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Enumeration of small Wilf classes avoiding 1342 and two other 4-letter patterns

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Abstract. This paper is one of a series whose goal is to enumerate the avoiders, in the sense of classical pattern avoidance, for each triple of 4-letter patterns. There are 317 symmetry classes of triples of 4-letter patterns, avoiders of 267 of which have already been enumerated. Here we enumerate avoiders for all small Wilf classes that have a representative triple containing the pattern 1342, giving 40 new enumerations and leaving only 13 classes still to be enumerated. In all but one case, we obtain an explicit algebraic generating function that is rational or of degree 2. The remaining one is shown to be algebraic of degree 3. We use standard methods, usually involving detailed consideration of the left to right maxima, and sometimes the initial letters, to obtain an algebraic or functional equation for the generating function.

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Keywords: pattern avoidance, Wilf-equivalence, generating function, kernel method.

1 Introduction

This paper is a sequel to [6] and continues the investigation of permutations avoiding a given triple of 4-letter patterns. All large Wilf classes (those consisting of more than one symmetry class) have been enumerated [8, 9]. Here we enumerate avoiders for all triples that contain the pattern 1342, lie in a small Wilf class, and are not amenable to the INSENC algorithm. The results are presented in Table 1, where the numbering follows that of Table 2 in the appendix of [7]. As a consequence, only 13 symmetry classes remain to be enumerated.

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In this abbreviated paper, we prove a selection of the results in Table 1, illustrating a variety of methods. The full paper, containing all proofs, is posted to the ArXiv [4]. Section 2 contains some preliminary remarks, and Section 3 contains our selection of proofs.

Table 1: Small Wilf classes of three 4-letter patterns not counted by INSENC that include the pattern 1342

No.	Pattern set T	Generating function $F_T(x)$	Thm./[Ref]
77	$\{1243, 2314, 3412\}$	$\frac{1-11x+53x^2-145x^3+248x^4-274x^5+192x^6-80x^7+17x^8}{(1-x)^6(1-2x)^3}$	[4]
86	$\{1324, 3412, 4132\}$	$\frac{1-7x+19x^2-24x^3+16x^4-4x^5-x^6+2x^7}{(1-x)^3(1-2x)(1-3x+x^2)}$	[6]
90	$\{1243, 2431, 3412\}$	$\frac{1-11x+51x^2-129x^3+195x^4-183x^5+104x^6-30x^7+3x^8}{(1-x)^4(1-2x)(1-3x+x^2)^2}$	[4]
103	$\{1423, 2341, 3124\}$	$\frac{1-9x+35x^2-77x^3+107x^4-97x^5+55x^6-17x^7+x^8}{(1-x)^5(1-4x+5x^2-3x^3)}C(x)$	Thm. 3.4
106	$\{1342, 2143, 3412\}$	$\frac{(1-2x)(1-6x+12x^2-9x^3+4x^4)}{(1-x)^3(1-3x)(1-3x+x^2)}$	[4]
118	$\{1423, 1234, 3412\}$	$\frac{1-12x+64x^2-198x^3+393x^4-521x^5+463x^6-269x^7+95x^8-17x^9}{(1-x)^7(1-2x)^3}$	[4]
130	$\{3412, 3124, 1342\}$	$\frac{1-9x+32x^2-58x^3+58x^4-33x^5+8x^6}{(1-x)^4(1-2x)(1-4x+2x^2)}$	[4]
131	$\{2134, 1423, 2341\}$	$\frac{2x^5+x^4-6x^3+7x^2-4x+1}{(1-2x)(1-x)^3}C(x)-\frac{x(2x^4-x^3+x^2-2x+1)}{(1-2x)(1-x)^4}$	Thm. 3.8
133	$\{1342, 2143, 2314\}$	$\frac{(1-2x)(1-3x+x^2)}{1-6x+11x^2-7x^3}$	Thm. 3.9
150	$\{4312, 4132, 1324\}$	$\frac{1-11x+52x^2-136x^3+214x^4-204x^5+111x^6-28x^7}{(1-x)^3(1-2x)^3(1-3x+2x^2)}$	[6]
151	$\{4312, 1324, 1423\}$	$\frac{1-12x+61x^2-169x^3+275x^4-263x^5+136x^6-29x^7+x^8}{(1-3x+x^2)(1-2x)^4(1-x)^2}$	[6]
153	$\{4231, 1324, 1423\}$	$\frac{1-10x+41x^2-87x^3+101x^4-61x^5+15x^6-x^7}{(1-x)^2(1-2x)^3(1-3x+x^2)}$	[6]
156	$\{1324, 2341, 2431\}$	$\frac{1-8x+23x^2-25x^3+3x^4+7x^5}{(1-2x)^2(1-3x+x^2)(1-2x-x^2)}$	[6]
158	$\{1324, 1342, 3412\}$	$\frac{1-10x+40x^2-81x^3+88x^4-50x^5+11x^6}{(1-x)^3(1-2x)(1-3x)(1-3x+x^2)}$	[6]
159	$\{1243, 1342, 3412\}$	$\frac{1-11x+48x^2-104x^3+115x^4-61x^5+13x^6}{(1-x)(1-2x)(1-3x)(1-3x+x^2)^2}$	[4]
162	$\{3412, 1423, 2341\}$	$\frac{1-7x+18x^2-21x^3+11x^4}{(1-2x)(1-6x+12x^2-11x^3+3x^4)}$	[4]
163	$\{1342, 2314, 3412\}$	$\frac{(1-3x+3x^2)^2C(x)-x(1-x)(1-3x+5x^2-4x^3)}{(1-x)^5(1-2x)}$	Thm. 3.11
164	$\{1432, 2431, 3214\}$	$\frac{(1-x)^4(1-2x)C(x)-x(1-4x+6x^2-5x^3)}{(1-x)(1-2x)(1-4x+5x^2-3x^3)}$	Thm. 3.13
165	$\{1342, 2314, 3421\}$	$\frac{(1-2x)(1-x)^4C(x)-x(1-4x+6x^2-5x^3+x^4)}{(1-x)^4(1-3x+x^2)}$	Thm. 3.14
172	$\{2143, 4132, 1324\}$	$\frac{(2-10x+16x^2-8x^3+x^4)C(x)-1+4x-5x^2+x^3}{(1-x)^2(1-3x+x^2)}$	[5]
175	$\{1423, 2341, 3142\}$	$\frac{1-6x+12x^2-11x^3+5x^4}{1-7x+17x^2-20x^3+12x^4-2x^5}$	Thm. 3.15
176	$\{1342, 2431, 3412\}$	$\frac{(1-x)^2(1-4x+6x^2-5x^3+x^4)C(x)-1+6x-14x^2+15x^3-8x^4+x^5}{x(1-3x+x^2)(1-x+x^3)}$	Thm. 3.16
178	$\{1342, 2314, 2431\}$	$\frac{(1-x)^2(1-4x+6x^2-5x^3+x^4)C(x)-1+6x-14x^2+15x^3-8x^4+x^5}{x(1-3x+x^2)(1-x+x^3)}$	Thm. 3.17
180	$\{1342, 2314, 4231\}$	$\frac{1-7x+18x^2-22x^3+16x^4-6x^5+x^6-\left(x-5x^2+8x^3-2x^4-2x^5+x^6\right)C(x)}{(1-2x)(1-x)^2\left(1-5x+4x^2-x^3\right)}$	[6]

No.	Pattern set T	Generating function $F_T(x)$	Thm./[Ref]
182	$\{2314, 2431, 3412\}$	$\frac{1+x^2(1-x)C(x)^4}{1-x(1-2x)C(x)^2}$	[4]
184	$\{1324, 2431, 3241\}$	$\frac{1-8x+24x^2-32x^3+19x^4-3x^5}{(1-x)(1-2x)(1-3x+x^2)^2}$	[6]
187	$\{1324, 2314, 2431\}$	$\frac{1-9x+31x^2-49x^3+34x^4-7x^5}{(1-3x+x^2)^2(1-2x)^2}$	[6]
190	$\{3142, 2314, 1423\}$	$\frac{(1-2x)(1-3x+x^2)^2}{(1-x)(1-8x+22x^2-24x^3+8x^4-x^5)}$	[4]
192	$\{1243, 1342, 2431\}$	$\frac{(1-5x+9x^2-6x^3)(C(x)-1)-x^3}{x(1-2x)(1-x)^2}$	[4]
193	$\{1324, 2431, 3142\}$	$\frac{x-1+(x^2-5x+2)C(x)}{1-3x+x^2}$	[6]
194	$\{3124, 4123, 1243\}$	$\frac{(1-5x+9x^2-8x^3+4x^4)C(x)-(1-5x+9x^2-6x^3+x^4)}{x(1-2x)^2}$	Thm. 3.22
197	$\{2413, 3241, 2134\}$	$\frac{1-5x+9x^2-7x^3+x^4+(1-5x+9x^2-9x^3+3x^4)\sqrt{1-4x}}{(1-x)(1-6x+12x^2-11x^3+3x^4+(1-4x+6x^2-5x^3+x^4)\sqrt{1-4x})}$	[4]
198	$\{1234, 1423, 2341\}$	$\frac{(1-7x+18x^2-19x^3+6x^4)C(x)-(1-6x+12x^2-8x^3+x^4)}{x^2(1-x)(1-2x)}$	[4]
199	$\{1243, 1423, 2341\}$	$\frac{x(x-1)^2(2x-1)C(x)+3x^4-7x^3+9x^2-5x+1}{(xC(x)-(x-1)^2)(x-1)^2(2x-1)}$	[4]
204	$\{1243, 1423, 2314\}$	$\frac{x(2x^2-2x+1)C(x)-(3x^2-3x+1)}{x(2x^2-2x+1)C(x)-(1-x)(3x^2-3x+1)}$	[4]
207	$\{2134, 1423, 1243\}$	$\frac{1-x(1-x)C(x)}{(1-x)(2-C(x))+x^2}$	[5]
208	$\{1234, 1342, 3124\}$	$\frac{(1-2x)(1-6x+12x^2-10x^3+2x^4)-x^2(1-2x+2x^2)^2C(x)}{1-9x+30x^2-49x^3+38x^4-8x^5-4x^6}$	[4]
210	$\{1243, 1324, 2431\}$	$\frac{1-6x+13x^2-11x^3+4x^4}{x^2(1-x)^2}C(x)-\frac{1-6x+12x^2-8x^3+2x^4}{x^2(1-x)(1-2x)}$	[6]
212	$\{1324, 2413, 2431\}$	$1+\frac{x(1-4x+4x^2-x^3-x(1-4x+2x^2)C(x))}{(1-3x+x^2)(1-3x+x^2-x(1-2x)C(x))}$	[6]
213	$\{2431, 1324, 1342\}$	$\frac{(1-5x+8x^2-5x^3)C(x)-1+4x-4x^2+x^3}{x^2(1-2x)}$	[6]
214	$\{1342, 2341, 3412\}$	$\frac{(1-2x)\left((1-5x+9x^2-6x^3)\sqrt{1-4x}-(1-9x+29x^2-38x^3+18x^4)\right)}{2(1-x)^2x(1-7x+14x^2-9x^3)}$	Thm. 3.25
217	$\{4132, 1342, 1243\}$	$\frac{(1-x)(1-3x+x^2)\sqrt{1-4x}-(1-8x+20x^2-15x^3+4x^4)}{2x(1-x)(1-5x+4x^2-x^3)}$	[4]
219	$\{1342, 2413, 3412\}$	$1+\frac{x(1-x)^2(1-2x)}{(x^2-3x+1)(2x^2-2x+1)-x(1-2x)(1-x)C(x)}$	[4]
220	$\{2431, 2314, 3142\}$	$1+\frac{x(1-x)^2(1-2x)}{(1-3x)(1-x)^3-x(1-2x)(1-x+x^2)(C(x)-1)}$	Thm. 3.26
222	$\{3412, 3421, 1342\}$	$\frac{2-11x+13x^2-6x^3+x(1-x)(1-6x+4x^2)(1-4x)^{-1/2}}{2(1-6x+8x^2-4x^3)}$	Thm. 3.30
223	$\{1243, 1342, 2413\}$	$\frac{(1-2x)(1-2x-\sqrt{1-8x+20x^2-20x^3+4x^4})}{2x(1-4x+5x^2-x^3)}$	Thm. 3.33
224	$\{4132, 1342, 1423\}$	$\frac{2-10x+9x^2-3x^3+x(1-x)(2-x)\sqrt{1-4x}}{2(1-5x+4x^2-x^4)}$	Thm. 3.34
226	$\{1342, 2143, 2413\}$	$\frac{1-3x+x^2-\sqrt{(1-7x+13x^2-8x^3)(1-3x+x^2)}}{2x(1-x)(1-2x)}$	[4]
231	$\{1324, 1342, 2341\}$	$\frac{(1-3x)(1-2x-xC(x))}{(1-4x)(1-3x+x^2)}$	[6]
232	$\{1234, 1342, 2341\}$	$\frac{1-4x+2x^2-(1-6x+9x^2)C(x)}{x(1-4x)}$	[4]
242	$\{2341, 2431, 3241\}$	$F_T(x)=1+\frac{xF_T(x)}{1-xF_T^2(x)}$	Thm. 3.35

2 Preliminaries

For every pattern set T considered, $F_T(x)$ denotes the generating function $\sum_{n\geq 0}|S_n(T)|x^n$ for T-avoiders and $G_m(x)$ the generating function for T-avoiders $\pi = i_1\pi^{(1)}i_2\pi^{(2)}\cdots i_m\pi^{(m)} \in S_n(T)$ with m left-right maxima $i_1, i_2, \ldots, i_m = n$; thus $F_T(x) = \sum_{m\geq 0} G_m(x)$. Then $G_0(x) = 1$ and whenever no pattern in T starts with a 4, we have $G_1(x) = xF_T(x)$. For most of the triples T, our efforts are directed toward finding an expression for $G_m(x)$, usually distinguishing the case $m = 2$ and sometimes $m = 3$ from larger values of m. As usual, $C(x)$ denotes the generating function for the Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$, which counts τ-avoiders for each 3-letter pattern τ (see [11]).

Given nonempty sets of numbers S and T, we will write $S < T$ to mean $\max(S) < \min(T)$ (with the inequality vacuously holding if S or T is empty). In this context, we will often denote singleton sets simply by the element in question. Also, for a number k, $S - k$ means the set $\{s - k : s \in S\}$.

3 Proofs

3.1 Case 103: $\{1423, 2341, 3124\}$

Let $G_m(x)$ denote the generating function for T-avoiders with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xF_T(x)$. The *skew sum* [2,3] of two permutations $\pi \in S_m$ and $\sigma \in S_n$ is defined by

$$(\pi \ominus \sigma)(i) \quad = \quad \begin{cases} \pi(i) + n & \text{if } 1 \leq i \leq m, \\ \sigma(i - m) & \text{if } m + 1 \leq i \leq m + n, \end{cases}$$

and a permutation is *skew indecomposable* if it is nonempty and cannot be written as the skew sum of two nonempty permutations. The skew sum operation is associative and every nonempty permutation π is uniquely expressible as a skew sum of skew indecomposables. For example, for $\pi = 564132$, we have $\pi = 12 \ominus 1 \ominus 132$, all skew indecomposable, and the corresponding segments of π, here $56, 4, 132$, are called the *skew components* of π. Note that all patterns in T contain 123.

Lemma 3.1 *If $\pi \in S_n(T)$ has only 2 left-right maxima, then the first skew component of π avoids 123.*

Proof. It suffices to show that if $\pi \in S_n(T)$ with 2 left-right maxima contains a 123 pattern, then the minimum 123 pattern abc (the positions of the letters a, b, c in π are minimum in lex order) does not lie in the first skew component. Consider the matrix diagram of π as illustrated in Figure 1, where shaded regions are empty for the reason indicated (min refers to the minimum property of abc). If a is a left-right maximum, then so are b and c, violating the hypothesis. Hence, a is not a left-right maximum, and so $A \neq \emptyset$. Also, $A > B$ (or ab is the 12 of a 3124). This forces A to consist of a square block at the upper left of the diagram and we are done. $\square$

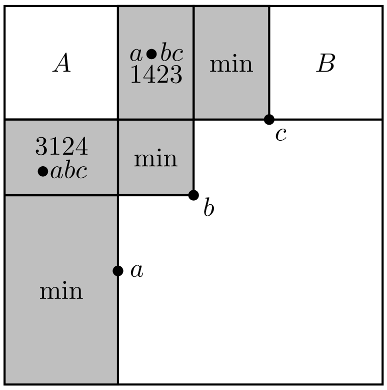


Figure 1: A T-avoider with 2 LR max and a 123

Lemma 3.2 *We have*

$$G_2(x) = \big(xC(x) - x\big)F_T(x)\,.$$

Proof. Suppose $\pi \in S_n(T)$ has 2 left-right maxima. The first skew component of π has length ≥ 2 and, from Lemma 3.1, avoids 123. The rest of π is an arbitrary T-avoider and so $G_2(x) = R(x)F_T(x)$, where $R(x)$ is the generating function for skew indecomposable 123-avoiders of length ≥ 2. We have $R(x) = xC(x) - x$: given the matrix diagram of a skew indecomposable $\{123\}$-avoider π of length n, the map "form lattice path from $(0, n)$ to $(n, 0)$ enclosing the right-left maxima of π" is a bijection to indecomposable Dyck paths, whose enumeration is well known. $\square$

Lemma 3.3 *We have*

$$G_3(x) = \frac{x^3}{(1-x)^5}\,, \textit{ and}$$

$$\sum_{m\geq 4} G_m(x) = \frac{1}{1-t-t^3} - 1 - t - t^2 - 2t^3 - 2t^4\,,$$

where $t = x/(1-x)$.

Proof. Suppose $\pi = i_1\pi^{(1)}i_2\pi^{(2)}\cdots i_m\pi^{(m)} \in S_n(T)$ has $m \geq 3$ left-right maxima. We have $\pi^{(s)} > i_{s-2}$ for $s = 3, 4, \ldots, m$ since $u \in \pi^{(s)}$ with $u < i_{s-2}$ makes $i_{s-2}i_{s-1}i_s u$ a 2341. Also, $\pi^{(s)}$ is decreasing for all s (a violator uv makes $i_s uv i_{s+1}$ a 3124 for $s = 1, 2, \ldots, m-1$ and $i_{m-2}i_m uv$ a 1423 for $s = m$). So π has the form illustrated for $m = 4$ in Figure 2a), where the down arrows indicate decreasing entries. Moreover, $\alpha_s\beta_{s+1}$ is decreasing for all s (a violator uv makes $i_s uv i_{s+2}$ a 3124 for $s = 1, 2, \ldots, m-2$ and $i_{s-1}i_s uv$ a 1423 for $s = 2, 3, \ldots, m-1$, covering all cases) and so π decomposes further as in Figure 2b).

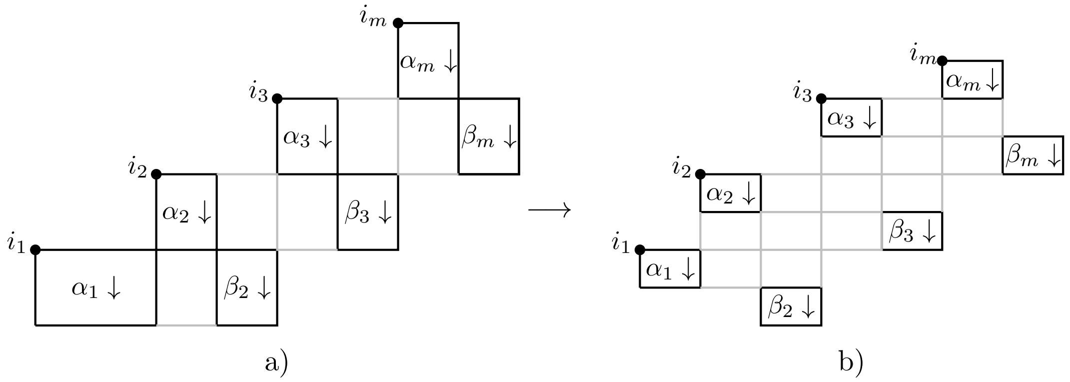


Figure 2: A T-avoider with $m \geq 3$ LR max

If $m = 3$, there are no further restrictions and so there are 5 boxes to be filled each with an arbitrary number of dots, and $G_3(x) = \frac{x^3}{(1-x)^5}$ as required.

However, if $m \geq 4$, then there is one more restriction: no two consecutively-indexed β's, say β_s and β_{s+1}, are both nonempty ($u \in \beta_s$ and $v \in \beta_{s+1}$ makes $i_{s-2}i_s uv$ a 1423 for $s = 3, 4, \ldots, m-1$ and $i_s uv i_m$ a 3124 for $s = 2, 3, \ldots, m-2$; note that neither condition says anything when $m = 3$). The contribution with k nonempty β's is $x^{m+k}/(1-x)^{m-k}$. A specification of empty/nonempty for each β corresponds to a binary string $w_1 w_2 \cdots w_m$ with $w_1 = 0$ and $w_s = 1$ if and only if $\beta_s \neq \emptyset$ for $s \geq 1$. Let $H_m(t)$ denote the generating function for binary strings $w_1 w_2 \cdots w_m$ with first bit 0 and no two consecutive 1's, where t marks the number of 1's. Then $G_m(x) = \left(\frac{x}{1-x}\right)^m H_m\left(\frac{x}{1-x}\right)$ for $m \geq 4$.

To find $\sum_{m\geq 4} G_m(x)$, set $H(t, y) = \sum_{m\geq 1} H_m(t) y^m$. We have the recurrence $H_m(t) = H_{m-1}(t) + tH_{m-2}(t)$ for $m \geq 3$ with initial conditions $H_1(t) = 1$ and $H_2(t) = 1 + t$. It follows routinely that $H(t, y) = y(1 + ty)/(1 - y - ty^2)$. Now, with $t := x/(1-x)$, $\sum_{m\geq 4} G_m(x) = \sum_{m\geq 4} t^m H_m(t) = H(t, t) - \sum_{m=1}^{3} t^m H_m(t)$, which simplifies to the stated expression. □

Lemmas 3.2 and 3.3 now give an expression for the right side in the identity $F_T(x) = \sum_{m\geq 0} G_m(x)$. Solving for $F_T(x)$ yields the following result.

THEOREM 3.4 *Let* $T = \{1423, 2341, 3124\}$. *Then*

$$F_T(x) = \frac{1 - 9x + 35x^2 - 77x^3 + 107x^4 - 97x^5 + 55x^6 - 17x^7 + x^8}{(1-x)^5(1 - 4x + 5x^2 - 3x^3)} C(x).$$

□

3.2 Case 131: $\{2134, 1423, 2341\}$

We focus on the first two letters of an avoider. Set $a(n) = |S_n(T)|$ and define $a(n; i_1, i_2, \ldots, i_m)$ to be the number of permutations $\pi = \pi_1\pi_2\cdots\pi_n$ in $S_n(T)$ such that $\pi_1\pi_2\cdots\pi_m = i_1 i_2 \cdots i_m$. Clearly, $a(n; 1) = |S_{n-1}(\{312, 2134, 2341\})|$. Let $g(x) = \sum_{n\geq 0} a(n; 1)x^n$ and $\ell_i = |S_i(\{213, 1423, 2341\})|$. It is known [1] that $g(x) = \frac{x^4 - x^3 + 4x^2 - 3x + 1}{(1-x)^4}$ and that $\ell_i = 2^i - i$. Set $b(n; i) = a(n; i, n-1)$ and $b'(n; i) = a(n; i, n)$. There are the following recurrences (proof omitted).

LEMMA 3.5 *We have* $\ell_i = |S_i(\{213, 1423, 2341\})| = 2^i - i$. *Then*

$$\begin{aligned}
a(n;i,j) &= a(n-1;i,j), \qquad \text{if } 2 \le i < j \le n-2,\\
a(n;i,j) &= a(n-1;i,j) + \sum_{k=1}^{j-1} a(n-1;j,k), \qquad \text{if } 1 \le j < i \le n-1 \text{ and } (i,j) \neq (n-1,1),\\
a(n;n,i) &= a(n-1;i), \qquad \text{if } 1 \le i \le n-1,\\
b(n;i) &= b(n-1;i) + \ell_{i-1}, \qquad 1 \le i \le n-2,\\
b'(n;i) &= b'(n-1;i) + \sum_{j=1}^{i-1} a(n-1;i,j), \qquad 1 \le i \le n-1,\\
a(n;n-1,n) &= a(n-2),\\
a(n;n-1,1) &= a(n-1;1).
\end{aligned}$$

□

Define $a'(n;i) = \sum_{j=1}^{i-1} a(n;i,j)$ and $a''(n;i) = \sum_{j=i+1}^{n} a(n;i,j)$. Let

$$B_n(v) = \sum_{i=1}^{n-2} b(n;i)v^{i-1}, \qquad B'_n(v) = \sum_{i=1}^{n-1} b'(n;i)v^{i-1},$$

$$A'_n(v) = \sum_{i=1}^{n} a'(n;i)v^{i-1}, \qquad A''_n(v) = \sum_{i=1}^{n} a''(n;i)v^{i-1},$$

$$A_n(v) = \sum_{i=1}^{n} a(n;i)v^{i-1}.$$

Define generating functions

$$B(x,v) = \sum_{n\ge 2} B_n(v)x^n, \qquad B'(x,v) = \sum_{n\ge 2} B'_n(v)x^n,$$

$$A(x,v) = 1 + x + \sum_{n\ge 2} A_n(v)x^n, \qquad A'(x,v) = \sum_{n\ge 2} A'_n(v)x^n,$$

$$A''(x,v) = \sum_{n\ge 2} A''_n(v)x^n.$$

LEMMA 3.6 *We have*

$$B(x,v) = \frac{(3v^2x^2 - 3vx + 1)x^3}{(1-x)^2(1-vx)^2(1-2vx)}, \text{ and}$$

$$B'(x,v) = \frac{x^2}{1-x} + \frac{x}{1-x}A'(x,v).$$

Proof. By Lemma 3.5,

$$b'(n;i) = a'(n-1;i) + b'(n-1;i), \quad 1 \le i \le n-1.$$

Multiplying by v^{i-1} and summing over $i = 1, 2, \ldots, n-1$ gives

$$B'_n(v) = A'_{n-1}(v) + B'_{n-1}(v),$$

with $B'_2(v) = 1$. Thus, the generating function for $B'_n(v)$ satisfies $B'(x,v) - x^2 = xA'(x,v) + xB'(x,v)$, which leads to $B'(x,v) = \frac{x^2}{1-x} + \frac{x}{1-x}A'(x,v)$.

Also by Lemma 3.5,

$$b(n;i) = b(n-1;i) + \ell_{i-1}, \quad 1 \le i \le n-2.$$

Multiplying by v^{i-1} and summing over $i = 1, 2, \ldots, n-2$, we have

$$B_n(v) = B_{n-1}(v) + \sum_{i=1}^{n-2} \ell_{i-1} v^{i-1},$$

which leads to $B(x,v) = \frac{1}{1-x} \sum_{n \ge 2} \sum_{i=1}^{n-2} (2^{i-1} - (i-1)) v^{i-1} x^n$ and the first assertion. □

Lemma 3.7 *We have*

$$A'(x,v) = \frac{x}{1-v}\left(A'(x,v) - \frac{1}{v}A'(vx,1)\right) + \frac{x(1+v)}{v}(A(xv,1) - 1) - x^2 A(xv,1), \text{ and}$$
$$A''(x,v) = xA''(x,v) + (1-x)x\big(g(x) - 1\big) + B(x,v) - B(x,0) + (1-x)(B'(x,v) - B'(x,0)).$$

Proof. Lemma 3.5 gives

$$a'(n;i) = a'(n-1;1) + a'(n-1;2) + \cdots + a'(n-1;i),$$

with $a'(n;n) = A_{n-1}(1)$ and $a'(n;n-1) = A_{n-1}(1) - A_{n-2}(1)$. Multiplying by v^{i-1} and summing over $i = 1, 2, \ldots, n-2$, we have

$$A'_n(v) = \frac{1}{1-v}(A'_{n-1}(v) - A'_{n-1}(1)v^{n-2}) + (A_{n-1}(1) - A_{n-2}(1))v^{n-2} + A_{n-1}(1)v^{n-1},$$

with $A'_1(v) = 1$. The first assertion follows by multiplying by x^n and summing over $n \ge 2$.

Lemma 3.5 also gives

$$a''(n;i) = \sum_{j=i+1}^{n-2} a(n;i,j) + b(n;i) + b'(n;i) = \sum_{j=i+1}^{n-2} a(n-1;i,j) + b(n;i) + b'(n;i),$$

which leads to $a''(n;i) = a''(n-1;i) + b(n;i) + b'(n;i) - b'(n-1;i)$ for all $i = 2, 3, \ldots, n-1$, where $a''(n;1) = a(n;1) = |S_{n-1}(\{312, 2134, 2341\})|$. Thus, multiplying by v^{i-1} and summing over $i = 2, 3, \ldots, n-1$, we have

$$A''_n(v) = A''_{n-1}(v) + A''_n(0) - A''_{n-1}(0) + B_n(v) - B_n(0) + B'_n(v) - B'_n(0) - B'_{n-1}(v) - B'_{n-1}(0),$$

with $A''_1(v) = 0$ and $B'_1(v) = B'_2(v) = 1$. Multiplying by x^n and summing over $n \ge 2$, we obtain

$$A''(x,v) = xA''(x,v) + (1-x)A''(x,0) + B(x,v) - B(x,0) + (1-x)\big(B'(x,v) - B'(x,0)\big),$$

where $A''(x,0) = x\sum_{n\geq 1}|S_n(\{312,2134,2341\})|x^n = x\big(g(x)-1\big)$, and the second assertion follows. $\square$

From Lemmas 3.6 and 3.7, we have

$$\left(1-\frac{x}{v(1-v)}\right)A'\Big(\frac{x}{v},v\Big) = -\frac{x}{v^2(1-v)}A'(x,1)+\frac{x(1+v)}{v^2}\big(A(x,1)-1\big)-\frac{x^2}{v^2}A(v,1).$$

Substituting $v = 1/C(x)$ implies

$$A'(x,1) = x(1-x)C(x)A(x,1)-xC(x)-x. \tag{1}$$

Moreover, we have

$$(1-x)A''(x,1) = x(1-x)\big(g(x)-1\big)+B(x,1)-B(x,0)+(1-x)\big(B'(x,1)-B'(x,0)\big), \tag{2}$$

where

$$B(x,1)=\frac{(3x^2-3x+1)x^3}{(1-x)^4(1-2x)},\quad B(x,0)=\frac{x^3}{(1-x)^2},\quad B'(x,1)=\frac{x^2}{1-x}+\frac{x}{1-x}A'(x,1)$$

and

$$B'(x,0)=\sum_{n\geq 2}a(n;1,n)x^n = \frac{x^2}{1-x}\ .$$

By solving the three equations (1), (2) and $A(x,1) = 1+x+A'(x,1)+A''(x,1)$ for $A(x,1)$, $A'(x,1)$, $A''(x,1)$, we obtain the following result.

Theorem 3.8 *Let $T = \{2134, 1423, 2341\}$. Then*

$$F_T(x) = \frac{2x^5+x^4-6x^3+7x^2-4x+1}{(1-2x)(1-x)^3}C(x)-\frac{x(2x^4-x^3+x^2-2x+1)}{(1-2x)(1-x)^4}.$$

3.3 Case 133: $\{1342, 2143, 2314\}$

Theorem 3.9 *Let $T = \{1342, 2143, 2314\}$. Then*

$$F_T(x) = \frac{(1-2x)(1-3x+x^2)}{1-6x+11x^2-7x^3}.$$

Proof. Let $G_m(x)$ be the generating function for members of $S_n(T)$ with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xF_T(x)$.

To find $G_2(x)$, consider $\pi = i\pi' n\pi'' \in S_n(T)$ with left-right maxima i and n. If π' is not empty, then $i = n-1$ (to avoid 2143) and i can be safely deleted, leaving nonempty T-avoiders with maximum entry not in first position. Hence, the contribution is $x\big(F_T(x)-1-xF_T(x)\big)$. If π' is empty, then $\pi_2 = n$ can be safely deleted and the contribution is $x\big(F_T(x)-1\big)$. Thus,

$$G_2(x) = x\big(F_T(x)-1-xF_T(x)\big)+x\big(F_T(x)-1\big)\,.$$

To find $G_m(x)$ with $m \ge 3$, consider a T-avoider $\pi = i_1\pi^{(1)} \cdots i_m\pi^{(m)}$ with m left-right maxima. It decomposes as in Figure 3, where the shaded regions are empty for the reason indicated and α_m lies to the left of β_m (or $i_1 i_{m-1}$ is the 23 of a 2314). If $\alpha_j \neq \emptyset$ for some $j \in [1, m-1]$, then $\alpha_i = \emptyset$ for all $i \neq j$ (2143), α_j avoids 231 and 2143, and β_m avoids T, and so the contribution is $x^m\big(K(x)-1\big)F_T(x)$, where $K(x) = \frac{1-2x}{1-3x+x^2}$ is the generating function for $\{231, 2143\}$-avoiders [14, Seq. A001519]. On the other hand, if $\alpha_i = \emptyset$ for all $i \in [1, m-1]$, then α_m avoids 231 and 2143 while β_m avoids T, giving $x^m K(x) F_T(x)$.

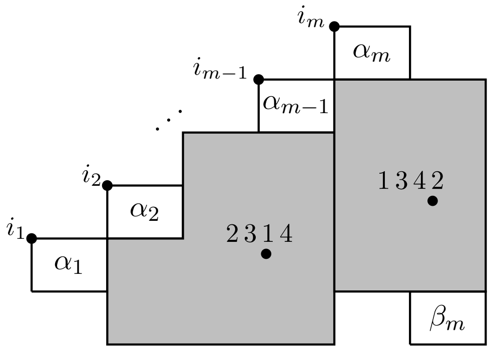


Figure 3: A T-avoider with $m \ge 3$ left-right maxima

Hence, $G_m(x) = (m-1)x^m\big(K(x)-1\big)F_T(x) + x^m K(x) F_T(x)$. Summing over $m \ge 3$ and substituting for $G_0(x), G_1(x)$ and $G_2(x)$, we obtain

$$F_T(x) = 1 + xF_T(x) + x\big(F_T(x) - 1 - xF_T(x)\big) + x\big(F_T(x) - 1\big) + \frac{x^3(1+x)F_T(x)}{1-3x+x^2},$$

and solving for $F_T(x)$ completes the proof. □

3.4 Case 163: $\{1342, 2314, 3412\}$

Note that all three patterns contain 231.

LEMMA 3.10 *The generating function for T-avoiders with 2 left-right maxima is given by*

$$H(x) = \frac{x\big((1-4x+7x^2-7x^3+4x^4)C(x) - 1 + 4x - 8x^2 + 9x^3 - 4x^4\big)}{(1-x)^4(1-2x)}.$$

Proof. Let $H_d(x)$ be the generating function for T-avoiders $i\pi' n\pi''$ with 2 left-right maxima where π'' has d letters smaller than i. If $d = 0$, then π' and π'' independently avoid 231, and so $H_0(x) = x^2C(x)^2$. Now let $d \ge 1$ and $j_1, j_2, \ldots, j_d$ be the letters in π'' smaller than i. These letters occur in decreasing order (to avoid 3412). Since π avoids 1342, we can write π as

$$\pi = i\alpha_0\alpha_1 \cdots \alpha_d n\beta_0 j_1\beta_1 \cdots j_d\beta_d\,,$$

where $i > \alpha_0 > j_1 > \alpha_1 > \cdots > j_d > \alpha_d$. Since π avoids 2314, we also have $\beta_0 > \beta_1 > \cdots > \beta_d$.

By considering the cases (i) $\alpha_d = \beta_d = \emptyset$, (ii) $\alpha_d \neq \emptyset, \beta_d = \emptyset$, (iii) $\alpha_d = \emptyset, \beta_d \neq \emptyset$, and (iv) $\alpha_d \neq \emptyset, \beta_d \neq \emptyset$, we obtain the respective contributions $xH_{d-1}(x)$, $\frac{x^{d+2}}{(1-x)^{d+1}}\big(C(x)-1\big)$, $\frac{x^{d+2}}{(1-x)^{d+1}}\big(C(x)-1\big)$, and $\frac{x^{d+2}}{(1-x)^2}\big(C(x)-1\big)^2$. Thus,

$$H_d(x) = xH_{d-1}(x) + \frac{2x^{d+2}}{(1-x)^{d+1}}\big(C(x)-1\big) + \frac{x^{d+2}}{(1-x)^2}\big(C(x)-1\big)^2.$$

Summing over $d \geq 1$ and using the expression for $H_0(x)$, we obtain

$$H(x) - x^2C(x)^2 = xH(x) + \frac{2x^3}{(1-x)(1-2x)}\big(C(x)-1\big) + \frac{x^3}{(1-x)^3}\big(C(x)-1\big)^2,$$

and the result follows by solving for $H(x)$. □

Theorem 3.11 *Let $T = \{1342, 2314, 3412\}$. Then*

$$F_T(x) = \frac{(1-3x+3x^2)^2C(x) - x(1-x)(1-3x+5x^2-4x^3)}{(1-x)^5(1-2x)}.$$

Proof. Let $G_m(x)$ be the generating function for T-avoiders with m left-right maxima. Clearly, $G_0(x) = 1$, $G_1(x) = xF_T(x)$, and Lemma 3.10 gives $G_2(x)$. For $G_m(x)$ with $m \geq 3$, suppose $\pi = i_1\pi^{(1)}i_2\pi^{(2)}\cdots i_m\pi^{(m)} \in S_n(T)$ has $m \geq 3$ left-right maxima. Since π avoids 1342, we see that $\pi^{(s)} > i_{s-1}$ for all $s = 2, 3, \ldots, m-1$, and $\pi^{(m)}$ can be written as $\alpha\beta$, where $\alpha > i_{m-1}$ and $\pi^{(1)} > \beta$ (to avoid 1342) and β is decreasing (to avoid 3412). Note that π avoids T if and only if each of $\pi^{(1)}, \ldots, \pi^{(m-1)}, \alpha$ avoids 231. Hence, $G_m(x) = \frac{x^mC(x)^m}{1-x}$. Summing over $m \geq 3$, we obtain

$$F_T(x) - 1 - xF_T(x) - G_2(x) = \frac{x^3C(x)^3}{(1-x)\big(1-xC(x)\big)} = \frac{x^3C(x)^4}{1-x}.$$

Substituting for $G_2(x)$, and solving for $F_T(x)$, completes the proof. □

3.5 Case 164: $\{1432, 2431, 3214\}$

We count by initial letters and define $a(n) = |S_n(T)|$ and $a(n; i_1, i_2, \ldots, i_m)$ to be the number of T-avoiders in S_n whose first m letters are $i_1, i_2, \ldots, i_m$. Clearly, $a(n; n) = a(n-1)$. Note that all three patterns in T contain 321.

Lemma 3.12 *The following two tables give a recurrence for $a(n; i, j)$ according as $i < j$ or $i > j$, valid whenever they make sense:*

$i=1$		$a(n;1,j) = \frac{j-1}{n-1}\binom{2n-2-j}{n-2}$
$i \geq 2$	$j=i+1$	$a(n;i,i+1) = a(n-1;i)$
	$j=i+2$	$a(n;i,i+2) = a(n;i-1,i+2)$
	$j \geq i+3$	$a(n;i,j) = \sum_{k=j-1}^{n-1} a(n-1;i,k)$

Recurrence for $a(n;i,j)$ when $i<j$

$j=1$		$a(n;i,1) = a(n;1,i) + 2^{i-2} + 1 - i$
$j \geq 2$	$i \leq n-2$	$a(n;i,j) = a(n-1;i,j)$
	$i=n-1$	$a(n;n-1,j) = a(n-1;n-2,j) + 2^{n-3-j}$
	$i=n$	$a(n;n,j) = a(n-1;j)$

Recurrence for $a(n;i,j)$ when $i>j$

Proof. We prove the first entry in each table and leave the other proofs to the reader. An avoider $\pi = 1j\pi'$ is counted by $a(n;1,j)$. Since π avoids 1432, $\mathrm{St}(j\pi')$ avoids 321, has length $n-1$ and first letter $j-1$. Such permutations are known to be counted by the "Catalan triangle" and so $a(n;1,j) = \frac{j-1}{n-1}\binom{2n-2-j}{n-2}$, the first item in the top table, see [10].

Now, consider $a(n;i,1)$. Let $\pi = i1\pi' \in S_n(T)$. Either there is no occurrence of 321 in π that starts with i, or there is such an occurrence. In the first case, the map $i1\pi' \to 1i\pi'$ is a bijection, so we have a contribution of $a(n;1,i)$. Thus, $a(n;i,1) = a(n;1,i) + b(n,i)$, where $b(n,i)$ is the number of permutations $i1\pi' \in S_n(T)$ containing an occurrence of 321 that starts with i.

Now let us find a formula for $b(n,i)$. Let $\pi = i1\pi' \in S_n(T)$ with $i\pi_p\pi_q$ an occurrence of 321 where p is minimal and $p+q$ is minimal. Say $\pi_p = u$ and $\pi_q = v$. Thus iuv is the leftmost (minimal) occurrence of 321 in π, and π has the form shown in Figure 4, where the shaded regions are empty for the reason indicated (min refers to the minimal property of iuv), n occurs before v (or n is the 4 of a 3214) and in fact immediately before v (or n is the 4 of a 1432), α is increasing by the minimal property of iuv, β is increasing (or v is the 2 of a 1432), and γ is increasing (or n is the 4 of a 1432). The generating function for the part of π below i is

$$\frac{x^3}{(1-x)^2(1-2x)}$$

and for the part at or above i is $x^2/(1-x)$. Hence,

$$b(n,i) = [x^{i-1}]\frac{x^3}{(1-x)^2(1-2x)} \times [x^{n-i+1}]\frac{x^2}{1-x},$$

which implies that $b(n,i) = 2^{i-2} + 1 - i$, as required. □

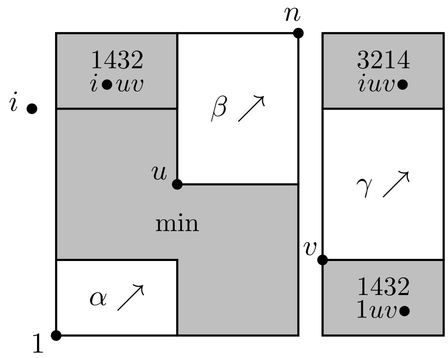


Figure 4: A T-avoider counted by $b(n,i)$

Define

$$A_n^+(i)=\sum_{j=i+1}^{n} a(n;i,j), \qquad A_n^+=\sum_{i=1}^{n-1} A_n^+(i), \qquad A^+(x)=\sum_{n\geq 2} A_n^+ x^n.$$

Similarly, define

$$A_n^-(i)=\sum_{j=1}^{i-1} a(n;i,j), \qquad A_n^-=\sum_{i=2}^{n} A_n^-(i), \qquad A^-(x)=\sum_{n\geq 2} A_n^- x^n.$$

Thus, with $A(x)=\sum_{n\geq 0} a(n)x^n$, we have

$$A(x)=1+x+A^+(x)+A^-(x).$$

From Lemma 3.12, we have

$$a(n;i,j)=\frac{j-1}{n-1}\binom{2n-2-j}{n-2},$$

for all $i\leq j-2$ (independent of i), and consequently,

$$\sum_{j=i+2}^{n} a(n;i,j)=\frac{i+2}{n}\binom{2n-i-3}{n-1}.$$

Hence, for $i\leq n-1$,

$$A_n^+(i)=a(n-1;i)+\frac{i+2}{n}\binom{2n-i-3}{n-1}.$$

Summing over i,

$$A_n^+=a(n-1)+\frac{n-2}{2n-1}C_n$$

for $n \geq 2$. Multiplying by x^n and summing over $n \geq 2$, we find

$$A^+(x) = xA(x) + (1-2x)C(x) - 1\,. \tag{3}$$

Finding $A^-(x)$ is a little more tedious. By Lemma 3.12, we have

$$\begin{aligned}
A_n^-(n-1) &= 2^{n-3} + \sum_{j=2}^{n-3} a(n; n-1, j) + a(n-3) \\
&= 2^{n-3} + a(n-3) + \sum_{j=2}^{n-3} a(n-1; n-2; j) + (2^0 + 2^1 + \cdots + 2^{n-5}) \\
&= 2^{n-3} + a(n-3) + A_{n-1}^-(n-2) - 2^{n-4} + 2^{n-4} - 1 \\
&= A_{n-1}^-(n-2) + a(n-3) + 2^{n-3} - 1,
\end{aligned}$$

and induction on n implies

$$A_n^-(n-1) = \sum_{j=0}^{n-3} a(j) + 2^{n-2} - n + 1\,. \tag{4}$$

Now for $n \geq 6$, using Lemma 3.12,

$$\begin{aligned}
A_n^- &= A_n^-(2) + A_n^-(3) + \sum_{i=4}^{n-2} A_n^-(i) + A_n^-(n-1) + A_n^-(n) \\
&= C_{n-2} + (C_{n-2} + 1) + \sum_{i=4}^{n-2}\sum_{j=1}^{i-1} a(n; i, j) + A_n^-(n-1) + a(n-1) \\
&= 2C_{n-2} + 1 + a(n-1) + A_n^-(n-1) + \sum_{i=4}^{n-2}\sum_{j=1}^{i-1} a(n-1; i, j) + \sum_{i=4}^{n-2}(a_n(n; i, 1) - a(n-1; i, 1)) \\
&= 2C_{n-2} + 1 + a(n-1) + A_n^-(n-1) + A_{n-1}^- - A_{n-1}^-(2) - A_{n-1}^-(3) - A_{n-1}^-(n-1) \\
&\quad + \sum_{i=4}^{n-2}\big(a_n(n; i, 1) - a(n-1; i, 1)\big) \\
&= 2C_{n-2} + 1 + a(n-1) + A_n^-(n-1) + A_{n-1}^- - 2C_{n-3} - 1 - a(n-2) \\
&\quad + \sum_{i=4}^{n-2}\big(a_n(n; 1, i) - a(n-1; 1, i)\big) \\
&= 2C_{n-2} + a(n-1) + A_n^-(n-1) + A_{n-1}^- - 2C_{n-3} - a(n-2) \\
&\quad + C_{n-1} - 3C_{n-2} + 2C_{n-3} - n + 2 \\
&= A_{n-1}^- + a(n-1) - a(n-2) + A_n^-(n-1) + C_{n-1} - C_{n-2} - n + 2.
\end{aligned}$$

Thus, by (4), we have

$$A_n^- = A_{n-1}^- - 2a(n-2) + \sum_{j=0}^{n-1} a(j) + C_{n-1} - C_{n-2} + 2^{n-2} - 2n + 3\,, \tag{5}$$

where $A_1^- = 0$ and $A_2^- = 1$, and this formula is also seen to hold for $n = 3, 4, 5$.

Multiplying (5) by x^n and summing over $n \geq 3$, we get

$$\begin{aligned} A^-(x) - x^2 = xA^-(x) - 2x^2\big(A(x)-1\big) + \frac{x}{1-x}\big(A(x)-1-2x\big) + x\big(C(x)-1-x\big) \\ - x^2\big(C(x)-1\big) - \frac{x^3(1-3x)}{(1-x)^2(1-2x)}, \end{aligned}$$

which leads to

$$A^-(x) = \frac{-x}{1-x} + \frac{x}{(1-x)^2}A(x) - \frac{2x^2}{1-x}A(x) + x\big(C(x)-1\big) - \frac{x^3(1-3x)}{(1-x)^3(1-2x)}. \qquad (6)$$

From (3) and (6), we have

$$\begin{aligned} A(x) - 1 - x = \frac{-x}{1-x} + \frac{x}{(1-x)^2}A(x) - \frac{2x^2}{1-x}A(x) + x\big(C(x)-1\big) - \frac{x^3(1-3x)}{(1-x)^3(1-2x)} \\ + xA(x) + (1-2x)C(x) - 1. \end{aligned}$$

Solving this equation for $A(x) = F_T(x)$ yields the following result.

Theorem 3.13 *Let $T = \{1432, 2431, 3214\}$. Then*

$$F_T(x) = \frac{(1-x)^4(1-2x)C(x) - x(1-4x+6x^2-5x^3)}{(1-x)(1-2x)(1-4x+5x^2-3x^3)}.$$

3.6 Case 165: $\{1342, 2314, 3421\}$

Note that all three patterns contain 231.

Theorem 3.14 *Let $T = \{1342, 2314, 3421\}$. Then*

$$F_T(x) = \frac{(1-2x)(1-x)^4C(x) - x(1-4x+6x^2-5x^3+x^4)}{(1-x)^4(1-3x+x^2)}.$$

Proof. Let $G_m(x)$ be the generating function for T-avoiders with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xF_T(x)$.

Let us first write an equation for $G_2(x)$. Let $\pi = i\pi' n\pi'' \in S_n(T)$ with 2 left-right maxima. Say there are k letters $j_1, j_2, \ldots, j_k$ in π'' that are smaller than i. Since π avoids 3421, we see that $j_1 < j_2 < \cdots < j_k$. Since π avoids 2314 and 1342, one can write π as

$$i\alpha^{(1)}\alpha^{(2)}\cdots\alpha^{(k+1)}n\beta^{(1)}j_1\beta^{(2)}\cdots j_k\beta^{(k+1)}$$

such that $i > \alpha^{(1)} > j_1 > \alpha^{(2)} > j_2 > \cdots > \alpha^{(k)} > j_k > \alpha^{(k+1)}$ and $n > \beta^{(1)} > \beta^{(2)} > \cdots > \beta^{(k+1)} > i$. Furthermore, each of $\alpha^{(1)}, \alpha^{(k+1)}, \beta^{(1)}, \beta^{(k+1)}$ avoids 231 and all other α's and β's avoid 12. We consider three cases:

- If $\alpha^{(1)}$ has a rise, then $\alpha^{(j)} = \beta^{(j)} = \emptyset$ for all $j = 2, 3, \ldots, k+1$. So we have a contribution of $x^{k+2}C(x)\big(C(x) - 1/(1-x)\big)$.

- If $\alpha^{(1)}$ is decreasing and $\beta^{(1)}$ has a rise, then $\alpha^{(j)} = \beta^{(j)} = \emptyset$ for all $j = 2, 3, \ldots, k+1$. So we have a contribution of $x^{k+2}/(1-x)\,C(x)\big(C(x) - 1/(1-x)\big)$.
- If $\alpha^{(1)}$ and $\beta^{(1)}$ are decreasing, then $\alpha^{(j)}, \beta^{(j)}$ are decreasing for all $j = 2, 3, \ldots, k$. So we have a contribution of $x^{k+2}/(1-x)^{2k}\, C(x)^2$.

Hence,

$$G_2(x) = \sum_{k\geq 1} x^{k+2} C(x)\left(C(x) - \frac{1}{1-x}\right) + \sum_{k\geq 1} \frac{x^{k+2}}{1-x} C(x)\left(C(x) - \frac{1}{1-x}\right) + \sum_{k\geq 0} \frac{x^{k+2}}{(1-x)^{2k}} C(x)^2 \,,$$

which implies

$$G_2(x) = \frac{x^2((1-x)(1-3x+2x^2)C(x)^2 - x(1-3x+x^2))}{(1-x)^3(1-3x+x^2)}.$$

For $G_m(x)$ with $m \geq 3$, a T-avoider π decomposes as in Figure 3 in Case 133 since that case also avoids 1342 and 2314, and furthermore, $\alpha_1, \ldots, \alpha_{m-1}$ all avoid 231 (or i_m is the 4 of a 2314), α_m avoids 231 (or i_1 is the 1 of a 1342), and β_m is increasing (or $i_1 i_m$ is the 34 of a 3421). Hence,

$$G_m(x) = \frac{x^m}{1-x} C(x)^m.$$

Summing over $m \geq 3$ and using the expressions for $G_0(x), G_1(x)$ and $G_2(x)$, we obtain

$$F_T(x) = 1 + xF_T(x) + \frac{x^2\big((1-x)(1-3x+2x^2)C(x)^2 - x(1-3x+x^2)\big)}{(1-x)^3(1-3x+x^2)} + \frac{x^3 C(x)^3}{(1-x)\big(1-xC(x)\big)}\,.$$

Solve for $F_T(x)$ and use the identity $C(x) = 1 + xC(x)^2$ repeatedly to complete the proof. □

3.7 Case 175: $\{1423, 2341, 3142\}$

The first and last patterns contain 312 and $\{312, 2341\}$-avoiders have generating function $L(x)$ given by [14, A116703]

$$L(x) = \frac{(1-x)^3}{1-4x+5x^2-3x^3}\,.$$

Let $L_m(x)$ denote the generating function for $\{312, 2341\}$-avoiders with m left-right maxima so that $L(x) = \sum_{m\geq 0} L_m(x)$.

THEOREM 3.15 *Let $T = \{1423, 2341, 3142\}$. Then*

$$F_T(x) = \frac{1-6x+12x^2-11x^3+5x^4}{1-7x+17x^2-20x^3+12x^4-2x^5}\,.$$

Proof. Let $G_m(x)$ be the generating function for T-avoiders with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xF_T(x)$.

For $G_2(x)$, define $G_2(x;r)$ to be the generating function for T-avoiders of the form $\pi = (n-r)\pi' n\pi''$ so that $G_2(x) = \sum_{r\geq 1} G_2(x;r)$. Since π avoids 1423, we see that $n-1, n-2, \ldots, n-r+1$ occur in that order and so π has the form $\pi = (n-r)\alpha_1 n\alpha_2(n-1)\cdots\alpha_r(n-r+1)\alpha_{r+1}$. Since π avoids 3142, we see that $\alpha_1 > \alpha_2 > \cdots > \alpha_r$.

If α_{r+1} is not empty, then α_j is decreasing for $j = 1, 2, \ldots, r$ since π avoids 2341, and α_{r+1} avoids T. So we have a contribution of $\frac{x^{r+1}}{(1-x)^r}\big(F_T(x) - 1\big)$. If α_{r+1} is empty, then by removing the letter $n-r+1$, we have a contribution of $x\,G_2(x; r-1)$. Thus, for $r \geq 2$,

$$G_2(x;r) = x\,G_2(x;r-1) + \frac{x^{r+1}\big(F_T(x)-1\big)}{(1-x)^r}\,. \tag{7}$$

Considering whether π'' is empty or not, we find that $G_2(x;1) = x^2F_T(x) + \frac{x^2}{1-x}\big(F_T(x)-1\big)$.

Summing (7) over $r \geq 2$, we obtain

$$G_2(x) = \frac{x^2}{1-x}F_T(x) + \frac{x^2}{(1-x)^2}F_T(x) + \frac{x^3}{(1-x)^2(1-2x)}\big(F_T(x)-1\big)\,.$$

For $G_m(x)$ with $m \geq 3$, suppose $\pi = i_1\pi^{(1)}i_2\pi^{(2)}\cdots i_m\pi^{(m)} \in S_n(T)$ has $m \geq 3$ left-right maxima. Since π avoids 2341, certainly $\pi^{(j)} > i_1$ for $j \geq 3$.

If $\pi^{(2)} > i_1$, then $i_2\pi^{(2)}\cdots i_m\pi^{(m)}$ avoids 312 (or i_1 is the 1 of a 1423) and the contribution is $xF_T(x)L_{m-1}(x)$.

If $\pi^{(2)} \not> i_1$, then $i_1 > 1$ and $\pi^{(j)} > i_2$ for $j \geq 3$ and $1 \in \pi^{(2)}$ (or $i_1 1 i_2$ is the 314 of a 3142). Thus, $i_1\pi^{(1)}i_2\pi^{(2)}$ and $i_3\pi^{(3)}\cdots i_m\pi^{(m)}$ respectively contribute factors of $G_2(x) - xF_T(x)L_1(x)$ and $L_{m-2}(x)$.

Hence, for all $m \geq 3$,

$$G_m(x) = xF_T(x)L_{m-1}(x) + \big(G_2(x) - xF_T(x)L_1(x)\big)L_{m-2}(x).$$

Summing over $m \geq 3$ gives

$$F_T(x) - G_2(x) - G_1(x) - 1 = xF_T(x)\big(L(x) - L_1(x) - 1\big) + \big(G_2(x) - xF_T(x)L_1(x)\big)\big(L(x)-1\big)\,.$$

Clearly, $L_1(x) = \frac{x}{1-x}$. Substitute for G_1, G_2, L, L_1 and solve for $F_T(x)$. □

3.8 Case 176: $\{1342, 2431, 3412\}$

Note that all three patterns contain 231, and the first two contain 132.

THEOREM 3.16 *Let* $T = \{1342, 2431, 3412\}$. *Then*

$$F_T(x) = \frac{(1-x)^2(1-4x+6x^2-5x^3+x^4)\,C(x) - 1 + 6x - 14x^2 + 15x^3 - 8x^4 + x^5}{x(1-3x+x^2)(1-x+x^3)}\,.$$

Proof. Let $G_m(x)$ be the generating function for T-avoiders with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xF_T(x)$.

For $G_2(x)$, suppose $\pi = i\pi' n\pi'' \in S_n(T)$ has 2 left-right maxima. If $\pi'' > i$, then π'' avoids 231 (or i is the 1 of a 1342) while π' avoids T, and the contribution is $x^2F_T(x)C(x)$. Otherwise, π'' has $d \geq 1$ letters smaller than i and these letters are decreasing left to right (or in is the 34 of a 3412) and form an interval of integers (or n is the 4 of a 2431). So π decomposes as in Figure 5, where γ is to the left of δ (or in is the 24 of a 2431) and is decreasing and nonempty, while α is to the left of β (or nj_1 is the 42 of a 1342). Also, δ avoids 231 (or i is the 1 of a 1342), while α avoids both 132 (or j_1 is the 4 of a 2431) and 3412 and β avoids T.

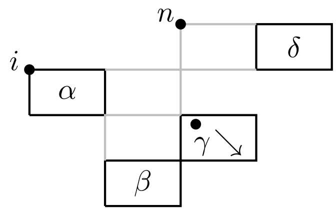


Figure 5: A T-avoider $i\pi' n\pi''$ with 2 left-right maxima and $\pi'' \not> i$

If α is decreasing, the contribution is $\frac{x^3}{(1-x)^2}C(x)F_T(x)$. If α is not decreasing, then β is decreasing (to avoid 3412) and the contribution is $\frac{x^3}{1-x}C(x)\big(K(x)-\frac{1}{1-x}\big)$, where $K(x)=\frac{1-2x}{1-3x+x^2}$ is the generating function for $\{132, 3412\}$-avoiders [14, Seq. A001519]. Hence,

$$G_2(x) = x^2C(x)F_T(x) + \frac{x^3}{(1-x)^2}C(x)F_T(x) + \frac{x^3}{1-x}C(x)\left(K(x) - \frac{1}{1-x}\right).$$

For $G_m(x)$ with $m \geq 3$, suppose $\pi = i_1\pi^{(1)}i_2\pi^{(2)}\cdots i_m\pi^{(m)} \in S_n(T)$ has $m \geq 3$ left-right maxima. If $\pi^{(m)} > i_{m-1}$, then π avoids T if and only if $i_1\pi^{(1)}\cdots i_{m-1}\pi^{(m-1)}$ avoids T and $\pi^{(m)}$ avoids 231, which gives a contribution of $xG_{m-1}(x)C(x)$.

If $\pi^{(m)} \not> i_{m-1}$, then π decomposes as in Figure 6, where the shaded regions are empty for the reason indicated, α_m is left of δ (or $i_{m-1}i_m$ is the 24 of a 2431), $\alpha_1 > \alpha_2$ (a violator uv and a in α_m makes ui_2va a 2431), and $\alpha_2\cdots\alpha_m$ is decreasing (or i_1i_2 is the 34 of a 3412). Also, α_1 avoids both 132 (since $\alpha_m \neq \emptyset$) and 3412, and δ avoids 231. Thus, we have a contribution of $\frac{x^{m+1}}{(1-x)^{m-1}}K(x)C(x)$.

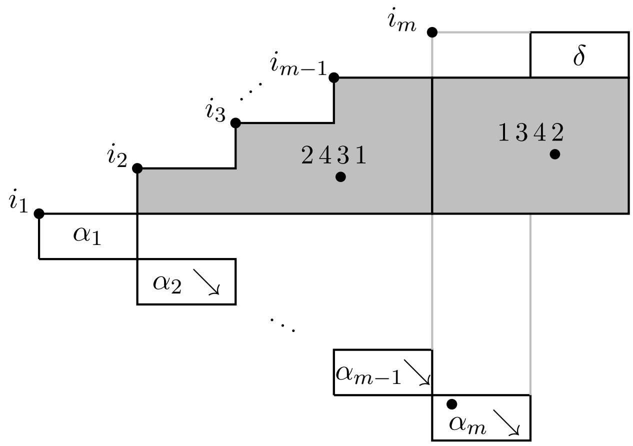


Figure 6: A T-avoider with $m \geq 3$ left-right maxima and $\pi^{(m)} \not> i_{m-1}$

Hence, for $m \geq 3$,

$$G_m(x) = xG_{m-1}(x)C(x) + \frac{x^{m+1}}{(1-x)^{m-1}}K(x)C(x).$$

Summing this recurrence over $m \geq 3$, we obtain

$$F_T(x) = 1 + xF_T(x) + G_2(x) + xC(x)\big(F_T(x) - 1 - xF_T(x)\big) + \frac{x^4}{(1-x)^2}\left(K(x) - \frac{1}{1-x}\right)C(x)\,,$$

and, substituting for $G_2(x)$, the result follows by solving for $F_T(x)$. □

3.9 Case 178: $\{1342, 2314, 2431\}$

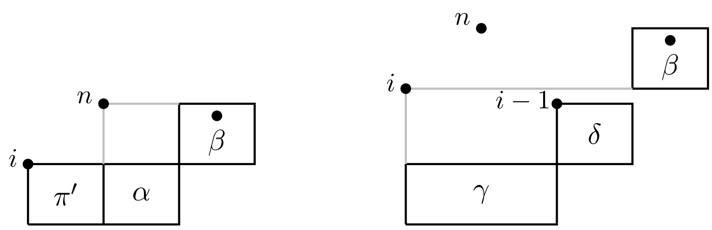
Note that all three patterns contain 231.

THEOREM 3.17 *Let* $T = \{1342, 2314, 2431\}$. *Then*

$$F_T(x) = \frac{(1-x)^2(1-4x+6x^2-5x^3+x^4)C(x) - 1 + 6x - 14x^2 + 15x^3 - 8x^4 + x^5}{x(1-3x+x^2)(1-x+x^3)}\,.$$

Proof. Let $G_m(x)$ be the generating function for T-avoiders with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xF_T(x)$.

For $G_2(x)$, suppose $\pi = i\pi' n\pi'' \in S_n(T)$. If $i = n-1$, the contribution is $x(F_T(x) - 1)$. Otherwise, we denote the contribution by H. So $G_2(x) = x\big(F_T(x) - 1\big) + H$. Now let us write a formula for H. Here, $i < n-1$ and π decomposes as in Figure 7a), where α is left of β (or in is the 24 of a 2431), $\beta \neq \emptyset$ and avoids 231 (or i is the 1 of a 1342), and $\pi'\alpha$ also avoids 231 (or n is the 4 of a 2314).



a) general form b) $i > 1$ and $i-1$ after n

Figure 7: A T-avoider $i\pi' n\pi''$ with 2 left-right maxima and $i < n-1$

We consider four cases:

- If $i = 1$, then $\pi'\alpha = \emptyset$ and the contribution is $x^2\big(C(x) - 1\big)$.
- If $i > 1$ and $i-1$ occurs in the second position, the contribution is xH.

- If $i>1$ and $i-1$ occurs before n but not in the second position, let a denote the smallest letter that occurs before $i-1$. We have $a=1$ (or $a(i-1)1b$ is a 2314 for b in β) and $\alpha=\emptyset$ (or $a(i-1)n$ is the 134 of a 1342). So π' is a 231-avoider of length ≥ 2 in which 1 precedes its maximal letter, and $\beta>i>\pi'$ is a nonempty 231-avoider, giving a contribution of $x^2\Big(\big(C(x)-1-x\big)-x\big(C(x)-1\big)\Big)\big(C(x)-1\big)=x^5C(x)^5$ in compact form.

- If $i>1$ and $i-1$ occurs after n, then π has the form in Figure 7b) where $\gamma<\delta$ because $\gamma(i-1)\delta=\pi'\alpha$ avoids 231. Since $(i-1)\delta$ is counted by $xC(x)$, the contribution is $xC(x)H$.

Thus
$$H=x^2(C(x)-1)+xH+x^5C(x)^5+xC(x)H,$$
which has solution
$$H=\frac{x^2(C(x)-1)+x^5C(x)^5}{1-x-xC(x)}=x(C(x)-1)^2+x^4C(x)^5(C(x)-1)\,.$$

For $G_m(x)$ with $m\geq 3$, a T-avoider $\pi=i_1\pi^{(1)}i_2\pi^{(2)}\cdots i_m\pi^{(m)}\in S_n$ decomposes as illustrated in Figure 8 for $m=5$, where α is left of β (or $i_{m-1}i_m$ is the 24 of a 2431).

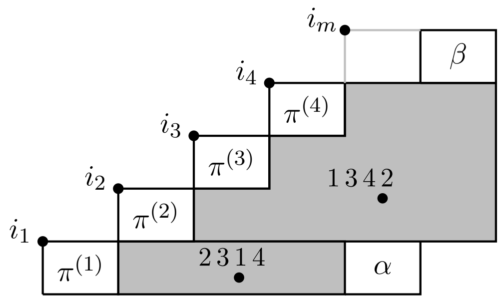


Figure 8: A T-avoider with $m\geq 3$ left-right maxima

If $\pi^{(2)}=\pi^{(3)}=\cdots=\pi^{(m)}=\emptyset$, then $\pi^{(1)}$ avoids 231, and the contribution is $x^mC(x)$. Otherwise, there is a maximal $p\in[2,m]$ such that $\pi^{(p)}\neq\emptyset$. There are two cases:

- $1\leq p\leq m-1$. Here, π avoids T if and only if $\pi^{(j)}$ avoids 231 for all $j=1,2,\ldots,p$, giving a contribution of $x^mC(x)^{p-1}\big(C(x)-1\big)$.

- $p=m$. Here, $\pi^{(m)}=\alpha\beta\neq\emptyset$. Hence, since $a\in\alpha$ and $b\in\beta$ makes i_1i_2ab a 2314, exactly one of α and β is nonempty.

  If $\alpha\neq\emptyset$, then $\pi^{(2)}\pi^{(3)}\cdots\pi^{(m-1)}=\emptyset$ (to avoid 2431), $\pi^{(1)}>\alpha$ (to avoid 1342), $\pi^{(1)}$ avoids $\{132,231\}$, and α is nonempty and avoids T, giving a contribution $x^mL\big(F_T(x)-1\big)$, where $L=\frac{1-x}{1-2x}$ is the generating function for $\{132,231\}$-avoiders [13].

  If $\beta\neq\emptyset$, then $\pi^{(j)}$ avoids 231 for $j=1,\ldots,m$, and we have a contribution of $x^mC(x)^{m-1}\big(C(x)-1\big)$.

Adding all the contributions, we have

$$G_m(x) = x^m C(x) + x^m\, L\,\big(F_T(x) - 1\big) + \sum_{p=2}^{m} x^m C(x)^{p-1}\big(C(x) - 1\big).$$

Summing over $m \geq 3$ and using the expressions for G_0, G_1, G_2, L, we obtain

$$F_T(x) = 1 - x + 2xF_T(x) + x\big(C(x) - 1\big)^2 + x^4 C(x)^5\big(C(x) - 1\big) + \frac{x^3(F_T(x) - 1)}{1 - 2x} + x^3 C^4(x)\,.$$

Solving for $F_T(x)$ gives an expression which can be written as in the statement of the theorem. $\square$

3.10 Case 194: $\{3124, 4123, 1243\}$

We define $a(n) = |S_n(T)|$ and define $a(n; i_1, i_2, \ldots, i_m)$ to be the number of permutations $\pi = \pi_1\pi_2\cdots\pi_n$ in $S_n(T)$ such that $\pi_1\pi_2\cdots\pi_m = i_1 i_2\cdots i_m$. Note that $a(n;1) = |S_{n-1}(\{3124, 4123, 132\})|$ and $H(x) := \sum_{n\geq 0} |S_n(\{132, 3124, 4123\})| x^n$ is given by $H(x) = 1 + \frac{x(1-x)^2}{(1-2x)^2}$.

Set $b(n;i) = a(n;i,i{+}1)$ and $b'(n;i) = a(n;i,n)$. As in the other cases, one can obtain the following relations.

Lemma 3.18 *For $n \geq 4$,*

$$\begin{aligned}
a(n;i,n) &= a(n;i,n-1), \qquad 1 \leq i \leq n-2,\\
a(n;i,j) &= a(n-1;i,j) + b(n;i-1), \qquad 2 \leq i < j \leq n-1,\\
a(n;i,j) &= \sum_{k=1}^{j} a(n-1;i-1,k), \qquad 1 \leq j < i-1 \leq n-2,\\
a(n;i,i-1) &= a(n-1;i-1), \qquad 2 \leq i \leq n,\\
b(n;i) &= \sum_{k=1}^{i} b(n-1;k), \qquad 1 \leq i \leq n-1,\\
a(n;n-1) &= a(n-1),\\
a(n;n) &= a(n-1),\\
b(n;n) &= 0.
\end{aligned}$$

Define $A^-(x;w,v) = \sum_{n\geq 2}\sum_{i=1}^{n}\sum_{j=1}^{i-1} a(n;i,j) w^i v^{j-1}$.

Proposition 3.19

$$A^-(x;1,1) = xC(x)\big(F_T(x) - 1\big).$$

Proof. By Lemma 3.18, we have $a(n;i,j) = \sum_{k=1}^{j} a(n-1;i-1,k)$ for $1 \leq j < i-1 \leq n-1$. Define $A^-_{n,i}(v) = \sum_{j=1}^{i-1} a(n;i,j) v^{j-1}$. Multiplying the last recurrence by v^{j-1} and summing over $j = 1, 2, \ldots, i-2$, we obtain

$$A^-_{n,i}(v) - a(n;i,i-1)v^{i-2} = \frac{1}{1-v}\big(A^-_{n-1,i-1}(v) - v^{i-2}A^-_{n-1,i-1}(1)\big),$$

which implies

$$A^-_{n,i}(v) = \frac{1}{1-v}(A^-_{n-1,i-1}(v) - v^{i-2}A^-_{n-1,i-1}(1)) + a(n;i-1)v^{i-2},$$

by Lemma 3.18, with $A^-_{n,1}(v) = 0$.

Define $A^-_n(w,v) = \sum_{i=1}^n A^-_{n;i}(v)w^i$ and $A_n(v) = \sum_{i=1}^n a(n;i)v^{i-1}$. Multiplying the last recurrence by w^i and summing over $i = 2, 3, \ldots, n$, we obtain

$$A^-_n(w,v) = \frac{w}{v(1-v)}(vA^-_{n-1}(w,v) - A^-_{n-1}(wv,1)) + w^2 A_{n-1}(wv),$$

with $A^-_1(w,v) = 0$. Hence,

$$A^-(x;w,v) = \frac{wx}{v(1-v)}(vA^-(x;w,v) - A^-(x;wv,1)) + xw^2(A(x;wv) - 1).$$

By taking $w = (1-v)/x$, we obtain

$$A^-(x;(1-v)v/x,1) = v(1-v)^2/x(A(x;(1-v)v/x) - 1),$$

which, by taking $v = 1/C(x)$, leads to $A^-(x;1,1) = xC(x)\big(A(x;1) - 1\big)$, as required. □

Define $B'_n(v) = \sum_{i=1}^{n-1} b'(n;i)v^{i-1}$ and $B_n(v) = \sum_{i=1}^{n-1} b(n;i)v^{i-1}$, and their generating functions by $B'(x;v) = \sum_{n\geq 3} B'_n(v)x^n$ and $B(x;v) = \sum_{n\geq 2} B_n(v)x^n$.

Lemma 3.20 *We have*

$$B(x;v) = \frac{x^2\big(1 - vC(xv)\big)}{1-x-v}$$

and

$$B'(x;v) = \frac{2x^3 + vB(x;v) - x^2(v+x)C(xv)}{1-2x}.$$

Proof. By Lemma 3.18, we have $b(n;n) = 0$ and $b(n;i) = b(n-1;1) + \cdots + b(n-1;i)$. Multiplying by v^{i-1} and summing over $i = 1, 2, \ldots, n-1$, we obtain

$$B_n(v) = \frac{1}{1-v}(B_{n-1}(v) - v^{n-1}B_{n-2}(1)),$$

where $B_2(v) = 1$. Hence,

$$B(x/v;v) = x^2/v^2 + \frac{x}{v(1-v)}(B(x/v;v) - B(x;1)).$$

By taking $v = 1/C(x)$, we have $B(x;1) = x^2C^2(x) = x(C(x) - 1)$, and thus

$$B(x;v) = \frac{x^2(1 - vC(xv))}{1-x-v}.$$

By Lemma 3.18, we have $b'(n;n-1) = b'(n;n-2) = C_{n-2}$ and

$$b'(n;i) = a(n;i,n) = a(n;i,n-1) = a(n-1;i,n-1) + a(n-1;i,n-2) + b(n;i-1),$$

which gives

$$b'(n;i) = 2b'(n-1;i) + b(n;i-1),$$

where $b'(n;n-1) = b'(n;n-2) = C_{n-2}$. By multiplying the last recurrence by v^{i-1} and summing over $i = 1, 2, \ldots, n-3$, we obtain

$$\begin{aligned} B'_n(v) &= C_{n-2}(v^{n-2} + v^{n-3}) + 2(B'_{n-1}(v) - C_{n-3}v^{n-3}) \\ &\quad + v(B_n(v) - b(n;n-3)v^{n-4} - b(n;n-2)v^{n-3} - b(n;n-1)v^{n-2}). \end{aligned}$$

From the first part of the proof, we get $b(n;n-1) = b(n;n-2) = C_{n-2}$ and $b(n;n-3) = C_{n-2} - C_{n-3}$. Thus, $B'_3(v) = 1 + v$ and

$$B'_n(v) = 2B'_{n-1}(v) + vB_n(v) - (C_{n-3} + C_{n-2}v^2)v^{n-3}.$$

Hence,

$$B'(x;v) = (1+v)x^3 + 2xB'(x;v) + v(B(x;v) - (1+v)x^3 - x^2) - x^3(C(xv) - 1) - vx^2(C(xv) - 1 - xv),$$

which leads to

$$B'(x;v) = \frac{2x^3 + vB(x;v) - x^2(v+x)C(xv)}{1-2x},$$

as claimed. $\square$

Define $A^+(x;v) = \sum_{n\geq 2}\sum_{i=1}^{n}\sum_{j=i+1}^{n} a(n;i,j)v^{j-1-i}$.

Proposition 3.21 *We have*

$$A^+(x;1) = \frac{(x^4 - 2x^3 + 5x^2 - 4x + 1)C(x) - x^3 - 2x^2 + 3x - 1}{(1-2x)^2}.$$

Proof. By Lemma 3.18, we have $a(n;i,n) = b'(n;i)$ and

$$a(n;i,j) = a(n-1;i,j) + a(n-1;i,j-1) + b(n;i-1),$$

for all $2 \leq i < j \leq n-1$. Define $A^+_{n;i}(v) = \sum_{j=i+1}^{n} a(n;i,j)v^{j-1-i}$. Thus,

$$A^+_{n;i}(v) - b'(n;i)v^{n-1-i} = A^+_{n-1;i}(v) + v\left(A^+_{n-1;i}(v) - b'(n-1;i)v^{n-2-i}\right) + \frac{1-v^{n-1-i}}{1-v}b(n;i-1),$$

which leads to

$$A^+_{n;i}(v) = b'(n;i)v^{n-1-i} - b'(n-1;i)v^{n-1-i} + (1+v)A^+_{n-1;i}(v) + \frac{1-v^{n-1-i}}{1-v}b(n;i-1), \tag{8}$$

for all $i = 2, 3, \ldots, n-2$. Note that $A^+_{n;n-1}(v) = a(n;n-1,n) = C_{n-2}$. Moreover, $A^+_{n;n}(v) = 0$.

Define $A^+_n(v) = \sum_{i=1}^{n} A^+_{n;i}(v)$. By summing (8) over $i = 2, 3, \ldots, n-2$, using $b'(n;1) = 2^{n-3}$ and $b'(n;n-1) = b(n;n-2) = C_{n-2}$ (see Lemma 3.20), we have

$$\begin{aligned} A^+_n(v) &= A^+_{n;1}(v) + \left(B'_n(1/v) - 2^{n-3}\right)v^{n-2} - \left(B'_{n-1}(1/v) - 2^{n-4}\right)v^{n-2} \\ &\quad + (1+v)\left(A^+_{n-1}(v) - A^+_{n-1;1}(v)\right) + \frac{1}{1-v}\left(B_n(1) - v^{n-3}B_n(1/v)\right) + C_{n-2}/v. \end{aligned}$$

Let $\sum_{n\geq 2} A^{+}_{n;1}(v)x^n = G(x;v)$. Multiplying by x^n and summing $n \geq 4$, we obtain

$$\begin{aligned}\big(1-(1+v)x\big)A^{+}(x;v) = \big(1-(1+v)x\big)G(x;v) + \frac{1-vx}{v^2}B'(vx;1/v) - x^3v - \frac{v^2x^4}{1-2vx}\\ + \frac{1}{1-v}\Big(B(x;1) - \frac{1}{v^3}B(vx;1/v)\Big) + \frac{x^2}{v}C(x).\end{aligned}$$

By Lemma 3.20, we have

$$\lim_{v\to 1}\frac{1}{1-v}\Big(B(x;1) - \frac{1}{v^3}B(vx;1/v)\Big) + x^2C(x) = (1-3x+x^2)C(x) - 1 + 2x.$$

Since $G(x;1) = xH(x) - x - x^2$, we have

$$\begin{aligned}A^{+}(x;1) = xH(x) + \frac{1-x}{1-2x}B'(x;1) - \frac{x(1-x)^2}{1-2x} - \frac{x^4}{(1-2x)^2}\\ + \frac{1-3x+x^2}{1-2x}C(x) - 1.\end{aligned}$$

By Lemma 3.20 and the formula for $H(x)$, we complete the proof. □

THEOREM 3.22 *Let $T = \{3124, 4123, 1243\}$. Then*

$$F_T(x) = \frac{(1-5x+9x^2-8x^3+4x^4)C(x) - (1-5x+9x^2-6x^3+x^4)}{x(1-2x)^2}.$$

Proof. By Propositions 3.19 and 3.21, we have

$$\begin{aligned}F_T(x) - 1 - x = xC(x)\big(F_T(x)-1\big)\\ + \frac{(x^4-2x^3+5x^2-4x+1)C(x) - x^3 - 2x^2 + 3x - 1}{(1-2x)^2}.\end{aligned}$$

Solving for $F_T(x)$ completes the proof. □

3.11 Case 214: $\{1342, 2341, 3412\}$

Most of the work is in finding an equation for $G_2(x)$.

LEMMA 3.23

$$G_2(x) = x^2F_T(x)C(x) + \frac{x^3C(x)^2\big(F_T(x)-1\big)}{1-2x} + \frac{x^3C(x)^2}{1-2x} - \frac{x^3C(x)^2}{(1-x)^2} + \frac{x^3C(x)}{(1-x)\big(1-x-xC(x)\big)}.$$

Proof. Refine $G_2(x)$ to $G_2(x;d)$, the generating function for permutations $\pi = i\pi' n\pi'' \in S_n(T)$ with 2 left-right maxima and such that π'' has $d \geq 0$ letters smaller than i. For $d = 0$, $\pi'' > i$ and π'' avoids 231 (or i is the "1" of a 1342), while π' avoids T. Hence, $G_2(x;0) = x^2F_T(x)C(x)$.

Now assume $d \geq 1$. The letters in π'' smaller than i, say $j_1, j_2, \ldots, j_d$, are decreasing (or in is the 34 of a 3412) and π has the form illustrated in Figure 9, where $\alpha_1\alpha_2\cdots\alpha_d$ is decreasing (or nj_d is the

41 of a 2341) and α_m lies to the left of α_{m+1} for $1 \le m \le d$ (or nj_m is the 42 of a 1342) and $\beta_1\beta_2\cdots\beta_d$ is decreasing (or ij_d is the 21 of a 2341).

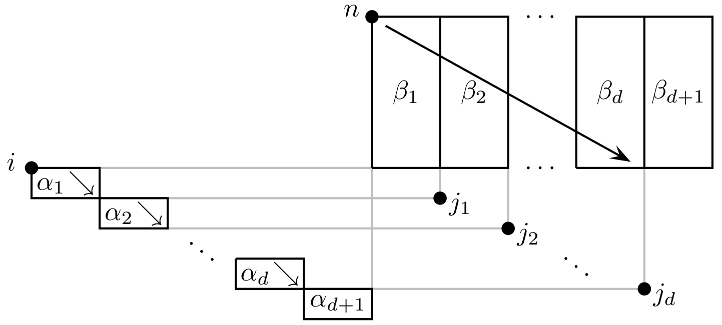


Figure 9: A general T-avoider with 2 left-right maxima

If $\alpha_2\cdots\alpha_d\alpha_{d+1} \ne \emptyset$, then $\beta_2\cdots\beta_d = \emptyset$ (since $u \in \alpha_2\cdots\alpha_d$ and $v \in \beta_2\cdots\beta_d$ makes uj_1vj_d a 2341, while $u \in \alpha_{d+1}$ and $v \in \beta_2\cdots\beta_d$ makes uj_1vj_d a 1342). Thus, with $\beta_1 = k_1k_2\cdots k_p$ where $p \ge 0$, π has the form illustrated in Figure 10, where γ_{m+1} lies to the left of γ_m for $1 \le m \le p$ (or ik_m is the 13 of a 1342) and all the γ's avoid 231 (or i is the "1" of a 1342).

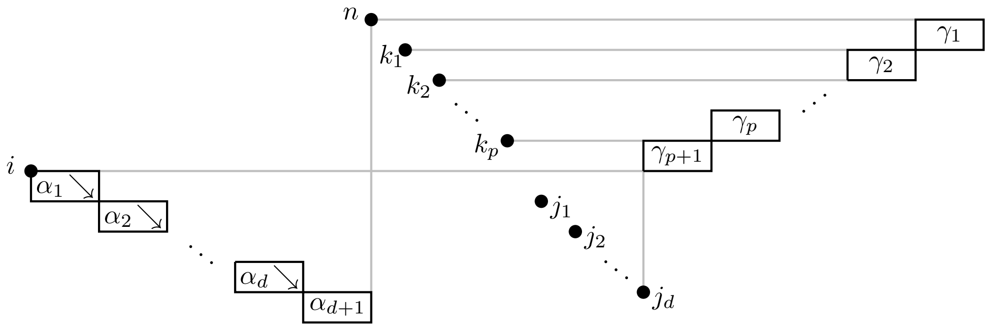


Figure 10: A T-avoider with 2 left-right maxima and $\alpha_2\cdots\alpha_d\alpha_{d+1} \ne \emptyset$

Now we consider three cases (Figure 10 applies to the first two of them):

- $\alpha_{d+1} \ne \emptyset$. In this case, we get a contribution of x^{2+d+p} from i, n and the j's and k's, of $F_T(x)-1$ from α_{d+1}, of $1/(1-x)^d$ from the other α's, and of $C(x)^{p+1}$ from the γ's. Summing over $p \ge 0$ gives a total contribution of
$$\frac{x^{2+d}\big(F_T(x)-1\big)C(x)}{(1-x)^d\big(1-xC(x)\big)}.$$
- $\alpha_m \ne \emptyset$ and $\alpha_{m+1} = \cdots = \alpha_{d+1} = \emptyset$ for some $m \in \{2,3,\dots,d\}$. This case is similar to the previous except that we have m decreasing α's to consider and the last of these is nonempty.

Thus, we get a contribution of
$$\frac{x^{3+d}C(x)}{(1-x)^m(1-xC(x))}\,.$$

- $\alpha_2=\cdots=\alpha_{d+1}=\emptyset$. In this case, π has the form $i(i-1)\cdots(d+1)n\beta_1 d\cdots\beta_d 1\beta_{d+1}$, where $\beta_1\cdots\beta_d=k_1k_2\cdots k_p$ is decreasing and $\beta_{d+1}=\gamma_{p+1}\gamma_p\cdots\gamma_1$ with γ's separated by the k's and γ_s avoiding 231 for all s. This leads, by a similar analysis, to a contribution of
$$\frac{x^{2+d}C(x)}{(1-x)\big(1-xC(x)\big)^d}\,.$$

Hence, for $d\geq 1$,
$$\begin{aligned}G_2(x;d)&=\frac{x^{2+d}(F_T(x)-1)C(x)}{(1-x)^d\big(1-xC(x)\big)}+\left(\sum_{m=2}^{d}\frac{x^{3+d}C(x)}{(1-x)^m\big(1-xC(x)\big)}\right)+\frac{x^{2+d}C(x)}{(1-x)\big(1-xC(x)\big)^d}\\&=\frac{x^{2+d}(F_T(x)-1)C(x)^2}{(1-x)^d}+\left(\sum_{m=2}^{d}\frac{x^{3+d}C(x)^2}{(1-x)^m)}\right)+\frac{x^{2+d}C(x)}{(1-x)\big(1-xC(x)\big)^d}.\end{aligned}$$

Since $G_2(x)=\sum_{d\geq 0}G_2(x;d)$ and $G_2(x;0)=x^2F_T(x)C(x)$, the result follows. □

Lemma 3.24 *For $m\geq 3$,*
$$G_m(x)=x^{m-2}C(x)^{m-2}G_2(x)\,.$$

Proof. Suppose $m\geq 3$ and $\pi=i_1\pi^{(1)}i_2\pi^{(2)}\cdots i_m\pi^{(m)}\in S_n(T)$ has m left-right maxima. Since π avoids 2341 and 1342, we have $\pi^{(s)}>i_{s-1}$ for all $s=3,4,\ldots,m$. Thus, π avoids T if and only if (i) $i_1\pi^{(1)}i_2\pi^{(2)}$ avoids T and has exactly 2 left-right maxima, and (ii) $\pi^{(s)}$ avoids 231 for all $s=3,4,\ldots,m$. The result follows. □

Theorem 3.25 *Let $T=\{1342,2341,3412\}$. Then*
$$F_T(x)=\frac{(1-2x)\big((1-5x+9x^2-6x^3)\sqrt{1-4x}-(1-9x+29x^2-38x^3+18x^4)\big)}{2(1-x)^2x(1-7x+14x^2-9x^3)}\,.$$

Proof. Using Lemma 3.24 and summing over $m\geq 3$ leads to
$$F_T(x)-1-xF_T(x)=G_2(x)+\frac{xC(x)G_2(x)}{1-xC(x)}=\frac{G_2(x)}{1-xC(x)}=G_2(x)C(x)\,.$$

Lemma 3.23 now implies
$$\begin{aligned}&F_T(x)-1-xF_T(x)\\&=x^2F_T(x)C(x)^2+\frac{x^3C(x)^3(F_T(x)-1)}{1-2x}+\frac{x^3C(x)^3}{1-2x}-\frac{x^3C(x)^3}{(1-x)^2}+\frac{x^3C(x)^4}{1-x}\,.\end{aligned}$$

Solving for $F_T(x)$ and simplifying completes the proof. □

3.12 Case 220: $\{2431, 2314, 3142\}$

THEOREM 3.26 *Let* $T = \{2431, 2314, 3142\}$. *Then*

$$F_T(x) = 1 + \frac{x(1-x)^2(1-2x)}{(1-3x)(1-x)^3 - x(1-2x)(1-x+x^2)\big(C(x)-1\big)}.$$

Proof. To write an equation for $G_m(x)$ where $m \geq 2$, suppose $\pi = i_1\pi^{(1)}i_2\pi^{(2)}\cdots i_m\pi^{(m)} \in S_n(T)$ with m left-right maxima. Then π has the form illustrated in Figure 11a) below where the shaded region is empty (a letter u in the shaded region implies $i_1 i_2\, u\, i_m$ is a 2314).

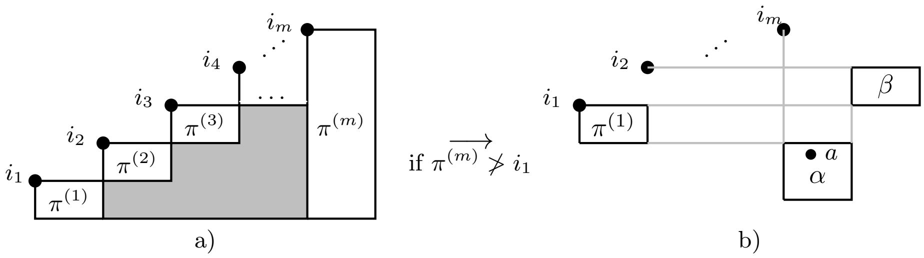


Figure 11: A T-avoider with $m \geq 2$ left-right maxima

Note that $\pi^{(1)}$ avoids 231 (or i_m is the "4" of a 2314). If $\pi^{(m)} > i_1$, then we get a contribution of $xC(x)G_{m-1}(x)$ where $C(x)$ accounts for $\pi^{(1)}$.

Now suppose $\pi^{(m)}$ has a letter a smaller than i_1. Then $\pi^{(2)}\pi^{(3)}\cdots\pi^{(m-1)} = \emptyset$ (if $u \in \pi^{(s)}$, $2 \leq s \leq m-1$, then $i_{s-1}\, i_s\, u\, a$ is a 2431) and $\pi^{(1)} > a$ (if $u \in \pi^{(1)}$ with $u < a$, then $i_1\, u\, i_2\, a$ is 3142). Also, $\pi^{(m)} < i_2$. To see this, suppose $u \in \pi^{(m)}$ with $u > i_2$. If u occurs before a in π, then $i_2\, i_m\, u\, a$ is a 2431, while if u occurs after a, then $i_1\, i_2\, a\, u$ is a 2314. So π has the form illustrated in Figure 11b), where all entries in α lie to the left of all entries in β (or uv with $u \in \beta$ and $v \in \alpha$ implies $i_1\, i_2\, u\, v$ is a 2431).

We now consider two cases:

- $\beta = \emptyset$. Here, π avoids T if and only if $\pi^{(1)}$ avoids 132 (to avoid 2431) and 231, and α avoids T, giving a contribution of $x^m\big(F_T(x)-1\big)L(x)$, where $L(x) = \frac{1-x}{1-2x}$ is the generating function for $\{132, 231\}$-avoiders.
- $\beta \neq \emptyset$. Here, $\pi^{(1)}$ is decreasing (uv in $\pi^{(1)}$ with $u < v \Rightarrow uvab$ is a 2314 for $b \in \beta$), α avoids 231 (to avoid 2314), and β avoids T, giving a contribution of $\frac{x^m}{1-x}\big(C(x)-1\big)\big(F_T(x)-1\big)$.

Thus, for all $m \geq 2$,

$$G_m(x) = xC(x)G_{m-1}(x) + x^m(F_T(x)-1)L(x) + \frac{x^m}{1-x}\big(C(x)-1\big)\big(F_T(x)-1\big).$$

Summing over $m \geq 2$, we obtain

$$F_T(x) - 1 - xF_T(x) = xC(x)\big(F_T(x)-1\big) + \frac{x^2}{1-x}\left(L(x) + \frac{1}{1-x}\big(C(x)-1\big)\right)\big(F_T(x)-1\big),$$

which implies $F_T(x)$ is as stated. $\square$

3.13 Case 222: $\{3412, 3421, 1342\}$

Here, we use the notion of a generating tree (West [15]), and consider the generating forest whose vertices are identified with $S := \bigcup_{n\geq 2} S_n(T)$, where 12 and 21 are the roots and each non-root $\pi \in S$ is a child of the permutation obtained from π by deleting its largest element. We will show that it is possible to label the vertices so that if v_1 and v_2 are any two vertices with the same label and ℓ is any label, then v_1 and v_2 have the same number of children with label ℓ. Indeed, we will specify (i) the labels of the roots, and (ii) a set of succession rules explaining how to derive from the label of a parent the labels of all of its children. This will determine a labelled generating forest.

A permutation $\pi = \pi_1\pi_2\cdots\pi_n \in S_n$ determines $n+1$ positions, called *sites*, between its entries. The sites are denoted $1, 2, \ldots, n+1$ from left to right. In particular, site i is the space between π_{i-1} and π_i for $2 \leq i \leq n$. Site i of π is said to be *active* (with respect to T) if, by inserting $n+1$ into site i of π, we get a permutation in $S_{n+1}(T)$, otherwise it is *inactive*. For $\pi \in S_n(T)$, sites 1 and $n+1$ are always active, and if $\pi_n = n$, then site n is active.

For $\pi \in S_n(T)$, define $A(\pi)$ to be the set of all active sites for π and $L(\pi)$ to be the set of active sites lying to the left of n. For example, $L(13254) = \{1, 2, 4\}$ since there are 4 possible sites in which to insert 6 to the left of $n = 5$ and, of these insertions, only 136254 is not in $S_6(T)$.

LEMMA 3.27 *For $\pi \in S_n(T)$, we have $A(\pi) = L(\pi) \cup \{n+1\}$ unless $\pi_n < n$ and site n is active, in which case $A(\pi) = L(\pi) \cup \{n, n+1\}$.*

Proof. No site to the right of n is active except (possibly) site n and (definitely) site $n+1$, for if $n+1$ is inserted after n in a site $\leq n-1$, then $n\,(n+1)\,\pi_{n-1}\,\pi_n$ is a 3412 or a 3421, both forbidden. □

If site n is inactive, then 1 and $n+1$ are the only active sites iff $\pi_1 = n$. In particular, there are at least 3 active sites unless site n is inactive and $\pi_1 = n$.

For $n \geq 2$, say $\pi \in S_n$ is *special* if it has the form $\pi = n(n-1)\cdots(j+1)\pi' j$ for some j with $2 \leq j \leq n$, where $j = n$ means $\pi = \pi' n$.

We now assign labels. Suppose $n \geq 2$ and $\pi \in S_n(T)$ has k active sites. Then π is labelled k, $\overline{k}$, $\overline{\overline{k}}$ according to whether site n is active and whether π is special as follows. If site n is inactive, then label π by $\overline{\overline{k}}$. Otherwise, if π is special, then label π by $\overline{k}$, and if π is not special, then label it by k.

For instance, all 3 sites are active for both 12 and 21 and only the former is special, so their labels are $\bar{3}$ and 3, respectively; 12 has three children 312, 132 and 123 with active sites $\{1, 3, 4\}$, $\{1, 2, 4\}$ and $\{1, 2, 3, 4\}$, respectively, hence labels $\bar{3}$, $\bar{\bar{3}}$ and $\bar{4}$ because only the first and third are special; 21 has three children 321, 231 and 213 with active sites $\{1, 3, 4\}$, $\{1, 2, 3, 4\}$ and $\{1, 2, 3, 4\}$, respectively, hence labels 3, 4 and $\bar{4}$ because only the last is special.

To establish the succession rules, we have the following proposition. The proof is left to the reader.

PROPOSITION 3.28 *Fix $n \geq 2$. Suppose $\pi \in S_n(T)$ has k active sites and site n is active so that*

$$A(\pi) = \{1 = L_1 < L_2 < \cdots < L_{k-1} = n\} \cup \{n+1\}\,.$$

If π is special, then $A(\pi^{L_1}) = \{L_1, n+1, n+2\}$ and $A(\pi^{L_i}) = \{L_1, \ldots, L_i, n+2\}$ for $2 \leq i \leq k-1$.
If π is not special, then $A(\pi^{L_i}) = \{L_1, \ldots, L_i, n+1, n+2\}$ for $1 \leq i \leq k-1$.
In both cases, $A(\pi^{n+1}) = \{L_1, \ldots, L_{k-1}, n+1, n+2\}$.
Next, suppose $\pi \in S_n(T)$ has k active sites and site n is inactive so that

$$A(\pi) = \{1 = L_1 < L_2 < \cdots < L_{k-1} \leq n-1\} \cup \{n+1\}\,.$$

Then $A(\pi^{L_i}) = \{L_1, \ldots, L_i, n+2\}$ *for* $1 \le i \le k-1$ *and* $A(\pi^{n+1}) = \{L_1, \ldots, L_{k-1}, n+1, n+2\}$.

As an immediate consequence, we obtain the following result.

COROLLARY 3.29 *The labelled generating forest* $\mathcal{F}$ *is given by*

$$\begin{array}{lll} \textbf{Roots:} & 3, \bar{3} & \\ \textbf{Rules:} & k \rightsquigarrow 3, 4, \ldots, k, k+1, \overline{k+1} & \text{for } k \ge 3, \\ & \bar{k} \rightsquigarrow \bar{3}, \bar{\bar{3}}, \bar{\bar{4}}, \ldots, \bar{\bar{k}}, \overline{k+1} & \text{for } k \ge 3, \\ & \bar{\bar{k}} \rightsquigarrow \bar{\bar{2}}, \bar{\bar{3}}, \ldots, \bar{\bar{k}}, \overline{k+1} & \text{for } k \ge 2. \end{array}$$

□

THEOREM 3.30 *Let* $T = \{3412, 3421, 1342\}$. *Then*

$$F_T(x) = \frac{2 - 11x + 13x^2 - 6x^3 + (1-x)x(1-6x+4x^2)(1-4x)^{-1/2}}{2(1-6x+8x^2-4x^3)}.$$

Proof. Let $a_k(x)$, $b_k(x)$ and $c_k(x)$ denote the generating functions for the number of permutations in the nth level of the labelled generating forest $\mathcal{F}$ with label k, $\bar{k}$ and $\bar{\bar{k}}$, respectively. By Corollary 3.29, we have

$$\begin{aligned} a_k(x) &= x \sum_{j \ge k-1} a_j(x), \\ b_k(x) &= x\big(a_{k-1}(x) + b_{k-1}(x) + c_{k-1}(x)\big), \\ c_k(x) &= x \sum_{j \ge k} \big(b_j(x) + c_j(x)\big), \end{aligned}$$

with $a_3(x) = x^2 + x\sum_{j\ge3} a_j(x)$, $b_3(x) = x^3 + xc_2(x) + x\sum_{j\ge3} b_j(x)$ and $c_2(x) = x\sum_{j\ge2} c_j(x)$.

Now let $A(x,v) = \sum_{k\ge3} a_k(x)v^k$, $B(x,v) = \sum_{k\ge2} b_k(x)v^k$ and $C(x,v) = \sum_{k\ge3} c_k(x)v^k$. The above recurrences can then be written as

$$A(x,v) = a_3(x)v^3 + \frac{x}{1-v}(v^4A(x,1) - v^2A(x,v)), \tag{9}$$

$$B(x,v) = b_3(x)v^3 + xv(v^3A(x,v) + B(x,v) + C(x,v) - c_2(x)v^2), \tag{10}$$

$$C(x,v) = c_2(x)v^2 + \frac{x}{1-v}(v^3B(x,1) - vB(x,v)) + \frac{x}{1-v}(v^3C(x,1) - vC(x,v)), \tag{11}$$

where $a_3(x) = x^2 + xA(x,1)$, $b_3(x) = x^2 + xB(x,1) + x^2C(x,1)$ and $c_2(x) = xC(x,1)$.

By finding $A(x,v)$ from (9) and $B(x,v)$ from (10) and then substituting into (11), we obtain

$$\begin{aligned} &\frac{(1-v+xv^2)^2}{(1-xv)(1-v)^2}C(x,t) \\ &\quad + \frac{xv^2(1-v+xv^2)(2v^2x^2-2x^2v+2x-1)}{(1-v)^2(1-2x)(1-xv)}C(x,1) \\ &\quad - \frac{(2v^2x^2-2xv+1)x^2v^3}{(1-xv)(1-2x)(1-v)}A(x,1) - \frac{(2v^2x^2-v+1)v^3x^3}{(1-xv)(1-2x)(1-v)} = 0. \end{aligned}$$

To solve the preceding functional equation, we apply the kernel method and take $v = C(x)$. This gives

$$A(x,1) = xC(x) - x.$$

Now, by differentiating the functional equation with respect to t and then substituting $t = C(x)$ and $A(x,1) = xC(x) - x$, we obtain

$$C(x,1) = -\frac{1-7x+12x^2-8x^3}{2(1-6x+8x^2-4x^3)} + \frac{1-9x+24x^2-20x^3+8x^4}{2(1-6x+8x^2-4x^3)\sqrt{1-4x}}.$$

Thus, by (10), we have

$$B(x,1) = \frac{x^3(1-2x-\sqrt{1-4x})}{(1-x)(1-4x)^2-(1-2x)(1-5x+2x^2)\sqrt{1-4x}}.$$

Since $F_T(x) = A(x,1) + B(x,1) + C(x,1)$, the result follows by adding the last three displayed expressions and simplifying. □

3.14 Case 223: $\{1243, 1342, 2413\}$

To find an explicit formula for $F_T(x)$, we define $A_m(x)$ (resp. $B_m(x)$) to be the generating function for T-avoiders $\pi = i_1\pi^{(1)}\cdots i_m\pi^{(m)}$ ($i_1,\ldots,i_m$ are the left-right maxima) such that $\pi^{(s)} < i_1$ for all $s \neq 2$ and $\pi^{(2)} < i_1$ (resp. $\pi^{(2)}$ has a letter greater than i_1). Also, we define $G_m(x)$ to be the generating function for T-avoiders with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xF_T(x)$. Recall $L(x) := \frac{1-x}{1-2x}$ is the generating function for $\{132, 231\}$-avoiders [13].

Lemma 3.31 *$A_1(x) = G_1(x)$ and for all $m \geq 2$,*

$$A_m(x) = xA_{m-1}(x) + x\sum_{j\geq m} G_j(x).$$

Proof. Clearly, $A_1(x) = G_1(x)$. To find an equation for $A_m(x)$ where $m \geq 2$, suppose $\pi = i_1\pi^{(1)}\cdots i_m\pi^{(m)} \in S_n(T)$ with m left-right maxima satisfies $\pi^{(s)} < i_1$ for all s. If $\pi^{(1)} = \emptyset$, we have a contribution of $xA_{m-1}(x)$. Otherwise, assume that $\pi^{(1)}$ has $d \geq 1$ left-right maxima. Since π avoids 1342 and 1243, we see that π can be written as

$$\pi = i_1 j_1\alpha^{(1)} j_2\alpha^{(2)}\cdots j_d\alpha^{(d)} i_2\pi^{(2)}\cdots i_m\pi^{(m)},$$

where $\alpha^{(s)} < j_1$ for all $s = 1, 3, \ldots, d$, $\alpha^{(2)} < j_2$, and $\pi^{(s)} < j_1$ for all $s = 2, 3, \ldots, m$. Thus, we have a contribution of $xG_{m+d-1}(x)$. Summing over all the contributions, we obtain

$$A_m(x) = xA_{m-1}(x) + x\sum_{d\geq 1} G_{m+d-1}(x),$$

as required. □

Lemma 3.32 *For all $m \geq 2$,*

$$B_m(x) = x\big(L(x)-1\big)A_{m-1}(x) + \frac{x^3}{(1-2x)^2}A_{m-1}(x).$$

Proof. Let us write an equation for $B_m(x)$, $m \geq 2$. Suppose $\pi = i_1\pi^{(1)} \cdots i_m\pi^{(m)} \in S_n(T)$ and that $\pi^{(1)}$ contains d letters. Note that $\pi^{(1)}$ is decreasing (π avoids 1243) and $\pi^{(2)}$ has the form $\alpha\beta$ with $\alpha > i_1$ and $\beta < i_1$.

If $d = 0$, then π avoids T if and only if α is nonempty and avoids 132 and 231, with the section $i_2\beta i_3\pi^{(3)} \cdots i_m\pi^{(m)}$ avoiding T. Thus, we have a contribution of $x\big(L(x) - 1\big)A_{m-1}(x)$.

For the case $d \geq 1$, since π avoids T, π has the form

$$\pi = i_1 j_1 j_2 \cdots j_d i_2 \alpha^{(0)}\alpha^{(1)} \cdots \alpha^{(d)}\beta i_3\pi^{(3)} \cdots i_m\pi^{(m)}$$

such that $j_0 := i_1 > j_1 > \cdots > j_d \geq 1$, $j_{s-1} > \alpha^{(s)} > j_s$ for all $s = 0, 1, \ldots, d$ with $j_{-1} = i_2$, and $\beta, \pi^{(3)}, \ldots, \pi^{(m)} < j_d$. Here, we consider three cases:

- $\alpha^{(s)}$ is decreasing for all $s = 0, 1, \ldots, d-1$. The contribution is $\frac{x^{d+2}}{(1-x)^d}L(x)A_{m-1}(x)$.
- $\alpha^{(0)}$ is not decreasing (i.e., has a rise). Then $\alpha^{(s)} = \emptyset$ for all $s = 1, 2, \ldots, d$. Hence, we have a contribution of $x^{d+1}\big(L(x) - 1/(1-x)\big)A_{m-1}(x)$.
- there is a minimal $p \in [1, d-1]$ such that $\alpha^{(p)}$ is not decreasing. Then $\alpha^{(s)}$ is decreasing for all $s = 0, 1, \ldots, p-1$ and $\alpha^{(s)} = \emptyset$ for all $s = p+1, \ldots, d$. Thus, we have a contribution of $\frac{x^{d+2}}{(1-x)^p}\big(L(x) - \frac{1}{1-x}\big)A_{m-1}(x)$.

Hence,

$$\begin{aligned}
B_m(x) &= x\big(L(x) - 1\big)A_{m-1}(x) \\
&\quad + \sum_{d\geq 1}\left(\frac{x^{d+2}}{(1-x)^d}L(x) + \left(x^{d+1} + \sum_{p=1}^{d-1}\frac{x^{d+2}}{(1-x)^p}\right)\big(L(x) - 1/(1-x)\big)\right)A_{m-1}(x) \\
&= x\big(L(x) - 1\big)A_{m-1}(x) + \frac{x^3}{(1-2x)^2}A_{m-1}(x),
\end{aligned}$$

as required. □

Now, we are ready to find a formula for $F_T(x)$.

THEOREM 3.33 *Let* $T = \{1243, 1342, 2413\}$. *Then*

$$F_T(x) = \frac{(1-2x)\big(1 - 2x - \sqrt{1 - 8x + 20x^2 - 20x^3 + 4x^4}\,\big)}{2x(1 - 4x + 5x^2 - x^3)}.$$

Proof. Define

$$G(x,y) = \sum_{m\geq 0} G_m(x)y^m,\ A(x,y) = \sum_{m\geq 1} A_m(x)y^m,\ B(x,y) = \sum_{m\geq 2} B_m(x)y^m.$$

From the definitions of A_m and B_m, we have $G_m(x) = A_m(x) + B_m(x)$ for all $m \geq 1$ with $G_0(x) = 1$ and $B_1(x) = 0$, which implies

$$G(x,y) = 1 + A(x,y) + B(x,y)\,. \tag{12}$$

Lemma 3.31 asserts that

$$A_m(x) = xA_{m-1}(x) + x\sum_{j\geq m} G_j(x), \quad A_1(x) = G_1(x).$$

Multiplying by y^m, summing over $m \geq 2$ and using the fact $G(x,1) = F_T(x)$ implies

$$(1-xy)A(x,y) = xy + \frac{xy}{1-y}(F_T(x) - G(x,y)). \tag{13}$$

Similarly, Lemma 3.32 yields

$$B(x,y) = \frac{x^2(1-x)y}{(1-2x)^2}A(x,y). \tag{14}$$

From (12), (13) and (14), we obtain

$$(1-xy)G(x,y) = 1 - xy + xy\left(1 + \frac{x^2(1-x)y}{(1-2x)^2}\right)\left(1 + \frac{F_T(x) - G(x,y)}{1-y}\right).$$

This equation can be solved by the kernel method, giving the stated result. □

3.15 Case 224: $\{4132, 1342, 1423\}$

Here, we consider (right-left) cell decompositions, which allow a useful characterization of R-avoiders, where $R := \{1342, 1423\}$ is a subset of T. So suppose

$$\pi = \pi^{(m)} i_m \pi^{(m-1)} i_{m-1} \cdots \pi^{(1)} i_1 \in S_n$$

has $m \geq 2$ right-left maxima $n = i_m > i_{m-1} > \cdots > i_1 \geq 1$. The right-left maxima determine a *cell decomposition* of the matrix diagram of π as illustrated in Figure 12 for $m = 4$. There are $\binom{m+1}{2}$ cells C_{ij}, where $i, j \geq 1$ and $i + j \leq m + 1$, indexed by (x, y) coordinates. For example, C_{21} and C_{32} are as shown.

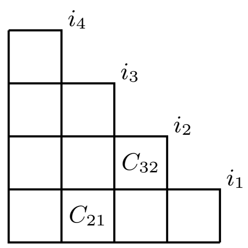


Figure 12: Cell decomposition

Cells with $i = 1$ or $j = 1$ are referred to as *boundary* cells, the others are *interior*. A cell is *occupied* if it contains at least one letter of π, otherwise it is *empty*. Let α_{ij} denote the subpermutation of entries in C_{ij}.

The reader may verify the following characterization of R-avoiders in terms of the cell decomposition. A permutation π is an R-avoider if and only if

1. For each occupied cell C, all cells that lie both strictly east and strictly north of C are empty.
2. For each pair of occupied cells C, D with D directly north of C (same column), all entries in C lie to the right of all entries in D.
3. For each pair of occupied cells C, D with D directly east of C (same row), all entries in C are larger than all entries in D.
4. α_{ij} avoids R for all i, j.

Condition (1) imposes restrictions on occupied cells as follows. A *major* cell for π is an interior cell C that is occupied and such that all cells directly north or directly east of C are empty. The set of major cells (possibly empty) determines a Dyck path of semilength $m-1$ such that cells in the first column correspond to vertices in the first ascent of the path and major cells correspond to valley vertices as illustrated in Figure 13. (If there are no major cells, the Dyck path covers the boundary cells and has no valleys.)

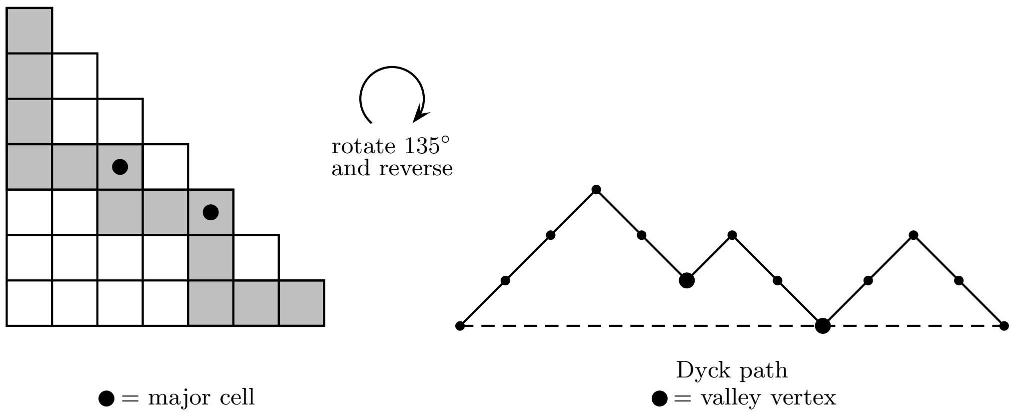


Figure 13: Dyck path from cell diagram

If π avoids R, then condition (1) implies that all cells not on the Dyck path are empty, and condition (4) implies $\mathrm{St}(\alpha_{ij})$ is an R-avoider for all i, j. Conversely, if $n = i_m > i_{m-1} > \cdots > i_1 \geq 1$ are given and we have a Dyck path in the associated cell diagram, and an R-avoider π_C is specified for each cell C on the Dyck path, with the additional proviso $\pi_C \neq \emptyset$ for valley cells, then conditions (2) and (3) imply that an R-avoider with this Dyck path is uniquely determined.

Now we can find an equation for the generating function $L_m(x)$ for T-avoiders with exactly $m+1$ right-left maxima. Clearly, $L_1(x) = xF_T(x)$. So assume $m \geq 2$. If an R-avoider also avoids 4132, then all cells not in the leftmost column avoid 132 and all cells in the leftmost column below the topmost nonempty cell also avoid 132. Also, the associated Dyck path has no valleys above the x-axis. Thus, we may assume that the Dyck path has the form

$$P = U^{a_1}D^{a_1}U^{a_2}D^{a_2}\cdots U^{a_{s+1}}D^{a_{s+1}},$$

with $s \geq 0$ valleys, all on the x-axis. By the cell decomposition each valley contributes $C(x) - 1$.

Consider the first a_1+1 cells, say $C_1, C_2, \dots, C_{a_1+1}$, from top to bottom in the leftmost column:

- if $C_1 = \cdots = C_{a_1} = \emptyset$, then we have a contribution of $x^m F_T(x)C(x)\big(C(x)-1\big)^s$;
- if $C_1 = \cdots = C_{j-1} = \emptyset$ and $C_j \neq \emptyset$, then $C_{j+2} = \cdots = C_{a_1+1} = \emptyset$, which gives a contribution $x^m\big(F_T(x)-1\big)C(x)^2\big(C(x)-1\big)^s$.

Summing over all Dyck paths of form P with fixed m, a_1 and s, we find that the generating function for T-avoiders having a fixed diagram associated with a Dyck path of $2m-2$ steps, no valleys above x-axis, first ascent of length a_1 steps and s valleys is given by

$$\begin{cases} x^m F_T(x)C(x) + (m-1)x^m\big(F_T(x)-1\big)C(x)^2, & a_1 = m-1 \text{ (i.e., } s=0) \\ x^m F_T(x)C(x)\big(C(x)-1\big)^s + a_1 x^m\big(F_T(x)-1\big)C(x)^2\big(C(x)-1\big)^s, & 1 \leq a_1 \leq m-2\,. \end{cases}$$

Thus, by summing over all s and a_1, we obtain

$$\begin{aligned} L_m(x) &= \sum_{s=0}^{m-2}\binom{m-2}{s}x^m F_T(x)C(x)\big(C(x)-1\big)^s \\ &\quad + \sum_{s=1}^{m-1}\binom{m-1}{s}x^m\big(F_T(x)-1\big)C^2(x)\big(C(x)-1\big)^{s-1} \\ &= x^m C^{m-1}(x)F_T(x) + \frac{x^m C(x)^{m+1}\big(F_T(x)-1\big)}{C(x)-1} - x^m\big(F_T(x)-1\big)C(x)^2\,, \end{aligned}$$

which implies

$$L_m(x) = x^m\big(xC^m F_T(x) + (C^m(x)-1)(F_T(x)-1)\big)\,.$$

Summing over $m \geq 1$, and noting $L_0(x) = 1$, gives

$$F_T(x) = 1 + xC(x)F_T(x) + \left(C(x) - \frac{1}{1-x}\right)(F_T(x)-1)\,.$$

Solving for $F_T(x)$ and simplifying leads to the following result.

Theorem 3.34 *Let $T = \{4132, 1342, 1423\}$. Then*

$$F_T(x) = \frac{2 - 10x + 9x^2 - 3x^3 + x(1-x)(2-x)\sqrt{1-4x}}{2(1-5x+4x^2-x^4)}.$$

3.16 Case 242: $\{2341, 2431, 3241\}$

Theorem 3.35 *Let $T = \{2341, 2431, 3241\}$. Then $F_T(x)$ satisfies*

$$F_T(x) = 1 + \frac{xF_T(x)}{1 - xF_T^2(x)}.$$

Explicitly,

$$F_T(x) = 1 + \sum_{n\geq 1} x^n\left(\sum_{i=1}^{n}\frac{1}{i}\binom{n-1}{i-1}\binom{2n-i}{i-1}\right).$$

Proof. Let $G_m(x)$ be the generating function for T-avoiders with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xF_T(x)$. Now let $m \geq 2$. To find an equation for $G_m(x)$, write $\pi \in S_n(T)$ with m left-right maxima as $i_1\pi^{(1)}i_2\pi^{(2)} \cdots i_m\pi^{(m)}$. Since π avoids 2341, $i_j < \pi^{(j+2)}$ for $j = 1, 2, \ldots, m-2$. Since π avoids 2431 and 3241, π is further restricted to have the form illustrated in Figure 14 for $m = 5$ (blank regions empty).

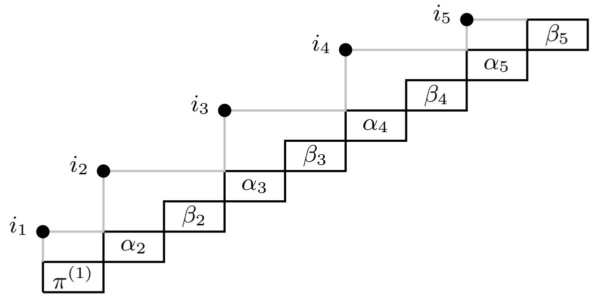


Figure 14: A T-avoider with $m = 5$

Conversely, within a permutation of this form, if all $2m-1$ labelled regions are T-avoiders, then so is the permutation. Hence, $G_m(x) = x^m F_T^{2m-1}(x)$ for $m \geq 2$, which is also seen to hold for $m = 1$. Summing over $m \geq 0$ yields the stated equation for $F_T(x)$.

Define a function $g(x, y)$ via $g(x, y) = \frac{xy(g(x,y)+1)}{1-x(g(x,y)+1)^2}$, where $g(x, 1) = F_T(x) - 1$. The Lagrange Inversion formula [16, Sec. 5.1] yields

$$g(x, y) = \sum_{i\geq 1} y^i \sum_{j\geq 0} \frac{x^{i+j}}{i} \binom{i-1+j}{i-1} \binom{i+2j}{i-1},$$

and extracting the coefficient of x^n in $g(x, 1)$ completes the proof. □

For other combinatorial objects with this counting sequence, see [14, Seq. A106228].

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