}(q,x)$ and $O_n(q,x) = \mathcal{W}_{2n+1}(q,x)$, then we have

THEOREM 3.4 *For every* $n \in \mathbb{N}$, *we have*

$$E_n^{(0,1)}(q,x) = \frac{O_{n-1}(q,qx)}{X(qx)} = \frac{\mathcal{W}_{2n-1}(q,qx)}{X(qx)} \tag{16}$$

$$O_n^{(0,1)}(q,x) = \frac{O_{n-1}(q,q^2x)}{X(q^2x)} = \frac{\mathcal{W}_{2n-1}(q,q^2x)}{X(q^2x)}\,. \tag{17}$$

Proof. Since $R_n^+(q,qx) = R_{n+1}(q,x)$ and $S_n^+(q,qx) = S_{n+1}(q,x)$, also the terms $\frac{O_{n-1}(q,qx)}{X(qx)}$ satisfy recurrence (12) with the initial values 0 and 1. So, we have identity (16). Similarly, since $R_n^+(q,q^2x) = R_{n+1}^+(q,x)$ and $S_n^+(q,q^2x) = S_{n+1}^+(q,x)$, also the terms $\frac{O_{n-1}(q,q^2x)}{X(q^2x)}$ satisfy recurrence (13) with the initial values 0 and 1. So, we have identity (17). $\square$

Now, we can prove next

THEOREM 3.5 *For every* $m,n \in \mathbb{N}$, $m,n \geq 1$, *we have the identities*

$$\begin{aligned}\sum_{k=1}^{m} Q_k(q^2,x)\,Q_k(q^2,x/q)\,\frac{\mathcal{W}_{2n-1}(q,q^{2k+1}x)}{\mathcal{W}_{2k}^{(a,b)}(q,x)\mathcal{W}_{2n+2k}^{(a,b)}(q,x)} &= \\ = \sum_{k=1}^{n} Q_k(q^2,x)\,Q_k(q^2,x/q)\,\frac{\mathcal{W}_{2m-1}(q,q^{2k+1}x)}{\mathcal{W}_{2k}^{(a,b)}(q,x)\mathcal{W}_{2m+2k}^{(a,b)}(q,x)}&\end{aligned} \tag{18}$$

and

$$\begin{aligned}\sum_{k=1}^{m} Q_k(q^2,qx)\,Q_k(q^2,x)\,\frac{\mathcal{W}_{2n-1}(q,q^{2k+2}x)}{\mathcal{W}_{2k+1}^{(a,b)}(q,x)\mathcal{W}_{2n+2k+1}^{(a,b)}(q,x)} &= \\ = \sum_{k=1}^{n} Q_k(q^2,qx)\,Q_k(q^2,x)\,\frac{\mathcal{W}_{2m-1}(q,q^{2k+2}x)}{\mathcal{W}_{2k+1}^{(a,b)}(q,x)\mathcal{W}_{2m+2k+1}^{(a,b)}(q,x)}&\end{aligned} \tag{19}$$

where $Q_k(q,x) = Y(qx)Y(q^2x)\cdots Y(q^{k-1}x)Y(q^kx)$. *In particular, we have the identities*

$$\begin{aligned}\sum_{k=1}^{m} Q_k(q^2,x)\,Q_k(q^2,x/q)\,\frac{\mathcal{W}_{2n-1}(q,q^{2k+1}x)}{\mathcal{W}_{2k}(q,x)\mathcal{W}_{2n+2k}(q,x)} &= \\ = \sum_{k=1}^{n} Q_k(q^2,x)\,Q_k(q^2,x/q)\,\frac{\mathcal{W}_{2m-1}(q,q^{2k+1}x)}{\mathcal{W}_{2k}(q,x)\mathcal{W}_{2m+2k}(q,x)}&\end{aligned} \tag{20}$$

and

$$\begin{aligned}\sum_{k=1}^{m} Q_k(q^2,qx)\,Q_k(q^2,x)\,\frac{\mathcal{W}_{2n-1}(q,q^{2k+2}x)}{\mathcal{W}_{2k+1}(q,x)\mathcal{W}_{2n+2k+1}(q,x)} &= \\ = \sum_{k=1}^{n} Q_k(q^2,qx)\,Q_k(q^2,x)\,\frac{\mathcal{W}_{2m-1}(q,q^{2k+2}x)}{\mathcal{W}_{2k+1}(q,x)\mathcal{W}_{2m+2k+1}(q,x)}&\end{aligned} \tag{21}$$

Proof. By recurrence (12), the terms $E_n(q,x)$ satisfy recurrence (5) with $p_n = R_n(q,x)$ and $q_n = S_n(q,x)$. So, by identity (15), we have

$$q_k^* = \prod_{i=1}^{k} S_k(q,x) = \prod_{i=1}^{k} \frac{X(q^{2i+1}x)}{X(q^{2i-1}x)}\,Y(q^{2i-1}x)Y(q^{2i+1}x)$$

$$
\begin{aligned}
&= \frac{X(q^{2k+1}x)}{X(qx)} \prod_{i=1}^{k} Y(q^{2i+1}x) \prod_{i=1}^{k} Y(q^{2i-1}x) \\
&= \frac{X(q^{2k+1}x)}{X(qx)} \, Q_k(q^2, qx) \, Q_k(q^2, x/q) \, .
\end{aligned}
$$

Moreover, by identities (14) and (15), the coefficients $b_n^{(k)} = b_n^{(k)}(q,x)$ appearing in the statement of Theorem 2.3 are defined by the recurrence

$$
\begin{aligned}
b_{n+2}^{(k)}(q,x) &= R_{n+k+1}(q,x) b_{n+1}^{(k)}(q,x) + S_{n+k+1}(q,x) b_n^{(k)}(q,x) \\
&= R_{n+1}(q,q^{2k}x) b_{n+1}^{(k)}(q,x) + S_{n+1}(q,q^{2k}x) b_n^{(k)}(q,x)
\end{aligned}
$$

with the initial values $b_0^{(k)}(q,x) = 0$ and $b_1^{(k)}(q,x) = 1$. So, by identity (16), we have

$$
b_n^{(k)}(q,x) = E_n^{(0,1)}(q,q^{2k}x) = \frac{\mathcal{W}_{2k-1}(q,q^{2k+1}x)}{X(q^{2k+1}x)} \, .
$$

Then, identity (6) becomes identity (20).

By recurrence (13), the terms $O_n(q,x)$ satisfy recurrence (5) with $p_n = R_n^{+}(q,x) = R_n(q,qx)$ and $q_n = S_n^{+}(q,x) = S_n(q,qx)$. So, as before, we have

$$
q_k^* = \frac{X(q^{2k+2}x)}{X(q^2x)} \, Q_k(q^2, qx) \, Q_k(q^2, x) \, .
$$

Moreover, the coefficients $b_n^{(k)} = b_n^{(k)}(q,x)$ are defined by the recurrence

$$
\begin{aligned}
b_{n+2}^{(k)}(q,x) &= R_{n+k+1}^{+}(q,x) b_{n+1}^{(k)}(q,x) + S_{n+k+1}^{+}(q,x) b_n^{(k)}(q,x) \\
&= R_{n+1}^{+}(q,q^{2k}x) b_{n+1}^{(k)}(q,x) + S_{n+1}^{+}(q,q^{2k}x) b_n^{(k)}(q,x)
\end{aligned}
$$

with the initial values $b_0^{(k)}(q,x) = 0$ and $b_1^{(k)}(q,x) = 1$. So, by identity (17), we have

$$
b_n^{(k)}(q,x) = O_n^{(0,1)}(q,q^{2k}x) = \frac{\mathcal{W}_{2k-1}(q,q^{2k+2}x)}{X(q^{2k+2}x)} \, .
$$

Then, identity (6) becomes identity (21). □

4 Specialization to q-polynomials

Now, we specialize the results obtained in the previous section to some q-polynomials of combinatorial interest. Specifically, we consider the q-polynomials $W_n(q,x)$ defined by the recurrence

$$
W_{n+2}(q,x) = (A + Bq^{n+2}x) W_{n+1}(q,x) + (C + Dq^{n+1}x) W_n(q,x) \tag{22}
$$

with the initial conditions $W_0(q,x) = 1$ and $W_1(q,x) = A + Bqx$, where $AB \neq 0$ and $CD \neq 0$. Notice that, by extending this recurrence to negative indices, we have $W_{-1}(q,x) = 0$. In particular, for

$x = 1$, we have the q-numbers $w_n(q) = W_n(q, 1)$. Furthermore, let $W_n^{(a,b)}(q, x)$ be the q-polynomials defined by recurrence (22) and by the initial values $W_0^{(a,b)}(q, x) = a$ and $W_1^{(a,b)}(q, x) = b$ (with $b \neq 0$).

First of all, we have

THEOREM 4.1 *For every $m, n \in \mathbb{N}$, $m, n \geq 1$, we have the identity*

$$\sum_{k=1}^{m} (-1)^k Q_k(q, x) \frac{W_{n-1}(q, q^{k+1}x)}{W_k^{(a,b)}(q, x) W_{n+k}^{(a,b)}(q, x)} = \sum_{k=1}^{n} (-1)^k Q_k(q, x) \frac{W_{m-1}(q, q^{k+1}x)}{W_k^{(a,b)}(q, x) W_{m+k}^{(a,b)}(q, x)} \tag{23}$$

where $Q_k(q, x) = (C + Dqx) \cdots (C + Dq^{k-1}x)(C + Dq^k x)$. In particular, we have the identity

$$\sum_{k=1}^{m} (-1)^k Q_k(q, x) \frac{W_{n-1}(q, q^{k+1}x)}{W_k(q, x) W_{n+k}(q, x)} = \sum_{k=1}^{n} (-1)^k Q_k(q, x) \frac{W_{m-1}(q, q^{k+1}x)}{W_k(q, x) W_{m+k}(q, x)} \tag{24}$$

and for $x = 1$ and $Q_k(q) = Q_k(q, 1)$, we have the identity

$$\sum_{k=1}^{m} (-1)^k Q_k(q) \frac{W_{n-1}(q, q^{k+1})}{w_k(q) w_{n+k}(q)} = \sum_{k=1}^{n} (-1)^k Q_k(q) \frac{W_{m-1}(q, q^{k+1})}{w_k(q) w_{m+k}(q)} . \tag{25}$$

Proof. Apply Theorem 3.2, with $X(x) = A + Bqx$ and $Y(x) = C + Dx$. □

Then, we have

THEOREM 4.2 *For every $m, n \in \mathbb{N}$, $m, n \geq 1$, we have the identities*

$$\begin{aligned} &\sum_{k=1}^{m} Q_k(q^2, x)\, Q_k(q^2, x/q) \frac{W_{2n-1}(q, q^{2k+1}x)}{W_{2k}^{(a,b)}(q, x) W_{2n+2k}^{(a,b)}(q, x)} = \\ &= \sum_{k=1}^{n} Q_k(q^2, x)\, Q_k(q^2, x/q) \frac{W_{2m-1}(q, q^{2k+1}x)}{W_{2k}^{(a,b)}(q, x) W_{2m+2k}^{(a,b)}(q, x)} \end{aligned} \tag{26}$$

and

$$\begin{aligned} &\sum_{k=1}^{m} Q_k(q^2, qx)\, Q_k(q^2, x) \frac{W_{2n-1}(q, q^{2k+2}x)}{W_{2k+1}^{(a,b)}(q, x) W_{2n+2k+1}^{(a,b)}(q, x)} = \\ &= \sum_{k=1}^{n} Q_k(q^2, qx)\, Q_k(q^2, x) \frac{W_{2m-1}(q, q^{2k+2}x)}{W_{2k+1}^{(a,b)}(q, x) W_{2m+2k+1}^{(a,b)}(q, x)} \end{aligned} \tag{27}$$

where $Q_k(q, x) = (C + Dqx) \cdots (C + Dq^{k-1}x)(C + Dq^k x)$. In particular, we have the identities

$$\begin{aligned} &\sum_{k=1}^{m} Q_k(q^2, x)\, Q_k(q^2, x/q) \frac{W_{2n-1}(q, q^{2k+1}x)}{W_{2k}(q, x) W_{2n+2k}(q, x)} = \\ &= \sum_{k=1}^{n} Q_k(q^2, x)\, Q_k(q^2, x/q) \frac{W_{2m-1}(q, q^{2k+1}x)}{W_{2k}(q, x) W_{2m+2k}(q, x)} \end{aligned} \tag{28}$$

and

$$\begin{aligned}\sum_{k=1}^{m} Q_k(q^2,qx)\,Q_k(q^2,x)\,\frac{W_{2n-1}(q,q^{2k+2}x)}{W_{2k+1}(q,x)W_{2n+2k+1}(q,x)} &= \\ = \sum_{k=1}^{n} Q_k(q^2,qx)\,Q_k(q^2,x)\,\frac{W_{2m-1}(q,q^{2k+2}x)}{W_{2k+1}(q,x)W_{2m+2k+1}(q,x)}&\,.\end{aligned} \tag{29}$$

Proof. Apply Theorem 3.5, with $X(x) = A + Bqx$ and $Y(x) = C + Dx$. □

Finally, we have

THEOREM 4.3 *The q-polynomials* $W_n^{(a,b)}(q,x)$ *have generating series*

$$\begin{aligned}W^{(a,b)}(q,x;t) &= \sum_{n\geq 0} W_n^{(a,b)}(q,x)\,t^n = \\ &= \sum_{k\geq 0} q^{\binom{k+1}{2}} x^k t^k \frac{(a+(b-aA-aBqx)q^k t)(B+Dt)(B+Dqt)\cdots(B+Dq^{k-1}t)}{(1-At-Ct^2)(1-Aqt-Cq^2t^2)\cdots(1-Aq^kt-Cq^{2k}t^k)}\,.\end{aligned} \tag{30}$$

In particular, the q-polynomials $W_n(q,x)$ *have generating series*

$$\sum_{n\geq 0} W_n(q,x)\,t^n = \sum_{k\geq 0} q^{\binom{k+1}{2}} x^k t^k \frac{(B+Dt)(B+Dqt)\cdots(B+Dq^{k-1}t)}{(1-At-Ct^2)(1-Aqt-Cq^2t^2)\cdots(1-Aq^kt-Cq^{2k}t^2)}\,. \tag{31}$$

Proof. Let $W(t) = W^{(a,b)}(q,x;t)$. By recurrence (22), we have

$$\frac{W(t)-a-bt}{t^2} = A\frac{W(t)-a}{t} + Bqx\frac{W(qt)-a}{t} + CW(t) + DqxW(qt)$$

from which we obtain the identity

$$W(t) = \frac{a+(b-aA-aBqx)t}{1-At-Ct^2} + \frac{qxt(B+Dt)}{1-At-Ct^2}\,W(qt)\,.$$

By applying this identity repeatedly, we obtain

$$\begin{aligned}W(t) = \sum_{k=0}^{n} q^{\binom{k+1}{2}} x^k t^k \frac{(a+(b-aA-aBqx)q^k t)(B+Dt)(B+Dqt)\cdots(B+Dq^{k-1}t)}{(1-At-Ct^2)(1-Aqt-Cq^2t^2)\cdots(1-Aq^kt-Cq^{2k}t^2)} + \\ + q^{\binom{n+2}{2}} x^{n+1} t^{n+1} \frac{(a+(b-aA-aBqx)q^{n+1} t)(B+Dt)(B+Dqt)\cdots(B+Dq^{n}t)}{(1-At-Ct^2)(1-Aqt-Cq^2t^2)\cdots(1-Aq^nt-Cq^{2n}t^2)}\,W(q^{n+1}t)\,.\end{aligned}$$

Now, by taking the limit of both sides for $n \to +\infty$, we get identity (30). Finally, since $W_0(q,x) = 1$ and $W_1(q,x) = A + Bqx$, identity (30) implies identity (31). □

REMARK 4.4. By identity (30), we also have

$$\sum_{n\geq 0} W_n^{(0,1)}(q,x)\,t^n = \sum_{k\geq 0} q^{\binom{k+1}{2}} (qx)^k t^{k+1} \frac{(B+Dt)(B+Dqt)\cdots(B+Dq^{k-1}t)}{(1-At-Ct^2)(1-Aqt-Cq^2t^2)\cdots(1-Aq^kt-Cq^{2k}t^k)}\,.$$

Notice that, by series (31), we have the identity $W^{(0,1)}(q,x;t) = tW(q,qx;t)$, from which we reobtain that $W_n^{(0,1)}(q,x) = W_{n-1}(q,qx)$.

5 Examples

Several q-polynomials are a specialization of the q-polynomials $W_n(q,x)$ considered in Section 4. Some of them can be defined in the following combinatorial setting. A *linear partition* of the linearly ordered set $[n] = \{1,2,\ldots,n\}$ is a family $\pi = \{B_1, B_2, \ldots, B_k\}$ of non-empty intervals B_i of $[n]$ such that $B_i \cap B_j = \varnothing$, for every $i \neq j$, and $B_1 \cup B_2 \cup \cdots \cup B_k = [n]$. A *2-filtering partition* of $[n]$ is a linear partition of $[n]$ where each block has size 1 or 2. Let $\Phi_n^{(2)}$ be the set of the 2-filtering partitions of $[n]$ where the blocks are of two types, say black or white. Given $\pi \in \Phi_n^{(2)}$, let $m(\pi) = m(B_1) + m(B_2) + \cdots + m(B_k)$, where $m(B_i) = 0$ if B_i is a block of the first kind (black), $m(B_i) = s$ if $B_i = \{s\}$ or $B_i = \{s, s+1\}$ is a block of the second kind (white); then, let $w(\pi)$ be the number of white blocks of π.

5.1 q-Fibonacci and q-Lucas polynomials

Let Φ_n be the subset of $\Phi_n^{(2)}$ consisting of the 2-filtering partitions with only 1-blocks of the first kind (black) and 2-blocks of the second kind (white). The *q-Fibonacci polynomials* are defined by

$$F_n(q,x) = \sum_{\pi \in \Phi_n} q^{m(\pi)} x^{w(\pi)}$$

and satisfy the recurrence

$$F_{n+2}(q,x) = F_{n+1}(q,x) + q^{n+1} x F_n(q,x)$$

with the initial values $F_0(q,x) = F_1(q,x) = 1$. In particular, for $x = 1$, we have the *q-Fibonacci numbers* $f_n(q) = F_n(q,1)$, [17, 10] [4, 5].

Similarly, we define the *q-Lucas polynomials* $L_n(q,x)$ by the recurrence

$$L_{n+2}(q,x) = L_{n+1}(q,x) + q^{n+1} x L_n(q,x)$$

with the initial values $L_0(q,x) = 1 + q$ and $L_1(q,x) = 1$. Then, for $x = 1$, we have the *q-Lucas numbers* $\ell_n(q) = L_n(q,1)$.

The q-Fibonacci polynomials are a special case of the q-polynomials $W_n(q,x)$. Indeed, we have $F_n(q,x) = W_n(q,x)$ for $A = 1$, $B = 0$, $C = 0$, $D = 1$. The q-Lucas polynomials satisfy the same recurrence, but with different initial values. Then, by identities (31) and (30), we have the generating series

$$\sum_{n\geq 0} F_n(q,x)\, t^n = \sum_{k\geq 0} \frac{q^{k^2} x^k t^{2k}}{(1-t)(1-qt)\cdots(1-q^k t)}$$

$$\sum_{n\geq 0} L_n(q,x)\, t^n = \sum_{k\geq 0} \frac{q^{k^2}(1+q-q^{k+1}t) x^k t^{2k}}{(1-t)(1-qt)\cdots(1-q^k t)}$$

from which we obtain $L_n(q,x) = (1+q)F_n(q,x) + qF_{n-1}(q,qx)$, for $n \geq 1$. Moreover, we have

THEOREM 5.1 *For every $m,n \in \mathbb{N}$, $m,n \geq 1$, we have the identities*

$$\sum_{k=1}^{m} (-1)^k q^{\binom{k+1}{2}} x^k \, \frac{F_{n-1}(q, q^{k+1}x)}{F_k(q,x) F_{n+k}(q,x)} = \sum_{k=1}^{n} (-1)^k q^{\binom{k+1}{2}} x^k \, \frac{F_{m-1}(q, q^{k+1}x)}{F_k(q,x) F_{m+k}(q,x)} \tag{32}$$

$$\sum_{k=1}^{m}(-1)^k q^{\binom{k+1}{2}} x^k \, \frac{F_{n-1}(q,q^{k+1}x)}{L_k(q,x)L_{n+k}(q,x)} = \sum_{k=1}^{n}(-1)^k q^{\binom{k+1}{2}} x^k \, \frac{F_{m-1}(q,q^{k+1}x)}{L_k(q,x)L_{m+k}(q,x)} \,. \tag{33}$$

In particular, for $x = 1$ *, we have the identities*

$$\sum_{k=1}^{m}(-1)^k q^{\binom{k+1}{2}} \, \frac{F_{n-1}(q,q^{k+1})}{f_k(q)f_{n+k}(q)} = \sum_{k=1}^{n}(-1)^k q^{\binom{k+1}{2}} \, \frac{F_{m-1}(q,q^{k+1})}{f_k(q)f_{m+k}(q)} \tag{34}$$

$$\sum_{k=1}^{m}(-1)^k q^{\binom{k+1}{2}} \, \frac{F_{n-1}(q,q^{k+1})}{\ell_k(q)\ell_{n+k}(q)} = \sum_{k=1}^{n}(-1)^k q^{\binom{k+1}{2}} \, \frac{F_{m-1}(q,q^{k+1})}{\ell_k(q)\ell_{m+k}(q)} \,. \tag{35}$$

Proof. Since $Q_k(q,x) = q^{k+(k-1)+\cdots+2+1}x^k = q^{\binom{k+1}{2}}x^k$, identity (24) becomes identity (32). Similarly, identity (23) becomes identity (33). □

Then, we have

THEOREM 5.2 *For every* $m, n \in \mathbb{N}$ *,* $m, n \geq 1$ *, we have the identities*

$$\sum_{k=1}^{m} q^{k(2k+1)} x^{2k} \, \frac{F_{2n-1}(q,q^{2k+1}x)}{F_{2k}(q,x)F_{2n+2k}(q,x)} = \sum_{k=1}^{n} q^{k(2k+1)} x^{2k} \, \frac{F_{2m-1}(q,q^{2k+1}x)}{F_{2k}(q,x)F_{2m+2k}(q,x)} \tag{36}$$

$$\sum_{k=1}^{m} q^{k(2k+1)} x^{2k} \, \frac{F_{2n-1}(q,q^{2k+1}x)}{L_{2k}(q,x)L_{2n+2k}(q,x)} = \sum_{k=1}^{n} q^{k(2k+1)} x^{2k} \, \frac{F_{2m-1}(q,q^{2k+1}x)}{L_{2k}(q,x)L_{2m+2k}(q,x)} \tag{37}$$

and

$$\sum_{k=1}^{m} q^{k(2k+3)} x^{2k} \, \frac{F_{2n-1}(q,q^{2k+2}x)}{F_{2k+1}(q,x)F_{2n+2k+1}(q,x)} = \sum_{k=1}^{n} q^{k(2k+3)} x^{2k} \, \frac{F_{2m-1}(q,q^{2k+2}x)}{F_{2k+1}(q,x)F_{2m+2k+1}(q,x)} \tag{38}$$

$$\sum_{k=1}^{m} q^{k(2k+3)} x^{2k} \, \frac{F_{2n-1}(q,q^{2k+2}x)}{L_{2k+1}(q,x)L_{2n+2k+1}(q,x)} = \sum_{k=1}^{n} q^{k(2k+3)} x^{2k} \, \frac{F_{2m-1}(q,q^{2k+2}x)}{L_{2k+1}(q,x)L_{2m+2k+1}(q,x)} \,. \tag{39}$$

Proof. Apply Theorem 4.2, noticing that

$$Q_k(q^2,x)Q_k(q^2,x/q) = q^{4\binom{k+1}{2}-k}x^{2k} = q^{k(2k+1)}x^{2k}$$
$$Q_k(q^2,qx)Q_k(q^2,x) = q^{4\binom{k+1}{2}+k}x^{2k} = q^{k(2k+3)}x^{2k} \,.$$

□

REMARK 5.3. In the literature, there are other q-analogues of the Fibonacci polynomials and numbers. For instance, we have the q-Fibonacci polynomials $\varphi_n(q,x)$ defined by the recurrence $\varphi_{n+2}(q,x) = q^{n+1}x\varphi_{n+1}(q,x) + q^n x\varphi_n(q,x)$ with the initial values $\varphi_0(q,x) = 1$ and $\varphi_1(q,x) = x$, and the q-Fibonacci numbers $\varphi_n(q) = \varphi_n(q,1)$ considered in [7]. In this case, we have $\varphi_n(q,x) = W_n(q,x)$ for $A = 0$, $B = 1/q$, $C = 0$, $D = 1/q$. So, we have the generating series

$$\sum_{n\geq 0}\varphi_n(q,x)\,t^n = \sum_{k\geq 0} q^{\binom{k}{2}}x^k t^k(1+t)(1+qt)\cdots(1+q^{k-1}t)$$

and the identity

$$\sum_{k=1}^{m}(-1)^k q^{\binom{k}{2}} x^k \, \frac{\varphi_{n-1}(q,q^{k+1}x)}{\varphi_k(q,x)\varphi_{n+k}(q,x)} = \sum_{k=1}^{n}(-1)^k q^{\binom{k}{2}} x^k \, \frac{\varphi_{m-1}(q,q^{k+1}x)}{\varphi_k(q,x)\varphi_{m+k}(q,x)} \,. \tag{40}$$

5.2 q-Pell polynomials

Let Ψ_n be the subset of $\Phi_n^{(2)}$ consisting of the 2-filtering partitions of $[n]$ where the 1-blocks are of both types (black and white), and the 2-blocks are only of the second type (white). The *q-Pell polynomials* are defined by

$$P_n(q,x) = \sum_{\pi \in \Psi_n} q^{m(\pi)} x^{w(\pi)}$$

and satisfy the recurrence

$$P_{n+2}(q,x) = (1 + q^{n+2}x)P_{n+1}(q,x) + q^{n+1}xP_n(q,x)$$

with the initial conditions $P_0(q,x) = 1$ and $P_1(q,x) = 1 + qx$. In particular, for $x = 1$, we have the *q-Pell numbers* $p_n(q) = P_n(q,1)$, [16, 15, 3]. For $q = 1$, we have the *Pell numbers* [18, A000129].

In this case, we have $P_n(q,x) = W_n(q,x)$ for $A = 1$, $B = 1$, $C = 0$, $D = 1$. Then, by identity (31), we have the generating series

$$\sum_{n \geq 0} P_n(q,x)\, t^n = \sum_{k \geq 0} q^{\binom{k+1}{2}} x^k t^k \, \frac{(1+t)(1+qt)\cdots(1+q^{k-1}t)}{(1-t)(1-qt)\cdots(1-q^k t)} \,.$$

Moreover, we have

Theorem 5.4 *For every $m,n \in \mathbb{N}$, $m,n \geq 1$, we have the identity*

$$\sum_{k=1}^{m}(-1)^k q^{\binom{k+1}{2}} x^k \, \frac{P_{n-1}(q,q^{k+1}x)}{P_k(q,x)P_{n+k}(q,x)} = \sum_{k=1}^{n}(-1)^k q^{\binom{k+1}{2}} x^k \, \frac{P_{m-1}(q,q^{k+1}x)}{P_k(q,x)P_{m+k}(q,x)} \,. \tag{41}$$

In particular, for $x = 1$, we have the identity

$$\sum_{k=1}^{m}(-1)^k q^{\binom{k+1}{2}} \, \frac{P_{n-1}(q,q^{k+1})}{p_k(q)p_{n+k}(q)} = \sum_{k=1}^{n}(-1)^k q^{\binom{k+1}{2}} \, \frac{P_{m-1}(q,q^{k+1})}{p_k(q)p_{m+k}(q)} \,. \tag{42}$$

Proof. Since $Q_k(q,x) = q^{k+(k-1)+\cdots+2+1}x^k = q^{\binom{k+1}{2}}x^k$, identity (24) becomes identity (41). □

Then, we have

Theorem 5.5 *For every $m,n \in \mathbb{N}$, $m,n \geq 1$, we have the identities*

$$\sum_{k=1}^{m} q^{k(2k+1)} x^{2k} \, \frac{P_{2n-1}(q,q^{2k+1}x)}{P_{2k}(q,x)P_{2n+2k}(q,x)} = \sum_{k=1}^{n} q^{k(2k+1)} x^{2k} \, \frac{P_{2m-1}(q,q^{2k+1}x)}{P_{2k}(q,x)P_{2m+2k}(q,x)} \tag{43}$$

$$\sum_{k=1}^{m} q^{k(2k+3)} x^{2k} \, \frac{P_{2n-1}(q,q^{2k+2}x)}{P_{2k+1}(q,x)P_{2n+2k+1}(q,x)} = \sum_{k=1}^{n} q^{k(2k+3)} x^{2k} \, \frac{P_{2m-1}(q,q^{2k+2}x)}{P_{2k+1}(q,x)P_{2m+2k+1}(q,x)} \,. \tag{44}$$

Proof. By Theorem 4.2, where $Q_k(q^2,x)Q_k(q^2,x/q) = q^{4\binom{k+1}{2}-k}x^{2k} = q^{k(2k+1)}x^{2k}$. □

REMARK 5.6. In [12], we have other two q-analogues of the Pell polynomials: the q-polynomials $a_n(q,x)$ defined by the recurrence $a_{n+2}(q,x) = (1+x)a_{n+1}(q,x) + q^n x a_n(q,x)$ with the initial values $a_0(q,x) = 0$ and $a_1(q,x) = x$, and the q-polynomials $b_n(q,x)$ defined by the recurrence $b_{n+2}(q,x) = (1+q^{n+1}x)b_{n+1}(q,x) + q^n x b_n(q,x)$ with the initial values $b_0(q,x) = 0$ and $b_1(q,x) = x$. The q-polynomials $b_{n+1}(q,x)$ satisfy the same recurrence of $P_n(q,x)$, but with different initial values, while the q-polynomials $a_{n+1}(q,x)$ do not satisfy an instance of recurrence (22).

5.3 q-Jacobsthal polynomials

Let Ξ_n be the subset of $\Phi_n^{(2)}$ consisting of the 2-filtering partitions of $[n]$ where the 1-blocks are only of the first type (black) and the 2-blocks are of both types (black and white). The *q-Jacobsthal polynomials* are defined by

$$J_n(q,x) = \sum_{\pi\in\Xi_n} q^{m(\pi)}x^{w(\pi)}$$

and satisfy the recurrence

$$J_{n+2}(q,x) = J_{n+1}(q,x) + (1+q^{n+1}x)J_n(q,x)$$

with the initial values $J_0(q,x) = J_1(q,x) = 1$. In particular, for $x = 1$, we have the *q-Jacobsthal numbers* $j_n(q) = J_n(q,1)$. Furthermore, for $q = 1$, we have the *Jacobsthal numbers* $j_n = (2^{n+1} + (-1)^n)/3$ [18, A001045].

In this case, we have $J_n(q,x) = W_n(q,x)$ for $A = 1$, $B = 0$, $C = 1$, $D = 1$. Then, by identity (31), we have the generating series

$$\sum_{n\geq 0} J_n(q,x)\, t^n = \sum_{k\geq 0} \frac{q^{k^2}x^k t^{2k}}{(1-t-t^2)(1-qt-q^2t^2)\cdots(1-q^kt-q^{2k}t^2)}\,.$$

Moreover, recalling that the *q-Pochhammer symbol* is defined by

$$(x;q)_k = (1-x)(1-qx)\cdots(1-q^{k-1}x)\,,$$

we have

THEOREM 5.7 *For every $m,n\in\mathbb{N}$, $m,n\geq 1$, we have the identity*

$$\sum_{k=1}^{m}(-1)^k(-qx;q)_k \frac{J_{n-1}(q,q^{k+1}x)}{J_k(q,x)J_{n+k}(q,x)} = \sum_{k=1}^{n}(-1)^k(-qx;q)_k \frac{J_{m-1}(q,q^{k+1}x)}{J_k(q,x)J_{m+k}(q,x)}\,. \tag{45}$$

In particular, for $x = 1$, we have the identity

$$\sum_{k=1}^{m}(-1)^k(-q;q)_k \frac{J_{n-1}(q,q^{k+1})}{j_k(q)j_{n+k}(q)} = \sum_{k=1}^{n}(-1)^k(-q;q)_k \frac{J_{m-1}(q,q^{k+1})}{j_k(q)j_{m+k}(q)}\,. \tag{46}$$

Proof. Since $Q_k(q,x) = (1+qx)\cdots(1+q^{k-1}x)(1+q^kx) = (-qx;q)_k$, identity (24) becomes identity (45). □

Then, we have

THEOREM 5.8 *For every* $m, n \in \mathbb{N}$, $m, n \geq 1$, *we have the identities*

$$\sum_{k=1}^{m}(-qx;q)_{2k}\,\frac{J_{2n-1}(q,q^{2k+1}x)}{J_{2k}(q,x)J_{2n+2k}(q,x)} = \sum_{k=1}^{n}(-qx;q)_{2k}\,\frac{J_{2m-1}(q,q^{2k+1}x)}{J_{2k}(q,x)J_{2m+2k}(q,x)} \tag{47}$$

$$\sum_{k=1}^{m}(-q^2x;q)_{2k}\,\frac{J_{2n-1}(q,q^{2k+2}x)}{J_{2k+1}(q,x)J_{2n+2k+1}(q,x)} = \sum_{k=1}^{n}(-q^2x;q)_{2k}\,\frac{J_{2m-1}(q,q^{2k+2}x)}{J_{2k+1}(q,x)J_{2m+2k+1}(q,x)}\,. \tag{48}$$

Proof. By Theorem 4.2, where $Q_k(q^2, x/q)Q_k(q^2, x) = (-qx;q^2)_k(-q^2x;q^2)_k = (-qx;q)_{2k}$. □

5.4 The q-polynomials $R_n(q,x)$

Let $R_n(q,x)$ be the q-polynomials associated to $\Phi_n^{(2)}$, i.e. the q-polynomials defined by

$$R_n(q,x) = \sum_{\pi \in \Phi_n^{(2)}} q^{m(\pi)} x^{w(\pi)}\,.$$

These q-polynomials satisfy the recurrence

$$R_{n+2}(q,x) = (1+q^{n+2}x)R_{n+1}(q,x) + (1+q^{n+1}x)R_n(q,x)$$

with the initial conditions $R_0(q,x) = 1$ and $R_1(q,x) = 1 + qx$. In particular, for $x = 1$, we have the *q-numbers* $r_n(q) = R_n(q,1)$. Furthermore, the coefficients of the polynomials $R_n(x) = R_n(1,x)$ form sequence A063967 in [18], while the numbers $r_n = r_n(1)$ form sequence A026150 in [18].

In this case, we have $R_n(q,x) = W_n(q,x)$ for $A = 1$, $B = 1$, $C = 1$, $D = 1$. Then, by identity (31), we have the generating series

$$\sum_{n\geq 0} R_n(q,x)\,t^n = \sum_{k\geq 0} q^{\binom{k+1}{2}} x^k t^k \frac{(1+t)(1+qt)\cdots(1+q^{k-1}t)}{(1-t-t^2)(1-qt-q^2t^2)\cdots(1-q^kt-q^{2k}t^2)}\,.$$

Moreover, we have

THEOREM 5.9 *For every* $m, n \in \mathbb{N}$, $m, n \geq 1$, *we have the identity*

$$\sum_{k=1}^{m}(-1)^k(-qx;q)_k\,\frac{R_{n-1}(q,q^{k+1}x)}{R_k(q,x)R_{n+k}(q,x)} = \sum_{k=1}^{n}(-1)^k(-qx;q)_k\,\frac{R_{m-1}(q,q^{k+1}x)}{R_k(q,x)R_{m+k}(q,x)}\,. \tag{49}$$

In particular, for $x = 1$, *we have the identity*

$$\sum_{k=1}^{m}(-1)^k(-q;q)_k\,\frac{R_{n-1}(q,q^{k+1})}{r_k(q)r_{n+k}(q)} = \sum_{k=1}^{n}(-1)^k(-q;q)_k\,\frac{R_{m-1}(q,q^{k+1})}{r_k(q)r_{m+k}(q)}\,. \tag{50}$$

Proof. Since $Q_k(q,x) = (1+qx)\cdots(1+q^{k-1}x)(1+q^kx) = (-qx;q)_k$, identity (24) becomes identity (49). □

Then, we have

THEOREM 5.10 *For every* $m, n \in \mathbb{N}$, $m, n \geq 1$, *we have the identities*

$$\sum_{k=1}^{m} (-qx;q)_{2k} \frac{R_{2n-1}(q, q^{2k+1}x)}{R_{2k}(q,x) R_{2n+2k}(q,x)} = \sum_{k=1}^{n} (-qx;q)_{2k} \frac{R_{2m-1}(q, q^{2k+1}x)}{R_{2k}(q,x) R_{2m+2k}(q,x)} \tag{51}$$

$$\sum_{k=1}^{m} (-q^2x;q)_{2k} \frac{R_{2n-1}(q, q^{2k+2}x)}{R_{2k+1}(q,x) R_{2n+2k+1}(q,x)} = \sum_{k=1}^{n} (-q^2x;q)_{2k} \frac{R_{2m-1}(q, q^{2k+2}x)}{R_{2k+1}(q,x) R_{2m+2k+1}(q,x)} . \tag{52}$$

Proof. By Theorem 4.2, where $Q_k(q^2, x/q) Q_k(q^2, x) = (-qx; q^2)_k (-q^2x; q^2)_k = (-qx; q)_{2k}$. □

5.5 q-Chebyshev polynomials

We define the *q-Chebyshev polynomials of the first kind* $T_n(q,x)$ by the recurrence

$$T_{n+2}(q,x) = 2q^{n+1} x\, T_{n+1}(q,x) - T_n(q,x)$$

with the initial conditions $T_0(q,x) = 1$ and $T_1(q,x) = x$. Similarly, we define the *q-Chebyshev polynomials of the second kind* $U_n(q,x)$ by the recurrence

$$U_{n+2}(q,x) = 2q^{n+1} x\, U_{n+1}(q,x) - U_n(q,x)$$

with the initial conditions $U_0(q,x) = 1$ and $U_1(q,x) = 2x$.

In this case, we have $U_n(q,x) = W_n(q,x)$ for $A = 0$, $B = 2/q$, $C = -1$, $D = 0$. Then, by identities (30) and (31), we have the generating series

$$T(q,x;t) = \sum_{n \geq 0} T_n(q,x)\, t^n = \sum_{k \geq 0} \frac{q^{\binom{k}{2}} 2^k x^k t^k (1 - q^k xt)}{(1+t^2)(1+q^2t^2) \cdots (1+q^{2k}t^2)}$$

$$U(q,x;t) = \sum_{n \geq 0} U_n(q,x)\, t^n = \sum_{k \geq 0} \frac{q^{\binom{k}{2}} 2^k x^k t^k}{(1+t^2)(1+q^2t^2) \cdots (1+q^{2k}t^2)} .$$

Notice that $T(q,x;t) = U(q,x;t) - xtU(q,qx;t)$, and consequently that $T_n(q,x) = U_n(q,x) - xU_{n-1}(q,qx)$. Moreover, we have

THEOREM 5.11 *For every* $m, n \in \mathbb{N}$, $m, n \geq 1$, *we have the identities*

$$\sum_{k=1}^{m} \frac{U_{n-1}(q, q^{k+1}x)}{T_k(q,x) T_{n+k}(q,x)} = \sum_{k=1}^{n} \frac{U_{m-1}(q, q^{k+1}x)}{T_k(q,x) T_{m+k}(q,x)} \tag{53}$$

$$\sum_{k=1}^{m} \frac{U_{n-1}(q, q^{k+1}x)}{U_k(q,x) U_{n+k}(q,x)} = \sum_{k=1}^{n} \frac{U_{m-1}(q, q^{k+1}x)}{U_k(q,x) U_{m+k}(q,x)} . \tag{54}$$

Proof. Since $Q_k(q,x) = (-1)^k$, identity (24) becomes identity (54). Similarly, identity (23) becomes identity (53). □

Then, we have

THEOREM 5.12 *For every* $m, n \in \mathbb{N}$, $m, n \geq 1$, *we have the identities*

$$\sum_{k=1}^{m} \frac{U_{2n-1}(q, q^{2k+1}x)}{T_{2k}(q,x)T_{2n+2k}(q,x)} = \sum_{k=1}^{n} \frac{U_{2m-1}(q, q^{2k+1}x)}{T_{2k}(q,x)T_{2m+2k}(q,x)} \tag{55}$$

$$\sum_{k=1}^{m} \frac{U_{2n-1}(q, q^{2k+2}x)}{T_{2k+1}(q,x)T_{2n+2k+1}(q,x)} = \sum_{k=1}^{n} \frac{U_{2m-1}(q, q^{2k+2}x)}{T_{2k+1}(q,x)T_{2m+2k+1}(q,x)} \tag{56}$$

and

$$\sum_{k=1}^{m} \frac{U_{2n-1}(q, q^{2k+1}x)}{U_{2k}(q,x)U_{2n+2k}(q,x)} = \sum_{k=1}^{n} \frac{U_{2m-1}(q, q^{2k+1}x)}{U_{2k}(q,x)U_{2m+2k}(q,x)} \tag{57}$$

$$\sum_{k=1}^{m} \frac{U_{2n-1}(q, q^{2k+2}x)}{U_{2k+1}(q,x)U_{2n+2k+1}(q,x)} = \sum_{k=1}^{n} \frac{U_{2m-1}(q, q^{2k+2}x)}{U_{2k+1}(q,x)U_{2m+2k+1}(q,x)} . \tag{58}$$

Proof. Apply Theorem 4.2. □

REMARK 5.13. In [11] we have the q-polynomials $U_n^{(a)}(q,x)$ (with a and x exchanged between them) defined by the recurrence

$$U_{n+2}^{(a)}(q,x) = (2a + q^{n+1}x)\, U_{n+1}^{(a)}(q,x) - U_n^{(a)}(q,x)$$

with the initial conditions $U_0^{(a)}(q,x) = 1$ and $U_1^{(a)}(q,x) = 2a + x$. So $U_n^{(a)}(q,x) = W_n(q,x)$ for $A = 2a$, $B = 1/q$, $C = -1$, $D = 0$. Consequently, we have the generating series

$$\sum_{n\geq 0} U_n^{(a)}(q,x)\, t^n = \sum_{k\geq 0} \frac{q^{\binom{k}{2}} x^k t^k}{(1-2at+t^2)(1-2aqt+q^2t^2)\cdots(1-2aq^kt+q^{2k}t^2)} .$$

and the same identities given by (54), (57) and (58).

5.6 q-Morgan-Voyce polynomials

We define the *q-Morgan-Voyce polynomials* $M_n(q,x)$ by the recurrence

$$T_{n+2}(q,x) = (2 + q^{n+1}x)\, M_{n+1}(q,x) - M_n(q,x)$$

with the initial conditions $M_0(q,x) = 1$ and $M_1(q,x) = 2 + x$. Similarly, we define the *q-Morgan-Voyce polynomials* $N_n(q,x)$ by the recurrence

$$N_{n+2}(q,x) = (2 + q^{n+1}x)\, N_{n+1}(q,x) - N_n(q,x)$$

with the initial conditions $N_0(q,x) = 1$ and $N_1(q,x) = 1 + x$.

In this case, we have $M_n(q,x) = W_n(q,x)$ for $A = 2$, $B = 1/q$, $C = -1$, $D = 0$. then, by identities (31) and (30), we have the generating series

$$M(q,x;t) = \sum_{n\geq 0} M_n(q,x)\, t^n = \sum_{k\geq 0} \frac{q^{\binom{k}{2}} x^k t^k}{(1-2t+t^2)(1-2qt+q^2t^2)\cdots(1-2q^kt+q^{2k}t^2)}$$

$$N(q,x;t) = \sum_{n\geq 0} N_n(q,x)\,t^n = \sum_{k\geq 0} \frac{q^{\binom{k}{2}} x^k t^k (1-q^k t)}{(1-2t+t^2)(1-2qt+q^2t^2)\cdots(1-2q^k t+q^{2k}t^2)}\,.$$

Notice that $N(q,x;t) = M(q,x;t) - tM(q,qx;t)$, and consequently that $N_n(q,x) = M_n(q,x) - M_{n-1}(q,qx)$. Moreover, we have

THEOREM 5.14 *For every $m,n \in \mathbb{N}$, $m,n \geq 1$, we have the identities*

$$\sum_{k=1}^{m} \frac{M_{n-1}(q,q^{k+1}x)}{M_k(q,x)M_{n+k}(q,x)} = \sum_{k=1}^{n} \frac{M_{m-1}(q,q^{k+1}x)}{M_k(q,x)M_{m+k}(q,x)} \tag{59}$$

$$\sum_{k=1}^{m} \frac{M_{n-1}(q,q^{k+1}x)}{N_k(q,x)N_{n+k}(q,x)} = \sum_{k=1}^{n} \frac{M_{m-1}(q,q^{k+1}x)}{N_k(q,x)N_{m+k}(q,x)}\,. \tag{60}$$

Proof. Since $Q_k(q,x) = (-1)^k$, identity (24) becomes identity (59). Similarly, identity (23) becomes identity (60). □

Then, we have

THEOREM 5.15 *For every $m,n \in \mathbb{N}$, $m,n \geq 1$, we have the identities*

$$\sum_{k=1}^{m} \frac{M_{2n-1}(q,q^{2k+1}x)}{M_{2k}(q,x)M_{2n+2k}(q,x)} = \sum_{k=1}^{n} \frac{M_{2m-1}(q,q^{2k+1}x)}{M_{2k}(q,x)M_{2m+2k}(q,x)} \tag{61}$$

$$\sum_{k=1}^{m} \frac{M_{2n-1}(q,q^{2k+2}x)}{M_{2k+1}(q,x)M_{2n+2k+1}(q,x)} = \sum_{k=1}^{n} \frac{M_{2m-1}(q,q^{2k+2}x)}{M_{2k+1}(q,x)M_{2m+2k+1}(q,x)} \tag{62}$$

and

$$\sum_{k=1}^{m} \frac{M_{2n-1}(q,q^{2k+1}x)}{N_{2k}(q,x)N_{2n+2k}(q,x)} = \sum_{k=1}^{n} \frac{M_{2m-1}(q,q^{2k+1}x)}{N_{2k}(q,x)N_{2m+2k}(q,x)} \tag{63}$$

$$\sum_{k=1}^{m} \frac{M_{2n-1}(q,q^{2k+2}x)}{N_{2k+1}(q,x)N_{2n+2k+1}(q,x)} = \sum_{k=1}^{n} \frac{M_{2m-1}(q,q^{2k+2}x)}{N_{2k+1}(q,x)N_{2m+2k+1}(q,x)}\,. \tag{64}$$

Proof. Apply Theorem 4.2. □

5.7 Two q-sums

As a final example, we consider the q-polynomials

$$S_n(q,x) = \sum_{k=0}^{n} q^{\binom{k}{2}} x^k \qquad \text{and} \qquad T_n(q,x) = \sum_{k=0}^{n} (x;q)_k\,.$$

LEMMA 5.16 *The q-polynomials $S_n(q,x)$ satisfy the recurrence*

$$S_{n+2}(q,x) = (1+q^{n+1}x)S_{n+1}(q,x) - q^{n+1}xS_n(q,x) \tag{65}$$

with the initial values $S_0(q,x)=1$ *and* $S_1(q,x)=1+x$ *, while the q-polynomials* $T_n(q,x)$ *satisfy the recurrence*

$$T_{n+2}(q,x)=(2-q^{n+1}x)T_{n+1}(q,x)-(1-q^{n+1}x)T_n(q,x) \tag{66}$$

with the initial values $T_0(q,x)=1$ *and* $T_1(q,x)=2-x$ *.*

Proof. In the first case, we have the identities

$$\begin{aligned} S_{n+1}(q,x)-S_n(q,x)&=q^{\binom{n+1}{2}}x^{n+1}\\ S_{n+2}(q,x)-S_{n+1}(q,x)&=q^{\binom{n+2}{2}}x^{n+2}\,, \end{aligned}$$

from which we obtain the equation

$$S_{n+2}(q,x)-S_{n+1}(q,x)=q^{n+1}x(S_{n+1}(q,x)-S_n(q,x))$$

equivalent to recurrence (65). Similarly, in the second case, we have the identities

$$\begin{aligned} T_{n+1}(q,x)-T_n(q,x)&=(x;q)_{n+1}\\ T_{n+2}(q,x)-T_{n+1}(q,x)&=(x;q)_{n+2}\,, \end{aligned}$$

from which we obtain the equation

$$T_{n+2}(q,x)-T_{n+1}(q,x)=(1-q^{n+1}x)(T_{n+1}(q,x)-T_n(q,x))$$

equivalent to recurrence (66). □

By Lemma 5.16, we have that also $S_n(q,x)$ and $T_n(q,x)$ are special cases of the q-polynomials $W_n(q,x)$. Specifically, we have $S_n(q,x)=W_n(q,x)$ for $A=1$, $B=1/q$, $C=0$, $D=-1$, and $T_n(q,x)=W_n(q,x)$ for $A=2$, $B=-1/q$, $C=-1$, $D=1$. So, we have

Theorem 5.17 *For every* $m,n\in\mathbb{N}$, $m,n\geq 1$, *we have the identities*

$$\sum_{k=1}^{m} q^{\binom{k+1}{2}}x^k\,\frac{S_{n-1}(q,q^{k+1}x)}{S_k(q,x)S_{n+k}(q,x)}=\sum_{k=1}^{n} q^{\binom{k+1}{2}}x^k\,\frac{S_{m-1}(q,q^{k+1}x)}{S_k(q,x)S_{m+k}(q,x)} \tag{67}$$

and

$$\sum_{k=1}^{m}(qx;q)_k\,\frac{T_{n-1}(q,q^{k+1}x)}{T_k(q,x)T_{n+k}(q,x)}=\sum_{k=1}^{n}(qx;q)_k\,\frac{T_{m-1}(q,q^{k+1}x)}{T_k(q,x)T_{m+k}(q,x)}\,. \tag{68}$$

Proof. In the first case, we have $Q_k(q,x)=q^{\binom{k+1}{2}}(-x)^k$ and identity (24) becomes identity (67). In the second case, we have $Q_k(q,x)=(-1)^k(qx;q)_k$ and identity (24) becomes identity (68). □

Finally, by Theorem (4.2), we have

Theorem 5.18 *For every* $m,n\in\mathbb{N}$, $m,n\geq 1$, *we have the identities*

$$\sum_{k=1}^{m} q^{k(2k+1)}x^{2k}\,\frac{S_{2n-1}(q,q^{2k+1}x)}{S_{2k}(q,x)S_{2n+2k}(q,x)}=\sum_{k=1}^{n} q^{k(2k+1)}x^{2k}\,\frac{S_{2m-1}(q,q^{2k+1}x)}{S_{2k}(q,x)S_{2m+2k}(q,x)} \tag{69}$$

$$\sum_{k=1}^{m} q^{k(2k+3)} x^{2k} \frac{S_{2n-1}(q, q^{2k+2}x)}{S_{2k+1}(q,x)S_{2n+2k+1}(q,x)} = \sum_{k=1}^{n} q^{k(2k+3)} x^{2k} \frac{S_{2m-1}(q, q^{2k+2}x)}{S_{2k+1}(q,x)S_{2m+2k+1}(q,x)} \tag{70}$$

and

$$\sum_{k=1}^{m} (qx;q)_{2k} \frac{T_{2n-1}(q, q^{2k+1}x)}{T_{2k}(q,x)T_{2n+2k}(q,x)} = \sum_{k=1}^{n} (qx;q)_{2k} \frac{T_{2m-1}(q, q^{2k+1}x)}{T_{2k}(q,x)T_{2m+2k}(q,x)} \tag{71}$$

$$\sum_{k=1}^{m} (q^2x;q)_{2k} \frac{T_{2n-1}(q, q^{2k+2}x)}{T_{2k+1}(q,x)T_{2n+2k+1}(q,x)} = \sum_{k=1}^{n} (q^2x;q)_{2k} \frac{T_{2m-1}(q, q^{2k+2}x)}{T_{2k+1}(q,x)T_{2m+2k+1}(q,x)}\,. \tag{72}$$

REMARK 5.19. The *Al-Salam and Ismail polynomials* $U_n(x;a,b)$, [1], are defined by the recurrence

$$U_{n+2}(x;a,b) = (1+q^{n+1}a)x\,U_{n+1}(x;a,b) - q^{n+1}b\,U_n(x;a,b)$$

with the initial values $U_0(x;a,b) = 1$ and $U_1(x;a,b) = (1+a)x$. These polynomials do not satisfy an instance of recurrence (22). However, if we consider the q-polynomials $u_n(q,x) = U_n(1;x,x)$, then they satisfy the recurrence $u_{n+2}(q,x) = (1+q^{n+1}x)\,u_{n+1}(q,x) - q^{n+1}x\,u_n(q,x)$ with the initial values $u_0(q,x) = 1$ and $u_1(q,x) = 1+x$. This means that $S_n(q,x) = u_n(q,x) = U_n(1;x,x)$.

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