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On certain functional equations related to Jordan $*$-derivations in semiprime $*$-rings and standard operator algebras

Mohammad Ashraf
Department of Mathematics
Aligarh Muslim University
Aligarh-202002 India
email: mashraf80@hotmail.com

and

Bilal Ahmad Wani
Department of Mathematics
Aligarh Muslim University
Aligarh-202002 India
email: bilalwanikmr@gmail.com

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Abstract. The purpose of this paper is to investigate identities with Jordan $*$-derivations in semiprime $*$-rings. Let $\mathcal{R}$ be a 2-torsion free semiprime $*$-ring. In this paper it has been shown that, if $\mathcal{R}$ admits an additive mapping $D:\mathcal{R}\to\mathcal{R}$ satisfying either $D(xyx)=D(xy)x^*+xyD(x)$ for all $x,y\in\mathcal{R}$, or $D(xyx)=D(x)y^*x^*+xD(yx)$ for all pairs $x,y\in\mathcal{R}$, then D is a $*$-derivation. Moreover this result makes it possible to prove that if $\mathcal{R}$ satisfies $2D(x^n)=D(x^{n-1})x^*+x^{n-1}D(x)+D(x)(x^*)^{n-1}+xD(x^{n-1})$ for all $x\in\mathcal{R}$ and some fixed integer $n\geq 2$, then D is a Jordan $*$-derivation under some torsion restrictions. Finally, we apply these purely ring theoretic results to standard operator algebras $\mathcal{A}(\mathcal{H})$. In particular, we prove that if $\mathcal{H}$ be a real or complex Hilbert space, with $dim(\mathcal{H})>1$, admitting a linear mapping $D:\mathcal{A}(\mathcal{H})\to\mathcal{B}(\mathcal{H})$ (where $\mathcal{B}(\mathcal{H})$ stands for the bounded linear operators) such that

$$2D(A^n)=D(A^{n-1})A^*+A^{n-1}D(A)+D(A)(A^*)^{n-1}+AD(A^{n-1})$$

for all $A\in\mathcal{A}(\mathcal{H})$. Then D is of the form $D(A)=AB-BA^*$ for all $A\in\mathcal{A}(\mathcal{H})$ and some fixed $B\in\mathcal{B}(\mathcal{H})$, which means that D is Jordan $*$-derivation.

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1 Introduction

Throughout this paper $\mathcal{R}$ will denote an associative ring with center $Z(\mathcal{R})$. Recall that a ring $\mathcal{R}$ is said to be n-torsion free, where $n>1$ is an integer, if $nx=0$ implies $x=0$ for all $x\in\mathcal{R}$. For any $x,y\in\mathcal{R}$, the symbol $[x,y]$ will denote the commutator $xy-yx$. A ring $\mathcal{R}$ is said to be prime if for any $a,b\in\mathcal{R}$, $a\mathcal{R}b=\{0\}$ implies $a=0$ or $b=0$, and $\mathcal{R}$ is semiprime if for any $a\in\mathcal{R}$, $a\mathcal{R}a=\{0\}$ implies $a=0$. An additive mapping $x\mapsto x^*$ on a ring $\mathcal{R}$ is called involution in case $(xy)^*=y^*x^*$ and $(x^*)^*=x$ hold for all $x,y\in\mathcal{R}$. A ring equipped with an involution is called a ring with involution or $*$-ring (see

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[13]). An additive mapping $D : \mathcal{R} \to \mathcal{R}$ is said to be a derivation on $\mathcal{R}$ if $D(xy) = D(x)y + xD(y)$ for all $x, y \in \mathcal{R}$ and is called a Jordan derivation if $D(x^2) = D(x)x + xD(x)$ for all $x \in \mathcal{R}$. A derivation D is inner if there exists $a \in \mathcal{R}$ such that $D(x) = [a, x]$ holds for all $x \in \mathcal{R}$. A Jordan triple derivation $D : \mathcal{R} \to \mathcal{R}$ is an additive mapping if $D(xyx) = D(x)yx + xD(y)x + xyD(x)$ for all $x, y \in \mathcal{R}$. It is clear that any derivation is a Jordan derivation. The converse is in general not true. A classical result of Herstein [11] asserts that every Jordan derivation on a prime ring of characteristic different from two is a derivation. A brief proof of Herstein's result can be found in [6]. If $\mathcal{R}$ is 2-torsion free, it can be easily proved that any Jordan derivation $D : \mathcal{R} \to \mathcal{R}$ is a Jordan triple derivation (see [12]). A famous result due to Brešar [5, Theorem 4.3], asserts that a Jordan triple derivation on a 2-torsion free semiprime ring is a derivation. Following the same line, a number of results have been obtained by several authors (see [2], [3], [4], [9], [18],[19], [22], [23]), where further references can be found.

Let $\mathcal{R}$ be a $*$-ring. An additive mapping $D : \mathcal{R} \to \mathcal{R}$ is said to be a $*$-derivation on $\mathcal{R}$ if $D(xy) = D(x)y^* + xD(y)$ for all $x, y \in \mathcal{R}$ and is called a Jordan $*$-derivation if $D(x^2) = D(x)x^* + xD(x)$ holds for all $x \in \mathcal{R}$ Note that the mapping $x \mapsto xa - ax^*$, where a is a fixed element in $\mathcal{R}$, is a Jordan $*$-derivation. Such a Jordan $*$-derivation is said to be inner. The study of Jordan $*$-derivations has been motivated by the problem of the representativity of quadratic forms by bilinear forms (for the results concerning this problem we refer the reader to [8], [15], and [16]). It turns out that the question, whether each quadratic form can be represented by some bilinear form, is connected with the question, whether every Jordan $*$-derivation is inner, as shown by Šemrl [15].

A Jordan triple $*$-derivation is an additive mapping $D : \mathcal{R} \to \mathcal{R}$ with the property $D(xyx) = D(x)y^*x^* + xD(y)x^* + xyD(x)$ for all $x, y \in \mathcal{R}$. One might expect that any Jordan $*$-derivation on a 2-torsion free semiprime $*$-ring is a $*$-derivation, but this is not the case. It is easy to prove that there are no nonzero $*$-derivations on noncommutative prime $*$-rings (see [7] for the details). Any Jordan $*$-derivation $D : \mathcal{R} \to \mathcal{R}$ on a 2-torsion free $*$-ring $\mathcal{R}$ is a Jordan triple $*$-derivation. However, the converse of this statement is not true in general (see [1]). In [24], Vukman showed that the converse holds if $\mathcal{R}$ is 6-torsion free semiprime $*$-ring. Recently, Fošner and Ilišević [10] proved that every Jordan triple $*$-derivation on a 2-torsion free semiprime $*$-ring is a Jordan $*$-derivation. In view of these results we begin our investigation with additive mapping D on a semiprime $*$-ring $\mathcal{R}$ which satisfies either of the identities $D(xyx) = D(xy)x^* + xyD(x)$ or $D(xyx) = D(x)y^*x^* + xD(yx)$ and show that D is a $*$-derivation on $\mathcal{R}$. Further, it is shown that if the additive mapping D satisfies either of the properties $D(xyx) = D(xy)x^* - xyD(x)$ or $D(xyx) = D(x)y^*x^* - xD(yx)$, then D is a Jordan $*$-derivation. Finally, a result concerning the identity $2D(x^n) = D(x^{n-1})x^* + x^{n-1}D(x) + D(x)(x^*)^{n-1} + xD(x^{n-1})$ has also been obtained.

2 Results on semiprime $*$-ring

We begin with the following results which are crucial for developing the proof of our main results.

Lemma 2.1 *[22, Lemma 3] Let $\mathcal{R}$ be a semiprime ring and $f : \mathcal{R} \to \mathcal{R}$ be an additive mapping such that either $f(x)x = 0$ or $xf(x) = 0$ for all $x \in \mathcal{R}$. Then $f = 0$.*

Now we will prove the following main results.

THEOREM 2.2 *Let $\mathcal{R}$ be a 2-torsion free semiprime $*$-ring and $D : \mathcal{R} \to \mathcal{R}$ be an additive mapping such that either*

$$D(xyx) = D(xy)x^* + xyD(x) \text{ , for all } x, y \in \mathcal{R} \tag{1}$$

or

$$D(xyx) = D(x)y^*x^* + xD(yx) \text{ , for all } x, y \in \mathcal{R}. \tag{2}$$

Then D is a Jordan $*$-derivation, whence D is a $*$-derivation. In fact, we can conclude that $D(\mathcal{R})$ generates a central ideal of $\mathcal{R}$.

Proof. We will restrict our attention on the relation (1), the proof in case when $\mathcal{R}$ satisfies the relation (2) is similar and will therefore be omitted. Linearization of the relation (1) gives

$$D(xyz + zyx) = D(xy)z^* + D(zy)x^* + xyD(z) + zyD(x),$$

for all $x, y, z \in \mathcal{R}$. In particular for $z = x^2$, the above relation gives

$$D(xyx^2 + x^2yx) = D(xy)x^{*^2} + D(x^2y)x^* + xyD(x^2) + x^2yD(x), \tag{3}$$

for all $x, y, \in \mathcal{R}$. Putting $xy + yx$ for y in (1) and applying the relation (1), we obtain

$$\begin{aligned} D(xyx^2 + x^2yx) &= D(x^2y)x^* + D(xy)x^{*^2} + xyD(x)x^* \\ &\quad + x^2yD(x) + xyxD(x), \end{aligned} \tag{4}$$

for all $x, y \in \mathcal{R}$. By comparing (3) and (4), we have

$$xyA(x) = 0, \text{ for all } x, y \in \mathcal{R}, \tag{5}$$

where $A(x)$ stands for $D(x^2) - D(x)x^* - xD(x)$. Right multiplication of (5) by x and left multiplication by $A(x)$ gives,

$$A(x)xyA(x)x = 0, \text{ for all } x, y \in \mathcal{R}.$$

By the semiprimeness of $\mathcal{R}$, it follows that

$$A(x)x = 0, \text{ for all } x \in \mathcal{R}. \tag{6}$$

The substitution of $A(x)yx$ for y in the relation (5), gives $xA(x)yxA(x) = 0$ for all pairs $x, y \in \mathcal{R}$. Hence, we obtain

$$xA(x) = 0, \text{ for all } x \in \mathcal{R}. \tag{7}$$

The linearization of the relation (6) gives

$$B(x, y)x + A(x)y + B(x, y)y + A(y)x = 0$$

for all pairs $x, y \in \mathcal{R}$, where $B(x,y)$ denotes $D(xy+yx)-D(x)y^*-xD(y)-D(y)x^*-yD(x)$. Putting in the above relation $-x$ for x and comparing the relation so obtained with the above relation one obtains

$$B(x,y)x + A(x)y = 0, \text{ for all } x, y \in \mathcal{R}.$$

In view of the relation (7), right multiplication by $A(x)$ gives, $A(x)yA(x) = 0$ for all pairs $x, y \in \mathcal{R}$. Hence it follows that $A(x) = 0$ for all $x \in \mathcal{R}$. In other words, D is a Jordan $*$-derivation. Hence it follows that D is a Jordan triple $*$-derivation. Now, comparing the relation $D(xyx) = D(x)y^*x^* + xD(y)x^* + xyD(x)$, for all $x, y \in \mathcal{R}$, with the relation (1), we get

$$\big(D(xy) - D(x)y^* - xD(y)\big)x^* = 0, \text{ for all } x, y \in \mathcal{R}.$$

For any fixed $y \in \mathcal{R}$, we have an additive mapping $x \mapsto D(xy) - D(x)y^* - xD(y)$ on $\mathcal{R}$. Thus from the above relation and by the consequence of Lemma 2.1, it follows that $D(xy) - D(x)y^* - xD(y) = 0$, for all pairs $x, y \in \mathcal{R}$. In other words, D is a $*$-derivation. Hence by [14, Theorem 3.1], $D(\mathcal{R}) \subseteq Z(\mathcal{R})$. This completes the proof. $\square$

For the sake of brevity, we omit the proof of the following statement.

Theorem 2.3 *Let $\mathcal{R}$ be a 2-torsion free semiprime $*$-ring. Suppose $D : \mathcal{R} \to \mathcal{R}$ is an additive mapping such that either*

$$D(xyx) = D(xy)x^* - xyD(x) \text{ , for all } x, y \in \mathcal{R} \tag{8}$$

or

$$D(xyx) = D(x)y^*x^* - xD(yx) \text{ , for all } x, y \in \mathcal{R}. \tag{9}$$

Then D is a Jordan $*$-derivation. If, in addition, $1 \in \mathcal{R}$, then $D = 0$.

Disadvantage of Theorem 2.2 is that in identities (1) and (2) there is no symmetry. Therefore, Theorem 2.2, together with the desire for symmetry leads to the following conjecture.

Conjecture 2.4 Let $\mathcal{R}$ be a 2-torsion free semiprime $*$-ring and $D : \mathcal{R} \to \mathcal{R}$ be an additive mapping such that

$$2D(xyx) = D(xy)x^* + xyD(x) + D(x)y^*x^* + xD(yx), \tag{10}$$

holds for all pairs $x, y \in \mathcal{R}$. Then D is a Jordan $*$-derivation.

Note that in case a ring has the identity element, the proof of the above conjecture is immediate. The substitution $y = e$ in the relation (10), where e stands for the identity element, gives that D is a Jordan $*$-derivation.

The substitution of $y = x^{n-2}$ in the relation (10) gives

$$2D(x^n) = D(x^{n-1})x^* + x^{n-1}D(x) + D(x)(x^*)^{n-1} + xD(x^{n-1}),$$

which leads to the following conjecture.

CONJECTURE 2.5 Let $\mathcal{R}$ be a semiprime $*$-ring with a suitable torsion restriction and $D : \mathcal{R} \to \mathcal{R}$ be an additive mapping such that

$$2D(x^n) = D(x^{n-1})x^* + x^{n-1}D(x) + D(x)(x^*)^{n-1} + xD(x^{n-1}),$$

holds for all $x \in \mathcal{R}$ and some fixed integer $n \geq 2$. Then D is a Jordan $*$-derivation.

Now we prove the above conjecture in case a ring has the identity element.

THEOREM 2.6 *Let $\mathcal{R}$ be a $2(n-1)!$-torsion free semiprime $*$-ring with identity e and $D : \mathcal{R} \to \mathcal{R}$ be an additive mapping such that*

$$2D(x^n) = D(x^{n-1})x^* + x^{n-1}D(x) + D(x)(x^*)^{n-1} + xD(x^{n-1}),$$

for all $x \in \mathcal{R}$ and some fixed integer $n \geq 2$. Then D is a Jordan $*$-derivation.

Proof. We have the relation

$$2D(x^n) = D(x^{n-1})x^* + x^{n-1}D(x) + D(x)(x^*)^{n-1} + xD(x^{n-1}), \tag{11}$$

holds for all $x \in \mathcal{R}$. The substitution of $x = e$ in the relation (11) gives $D(e) = 0$. Let y be any element of the center $Z(\mathcal{R})$. Putting $x + y$ for x in the above relation, we obtain

$$\begin{aligned} 2\sum_{i=0}^{n}\binom{n}{i}D(x^{n-i}y^i) \quad = \quad & \left(\sum_{i=0}^{n-1}\binom{n-1}{i}D(x^{n-1-i}y^i)\right)(x^* + y^*) \\ & +\left(\sum_{i=0}^{n-1}\binom{n-1}{i}x^{n-1-i}y^i\right)D(x+y) \\ & +D(x+y)\left(\sum_{i=0}^{n-1}\binom{n-1}{i}(x^*)^{n-1-i}(y^*)^i\right) \\ & +(x+y)\left(\sum_{i=0}^{n-1}\binom{n-1}{i}D(x^{n-1-i}y^i)\right). \end{aligned}$$

Using (11) in the above relation and rearranging it in sense of collecting together terms involving equal number of factors of y, we obtain

$$\begin{aligned}
0 &= 2\binom{n}{1}D(x^{n-1}y) - \binom{n-1}{0}\Big\{D(x^{n-1}y^*) + x^{n-1}D(y) + D(y)(x^*)^{n-1} + yD(x^{n-1})\Big\} \\
&\quad -\binom{n-1}{1}\Big\{D(x^{n-2}y)x^* + x^{n-2}yD(x) + D(x)(x^*)^{n-2}y^* + xD(x^{n-2}y)\Big\} \\
&\quad +2\binom{n}{2}D(x^{n-2}y^2) - \binom{n-1}{1}\Big\{D(x^{n-2}y)y^* + x^{n-2}yD(y) + D(y)(x^*)^{n-2}y^* \\
&\quad +yD(x^{n-2}y)\Big\} - \binom{n-1}{2}\Big\{D(x^{n-3}y^2)x^* + x^{n-3}y^2D(x) + D(x)(x^*)^{n-3}(y^*)^2 \\
&\quad +xD(x^{n-3}y^2)\Big\} + \cdots + 2\binom{n}{n-1}D(xy^{n-1}) - \binom{n-1}{n-2}\Big\{D(xy^{n-2})y^* \\
&\quad +xy^{n-2}D(y) + D(y)x^*(y^*)^{n-2} + yD(xy^{n-2})\Big\} \\
&\quad -\binom{n-1}{n-1}\Big\{D(y^{n-1})x^* + y^{n-1}D(x) + D(x)(y^*)^{n-1} + xD(y^{n-1})\Big\}.
\end{aligned}$$

This can be written as

$$f_0(x,y) + f_1(x,y) + f_2(x,y) + \cdots + f_{n-1}(x,y) = 0, \tag{12}$$

where $f_i(x,y)$ stands for the expression of terms involving i factors of y. Replace x by $x+2y$, $x+3y$, $\dots$, $x+(n-1)y$ in the relation (11)and expressing the resulting system of $n-2$ homogeneous equations of variables $f_i(x,y)$ for $i = 1,2,\dots n-1$ together with (12), we see that the coefficient matrix of the system of $n-1$ homogenous equations is a Van-der Monde matrix

$$\begin{pmatrix}
1 & 1 & \dots & 1 \\
2 & 2^2 & \dots & 2^{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
n-1 & (n-1)^2 & \dots & (n-1)^{n-1}
\end{pmatrix}.$$

Since the determinant of this matrix is different from zero, it follows that the system has only a trivial solution, i.e., $f_i(x,y) = 0$ for $i = 1,2,\dots n-1$. In particular, if y is replaced with the identity element e in $f_{n-2}(x,y)$, we obtain

$$\begin{aligned}
f_{n-2}(x,e) &= 2\binom{n}{n-2}D(x^2) - \binom{n-1}{n-2}D(x)x^* - \binom{n-1}{n-3}D(x^2) \\
&\quad -\binom{n-1}{n-2}xD(x) - \binom{n-1}{n-3}x^2D(e) - \binom{n-1}{n-2}D(x)x^* \\
&\quad -\binom{n-1}{n-3}D(e)(x^*)^2 - \binom{n-1}{n-3}D(x^2) - \binom{n-1}{n-2}xD(x).
\end{aligned}$$

After few calculations and considering the relation $D(e) = 0$, we obtain

$$(n(n-1) - (n-1)(n-2))D(x^2) = 2(n-1)\big(D(x)x^* + xD(x)\big).$$

Since $\mathcal{R}$ is $2(n-1)!$-torsion free, it follows from the above relation that

$$D(x^2) = D(x)x^* + xD(x) \text{ for all } x \in \mathcal{R}.$$

Hence D is a Jordan $*$-derivation, which completes the proof. □

3 Results on standard operator algebra $\mathcal{A}(\mathcal{H})$

Let $\mathcal{H}$ be a real or complex Hilbert space, $dim(\mathcal{H}) > 1$. By $\mathcal{B}(\mathcal{H})$ we mean the algebra of all bounded linear operators on $\mathcal{H}$. Denote by $\mathcal{F}(\mathcal{H})$ the subalgebra of bounded finite rank operators. It is to be noted that $\mathcal{F}(\mathcal{H})$ forms a $*$-closed ideal in $\mathcal{B}(\mathcal{H})$. An algebra $\mathcal{A}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ is said to be standard operator algebra in case $\mathcal{F}(\mathcal{H}) \subset \mathcal{A}(\mathcal{H})$. Let us point out that any standard operator algebra is prime, which is a consequence of Hahn-Banach theorem.

The main result of the paper is related to the result below first proved by Šemrl [17](see also [8]).

THEOREM 3.1 *[17, Theorem] Let $\mathcal{H}$ be a real or complex Hilbert space, with $dim(\mathcal{H}) > 1$, and let $\mathcal{A}(\mathcal{H})$ be a standard operator algebra on $\mathcal{H}$. Suppose that $D : \mathcal{A}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is a linear Jordan $*$-derivation. Then there exists a unique linear operator $B \in \mathcal{B}(\mathcal{H})$ such that $D(A) = AB - BA^*$ for all $A \in \mathcal{A}(\mathcal{H})$.*

THEOREM 3.2 *Let $\mathcal{H}$ be a real or complex Hilbert space, with $dim(\mathcal{H}) > 1$, and let $\mathcal{A}(\mathcal{H})$ be a standard operator algebra on $\mathcal{H}$. Suppose there exists a linear mapping $D : \mathcal{A}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ such that either*

$$D(A^{2n+1}) = D(A^{2n})A^* + A^{2n}D(A) \text{ , for all } A \in \mathcal{A}(\mathcal{H}),$$

or

$$D(A^{2n+1}) = D(A)(A^*)^{2n} + AD(A^{2n}) \text{ , for all } A \in \mathcal{A}(\mathcal{H}).$$

In this case D is of the form $D(A) = AB - BA^*$ for all $A \in \mathcal{A}(\mathcal{H})$ and some fixed $B \in \mathcal{B}(\mathcal{H})$, which means that D is a Jordan $*$-derivation.

Proof. We have the relation

$$D(A^{2n+1}) = D(A^{2n})A^* + A^{2n}D(A) \tag{13}$$

for all $A \in \mathcal{A}(\mathcal{H})$. Let us first consider the restriction of D on $\mathcal{F}(\mathcal{H})$. Let A be from $\mathcal{F}(\mathcal{H})$ and let $P \in \mathcal{F}(\mathcal{H})$ be an idempotent operator with $AP = PA = A$. Putting P for A in the relation (13), we obtain

$$D(P) = D(P)P^* + PD(P) \tag{14}$$

Putting $A + P$ for A in the relation (13), we obtain

$$\begin{aligned}\sum_{i=0}^{2n+1} \binom{2n+1}{i} D(A^{2n+1-i}P^i) &= \left(\sum_{i=0}^{2n} \binom{2n}{i} D(A^{2n-i}P^i)\right)(A^* + P^*) \\ &+ \left(\sum_{i=0}^{2n} \binom{2n}{i} A^{2n-i}P^i\right) D(A+P).\end{aligned}$$

Rearranging the above relation in the sense of collecting together terms involving equal number of factors of P, we obtain $\sum_{i=1}^{2n} f_i(A,P) = 0$, where

$$\begin{aligned} f_i(A,P) &= \binom{2n+1}{i} D(A^{2n+1-i}P^i) \\ &\quad -\binom{2n}{i}\big(D(A^{2n-i}P^i)A^* + A^{2n-i}P^iD(A)\big) \\ &\quad -\binom{2n}{i-1}\big(D(A^{2n+1-i}P^{i-1})P^* + (A^{2n+1-i}P^{i-1})D(P)\big); \end{aligned}$$

Replacing A by $A+2P$, $A+3P$,..., $A+2nP$ in the relation (13) and expressing the resulting system of $2n$ homogeneous equations of variables $f_i(A,P)$ for $i = 1, 2, ...2n$, we see that the coefficient matrix of the system of $2n$ homogenous equations is a Van-der Monde matrix

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \\ 2 & 2^2 & \ldots & 2^{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 2n & (2n)^2 & \ldots & (2n)^{2n} \end{pmatrix}.$$

Since the determinant of this matrix is different from zero, it follows immediately that the system has only a trivial solution. In particular

$$\begin{aligned} f_{2n-1}(A,P) &= \binom{2n+1}{2n-1} D(A^2) - \binom{2n}{2n-1}\big(D(A)A^* + AD(A)\big) \\ &\quad +\binom{2n}{2n-2}\big(D(A^2)P^* + A^2D(P)\big) \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} f_{2n}(A,P) &= \binom{2n+1}{2n} D(A) - \binom{2n}{2n}\big(D(P)A^* + PD(A)\big) \\ &\quad +\binom{2n}{2n-1}\big(D(A)P^* + AD(P)\big) \\ &= 0, \end{aligned}$$

The above relations reduces to

$$\begin{aligned} n(2n+1)D(A^2) &= 2nD(A)A^* + 2nAD(A) + n(2n-1)D(A^2)P^* \\ &\quad +n(2n-1)A^2D(P), \end{aligned} \tag{15}$$

$$(2n+1)D(A) = D(P)A^* + PD(A) + (2n)D(A)P^* + (2n)AD(P). \tag{16}$$

Multiplying (15) by P^* and using (14), we have

$$D(A^2)P^* = D(A)A^* + AD(A)P^* \tag{17}$$

Applying (17) in the relation (15), we obtain

$$\begin{aligned} n(2n+1)D(A^2) &= n(2n+1)D(A)A^* + 2nAD(A) \\ &\quad + n(2n-1)\Big(AD(A)P^* + A^2D(P)\Big) \end{aligned} \tag{18}$$

Left multiplication by A in (16) gives

$$AD(A) = AD(A)P^* + A^2D(P).$$

Applying the above relation in (18), we get

$$D(A^2) = D(A)A^* + AD(A). \tag{19}$$

From the relation (16) one can conclude that D maps $\mathcal{F}(\mathcal{H})$ into itself. We have therefore a linear mapping which maps $\mathcal{F}(\mathcal{H})$ into itself satisfying the relation (19) for all $A \in \mathcal{F}(\mathcal{H})$. Hence D is a Jordan $*$-derivation on $\mathcal{F}(\mathcal{H})$. Applying Theorem 3.1 one can conclude that D is of the form

$$D(A) = AB - BA^* \tag{20}$$

for all $A \in \mathcal{F}(\mathcal{H})$ and some fixed $B \in \mathcal{B}(\mathcal{H})$. It remains to prove that the relation (20) holds for all $A \in \mathcal{A}(\mathcal{H})$ as well. For this purpose we define $D_0 : \mathcal{A}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by $D_0(A) = AB - BA^*$ and consider $D_1 = D - D_0$. Indeed, the mapping D_1 is linear, satisfies the relation (13) and it vanishes on $\mathcal{F}(\mathcal{H})$. Now we will prove that D_1 vanishes on $\mathcal{A}(\mathcal{H})$ also. Let $A \in \mathcal{A}(\mathcal{H})$ and P be an idempotent operator of rank one. Let us introduce $S \in \mathcal{A}(\mathcal{H})$ by $S = A + PAP - (AP + PA)$. We have $SP = PS = 0$. It is easy to prove that $D_1(S) = D_1(A)$ and $D_1(S^{2n}) = D_1(A^{2n})$. The relation (13) leads us to

$$\begin{aligned} D_1(S^{2n})S^* + S^{2n}D_1(S) &= D_1(S^{2n+1}) = D_1(S^{2n+1} + P) = D_1((S+P)^{2n+1}) \\ &= D_1(S^{2n})(S+P)^* + (S+P)^{2n}D_1(S+P) \\ &= D_1(S^{2n})S^* + D_1(S^{2n})P^* + (S^{2n} + P)D_1(S) \\ &= D_1(S^{2n})S^* + D_1(S^{2n})P^* + S^{2n}D_1(S) + PD_1(S). \end{aligned}$$

Therefore,

$$D_1(S^{2n})P^* + PD_1(S) = 0;$$

Since $D_1(S) = D_1(A)$ and $D_1(S^{2n}) = D_1(A^{2n})$, above relation can be written as

$$D_1(A^{2n})P^* + PD_1(A) = 0;$$

Replace A by $-A$ in the above relation and compare the relation so obtained with the above relation, we obtain

$$PD_1(A) = 0$$

for all $A \in \mathcal{A}(\mathcal{H})$. Since P is an arbitrary idempotent operator of rank one, we have $D_1(A) = 0$ for all $A \in \mathcal{A}(\mathcal{H})$, which completes the proof of our theorem. $\square$

THEOREM 3.3 *Let $\mathcal{H}$ be a real or complex Hilbert space, with $dim(\mathcal{H}) > 1$, and let $\mathcal{A}(\mathcal{H})$ be a standard operator algebra on $\mathcal{H}$. Suppose there exists a linear mapping $D : \mathcal{A}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ such that either*

$$D(A^{2n+1}) = D(A^{2n})A^* - A^{2n}D(A) \text{ , for all } A \in \mathcal{A}(\mathcal{H}),$$

or

$$D(A^{2n+1}) = D(A)(A^*)^{2n} - AD(A^{2n}) \text{ , for all } A \in \mathcal{A}(\mathcal{H})$$

and some integer $n \geq 1$. Then $D(A) = 0$ for all $A \in \mathcal{A}(\mathcal{H})$.

Proof. We have the relation

$$D(A^{2n+1}) = D(A^{2n})A^* - A^{2n}D(A) \tag{21}$$

for all $A \in \mathcal{A}(\mathcal{H})$. Let us first consider the restriction of D on $\mathcal{F}(\mathcal{H})$. Let A be from $\mathcal{F}(\mathcal{H})$ and let $P \in \mathcal{F}(\mathcal{H})$ be an idempotent operator with $AP = PA = A$. Putting P for A in the relation (21), we obtain

$$D(P) = D(P)P^* - PD(P) \tag{22}$$

A right multiplication by P^* in the above relation gives

$$PD(P)P^* \;=\; 0$$

Left multiplication by P in (22) and combining with the above relation yields

$$PD(P) \;=\; 0 \tag{23}$$

Putting $A + P$ for A in the relation (21), we obtain

$$\begin{aligned}\sum_{i=0}^{2n+1}\binom{2n+1}{i}D(A^{2n+1-i}P^i) \;&=\; \Big(\sum_{i=0}^{2n}\binom{2n}{i}D(A^{2n-i}P^i)\Big)(A^* + P^*)\\ &\quad -\Big(\sum_{i=0}^{2n}\binom{2n}{i}A^{2n-i}P^i\Big)D(A+P).\end{aligned}$$

Rearranging the above relation in the sense of collecting together terms involving equal number of factors of P, we obtain $\sum_{i=1}^{2n} f_i(A,P) = 0$, where

$$\begin{aligned}f_i(A,P) \;&=\; \binom{2n+1}{i}D(A^{2n+1-i}P^i)\\ &\quad -\binom{2n}{i}\big(D(A^{2n-i}P^i)A^* - A^{2n-i}P^iD(A)\big)\\ &\quad -\binom{2n}{i-1}\big(D(A^{2n+1-i}P^{i-1})P^* - (A^{2n+1-i}P^{i-1})D(P)\big)\end{aligned}$$

Replace A by $A+P$, $A+2P$, $A+3P$,..., $A+2nP$ in the relation (21) and express the resulting system of $2n$ homogeneous equations of variables $f_i(A,P)$ for $i=1,2,...2n$, we see that the coefficient matrix of the system of $2n$ homogenous equations is a Van-der Monde matrix

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ 2 & 2^2 & \dots & 2^{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 2n & (2n)^2 & \dots & (2n)^{2n} \end{pmatrix}.$$

Since the determinant of this matrix is different from zero, it follows immediately that the system has only a trivial solution. In particular

$$\begin{aligned} f_{2n}(A,P) &= \binom{2n+1}{2n}D(A) - \binom{2n}{2n}\big(D(P)A^* - PD(A)\big) \\ &\quad -\binom{2n}{2n-1}\big(D(A)P^* - AD(P)\big) \\ &= 0, \end{aligned}$$

The above relation reduces to

$$(2n+1)D(A) = D(P)A^* - PD(A) + (2n)D(A)P^* - (2n)AD(P).$$

Using (23) in the above relation, we get

$$(2n+1)D(A) = D(P)A^* - PD(A) + (2n)D(A)P^*. \tag{24}$$

Left multiplication by P and right multiplication by P^* in the above relation leads to

$$PD(A)P^* = 0.$$

Left multiplication by P in the relation (24) and combining (23) together with the above relation, gives

$$PD(A) = 0 \tag{25}$$

Left multiplication by A in the above relation yields

$$AD(A) = 0 \tag{26}$$

Using (25) in (24), we get

$$(2n+1)D(A) = D(P)A^* + (2n)D(A)P^*. \tag{27}$$

A right multiplication by P^* in the above relation yields $D(A)P^* = D(P)A^*$, and hence combining the latter relation with (27), we obtain

$$D(A) = D(A)P^*. \tag{28}$$

From the relation (28) one can conclude that D maps $\mathcal{F}(\mathcal{H})$ into itself. We have therefore a linear mapping which maps $\mathcal{F}(\mathcal{H})$ into itself satisfying the relation (26) for all $A \in \mathcal{F}(\mathcal{H})$. Applying Lemma 2.1 one can conclude that $D(A) = 0$ for all $A \in \mathcal{F}(\mathcal{H})$.

It remains to prove that $D(A) = 0$ holds for all $A \in \mathcal{A}(\mathcal{H})$ as well. Indeed, the mapping D on $\mathcal{A}(\mathcal{H})$ is linear and satisfies the relation (21). Our aim is to prove that D vanishes on $\mathcal{A}(\mathcal{H})$ also. Let $A \in \mathcal{A}(\mathcal{H})$ and P be an idempotent operator of rank one. Let us introduce $S \in \mathcal{A}(\mathcal{H})$ by $S = A + PAP - (AP + PA)$. We have $SP = PS = 0$. It is easy to prove that $D(S) = D(A)$ and $D(S^{2n}) = D(A^{2n})$. The relation (21) leads us to

$$
\begin{aligned}
D(S^{2n})S^* - S^{2n}D(S) &= D(S^{2n+1}) = D(S^{2n+1} + P) = D((S + P)^{2n+1}) \\
&= D(S^{2n})(S + P)^* - (S + P)^{2n}D(S + P) \\
&= D(S^{2n})S^* + D(S^{2n})P^* - (S^{2n} + P)D(S) \\
&= D(S^{2n})S^* + D(S^{2n})P^* - S^{2n}D(S) - PD(S).
\end{aligned}
$$

Therefore,

$$D(S^{2n})P^* - PD(S) = 0;$$

Since $D(S) = D(A)$ and $D(S^{2n}) = D(A^{2n})$, the above relation can be written as

$$D(A^{2n})P^* - PD(A) = 0;$$

Replace A by $-A$ in the above relation and compare the relation so obtained with the above relation, we obtain

$$PD(A) = 0$$

for all $A \in \mathcal{A}(\mathcal{H})$. Since P is an arbitrary idempotent operator of rank one, we have $D(A) = 0$ for all $A \in \mathcal{A}(\mathcal{H})$, which completes the proof of the theorem. □

THEOREM 3.4 *Let $\mathcal{H}$ be a real or complex Hilbert space, with $dim(\mathcal{H}) > 1$, and let $\mathcal{A}(\mathcal{H})$ be a standard operator algebra on $\mathcal{H}$. Suppose there exist a linear mappings $D, G : \mathcal{A}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ such that either the relations*

$$D(A^{2n+1}) = D(A^{2n})A^* + A^{2n}G(A),$$

$$G(A^{2n+1}) = G(A^{2n})A^* + A^{2n}D(A)$$

hold for all $A \in \mathcal{A}(\mathcal{H})$, or the relations

$$D(A^{2n+1}) = D(A)(A^*)^{2n} + AG(A^{2n}),$$

$$G(A^{2n+1}) = G(A)(A^*)^{2n} + AD(A^{2n})$$

hold for all $A \in \mathcal{A}(\mathcal{H})$. In both the cases $D(A) = G(A) = AB - BA^*$ for all $A \in \mathcal{A}(\mathcal{H})$ and some fixed $B \in \mathcal{B}(\mathcal{H})$, which means that D and G are Jordan $*$-derivations.

Proof. We will restrict our attention on the first system of relations, the proof in case when $\mathcal{A}(\mathcal{H})$ satisfies the second system of relations is similar and will therefore be omitted. We have

$$D(A^{2n+1}) = D(A^{2n})A^* + A^{2n}G(A),$$

$$G(A^{2n+1}) = G(A^{2n})A^* + A^{2n}D(A)$$

hold for all $A \in \mathcal{A}(\mathcal{H})$. Subtracting the above relations, we get

$$T(A^{2n+1}) = T(A^{2n})A^* - A^{2n}T(A), \tag{29}$$

where $T = D - G$. Using Theorem 3.3, we conclude that $T(A) = 0$ for all $A \in \mathcal{A}(\mathcal{H})$, which implies $D = G$. This assertion enables us to combine the given two relations into only one relation

$$D(A^{2n+1}) = D(A^{2n})A^* + A^{2n}D(A)$$

for all $A \in \mathcal{A}(\mathcal{H})$. From Theorem 3.2 it follows that $D(A) = G(A) = AB - BA^*$ for all $A \in \mathcal{A}(\mathcal{H})$, and hence the proof is complete. $\square$

Our next result is in the spirit of the conjecture 2.5.

THEOREM 3.5 *Let $\mathcal{H}$ be a real or complex Hilbert space, with $dim(\mathcal{H}) > 1$, and let $\mathcal{A}(\mathcal{H})$ be a standard operator algebra on $\mathcal{H}$. Suppose there exists a linear mapping $D : \mathcal{A}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ such that*

$$2D(A^n) = D(A^{n-1})A^* + A^{n-1}D(A) + D(A)(A^*)^{n-1} + AD(A^{n-1}) \tag{30}$$

for all $A \in \mathcal{A}(\mathcal{H})$. In this case D is of the form $D(A) = AB - BA^*$ for all $A \in \mathcal{A}(\mathcal{H})$ and some fixed $B \in \mathcal{B}(\mathcal{H})$, which means that D is a Jordan $*$-derivation.

Proof. Let us first consider the restriction of D on $\mathcal{F}(\mathcal{H})$. Let A be from $\mathcal{F}(\mathcal{H})$ and let $P \in \mathcal{F}(\mathcal{H})$ be an idempotent operator with $AP = PA = A$. Putting P for A in the relation (30), we obtain

$$D(P) = D(P)P^* + PD(P) \tag{31}$$

Putting $A + P$ for A in the relation (30), we obtain

$$\begin{aligned} 2\sum_{i=0}^{n}\binom{n}{i}D(A^{n-i}P^i) &= \Big(\sum_{i=0}^{n-1}\binom{n-1}{i}D(A^{n-1-i}P^i)\Big)(A^* + P^*) \\ &+\Big(\sum_{i=0}^{n-1}\binom{n-1}{i}A^{n-1-i}P^i\Big)D(A+P) \\ &+D(A+P)\Big(\sum_{i=0}^{n-1}\binom{n-1}{i}(A^*)^{n-1-i}(P^*)^i\Big) \\ &+(A+P)\Big(\sum_{i=0}^{n-1}\binom{n-1}{i}D(A^{n-1-i}P^i)\Big). \end{aligned}$$

Rearranging the above relation in the sense of collecting together terms involving equal number of factors of P, we obtain $\sum_{i=1}^{n-1} f_i(A,P)=0$, where

$$\begin{aligned} f_i(A,P) &= 2\binom{n}{i}D(A^{n-i}P^i)-\binom{n-1}{i}\big(D(A^{n-1-i}P^i)A^*+A^{n-1-i}P^iD(A)\big)\\ &-\binom{n-1}{i-1}\big(D(A^{n-i}P^{i-1})P^*+(A^{n-i}P^{i-1})D(P)\big)\\ &-\binom{n-1}{i}\big(D(A)(A^*)^{n-1-i}(P^*)^i+AD(A^{n-1-i}P^i)\big)\\ &-\binom{n-1}{i-1}\big(D(P)(A^*)^{n-i}(P^*)^{i-1}+PD(A^{n-i}P^{i-1})\big)\end{aligned}$$

Replacing A by $A+2P$, $A+3P$,..., $A+(n-1)P$ in the relation (30) and expressing the resulting system of $n-1$ homogeneous equations of variables $f_i(A,P)$ for $i=1,2,...n-1$, we see that the coefficient matrix of the system of $n-1$ homogenous equations is a Van-der Monde matrix

$$\begin{pmatrix} 1 & 1 & \dots & 1\\ 2 & 2^2 & \dots & 2^{n-1}\\ \vdots & \vdots & \ddots & \vdots\\ n-1 & (n-1)^2 & \dots & (n-1)^{n-1}\end{pmatrix}.$$

Since the determinant of this matrix is different from zero, it follows immediately that the system has only a trivial solution. In particular

$$\begin{aligned} f_{n-1}(A,P) &= 2\binom{n}{n-1}D(A)-\binom{n-1}{n-1}D(P)A^*-\binom{n-1}{n-2}D(A)P^*\\ &-\binom{n-1}{n-1}PD(A)-\binom{n-1}{n-2}AD(P)-\binom{n-1}{n-1}D(A)P^*\\ &-\binom{n-1}{n-2}D(P)A^*-\binom{n-1}{n-1}AD(P)-\binom{n-1}{n-2}PD(A).\end{aligned}$$

The above relation reduces to

$$2D(A) = D(A)P^*+AD(P)+D(P)A^*+PD(A) \tag{32}$$

Replace A by A^2 in the above relation, to obtain

$$2D(A^2) = D(A^2)P^*+A^2D(P)+D(P)(A^*)^2+PD(A^2) \tag{33}$$

As the previously mentioned system of $n-1$ homogeneous equtions has only a trivial solution, we also obtain

$$\begin{aligned} f_{n-2}(A,P) &= 2\binom{n}{n-2}D(A^2)-\binom{n-1}{n-2}D(A)A^*-\binom{n-1}{n-3}D(A^2)P^*\\ &-\binom{n-1}{n-2}AD(A)-\binom{n-1}{n-3}A^2D(P)-\binom{n-1}{n-2}D(A)A^*\\ &-\binom{n-1}{n-3}D(P)(A^*)^2\binom{n-1}{n-2}AD(A)-\binom{n-1}{n-3}PD(A^2)\\ &= 0.\end{aligned}$$

The above relation reduces to

$$\begin{aligned} n(n-1)D(A^2) &= 2(n-1)\Big(D(A)A^* + AD(A)\Big) \\ &\quad + \binom{n-1}{n-3}\Big(D(A^2)P^* + A^2D(P) + D(P)(A^*)^2 + PD(A^2)\Big). \end{aligned}$$

Applying the relation (33) in the above relation, we obtain

$$n(n-1)D(A^2) = 2(n-1)\big(D(A)A^* + AD(A)\big) + (n-1)(n-2)D(A^2),$$

which reduces to

$$D(A^2) = D(A)A^* + AD(A) \tag{34}$$

From the relation (32) one can conclude that D maps $\mathcal{F}(\mathcal{H})$ into itself. We therefore have a linear mapping D which maps $\mathcal{F}(\mathcal{H})$ into itself satisfying the relation (34) for all $A \in \mathcal{F}(\mathcal{H})$. Hence D is a Jordan $*$-derivation on $\mathcal{F}(\mathcal{H})$. Applying Theorem 3.1 one can conclude that D is of the form

$$D(A) = AB - BA^* \tag{35}$$

for all $A \in \mathcal{F}(\mathcal{H})$ and some fixed $B \in \mathcal{B}(\mathcal{H})$. It remains to prove that the relation (35) holds for all $A \in \mathcal{A}(\mathcal{H})$ as well. For this purpose we define $D_0 : \mathcal{A}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by $D_0(A) = AB - BA^*$ and consider $D_1 = D - D_0$. Indeed, the mapping D_1 is linear, satisfies the relation (30) and it vanishes on $\mathcal{F}(\mathcal{H})$. Now we prove that D_1 vanishes on $\mathcal{A}(\mathcal{H})$ also. Let $A \in \mathcal{A}(\mathcal{H})$ and P be an idempotent operator of rank one. Let us introduce $S \in \mathcal{A}(\mathcal{H})$ by $S = A + PAP - (AP + PA)$. We have $SP = PS = 0$. It is easy to prove that $D_1(S) = D_1(A)$ and $D_1(S^{n-1}) = D_1(A^{n-1})$. By the relation (30) we now have

$$\begin{aligned} &D_1(S^{n-1})S^* + S^{n-1}D_1(S) + D_1(S)(S^*)^{n-1} + SD_1(S^{n-1}) \\ &= 2D_1(S^n) = 2D_1(S^n + P) = 2D_1((S+P)^n) \\ &= D_1((S+P)^{n-1})(S+P)^* + (S+P)^{n-1}D_1(S+P) \\ &\quad + D_1(S+P)((S+P)^*)^{n-1} + (S+P)D_1((S+P)^{n-1}) \\ &= D_1(S^{n-1})S^* + D_1(S^{n-1})P^* + S^{n-1}D_1(S) + PD_1(S) \\ &\quad + D_1(S)(S^*)^{n-1} + D_1(S)P^* + SD_1(S^{n-1}) + PD_1(S^{n-1}). \end{aligned}$$

From the above relation it follows that

$$D_1(S^{n-1})P^* + PD_1(S) + D_1(S)P^* + PD_1(S^{n-1}) = 0.$$

Since $D_1(S) = D_1(A)$, we can rewrite the above relation as

$$D_1(A^{n-1})P^* + PD_1(A) + D_1(A)P^* + PD_1(A^{n-1}) = 0. \tag{36}$$

Putting $2A$ for A in the above relation, we obtain

$$2^{n-1}D_1(A^{n-1})P^* + 2PD_1(A) + 2D_1(A)P^* + 2^{n-1}PD_1(A^{n-1}) = 0. \tag{37}$$

In case $n = 2$, the relation (36) implies that

$$PD_1(A) + D_1(A)P^* = 0. \tag{38}$$

In case$n > 2$, the relations (36) and (37) give the relation (38). Multiplying the above relation from left side by P and right side by P^*, we obtain

$$PD_1(A)P^* \quad = \quad 0.$$

Left multiplication by P in the relation (38) and using using the above relation, we obtain

$$PD_1(A) \quad = \quad 0.$$

for all $A \in \mathcal{A}(\mathcal{H})$. Since P is an arbitrary an idempotent operator of rank one, we have $D_1(A) = 0$ for all $A \in \mathcal{A}(\mathcal{H})$, which completes the proof of the theorem. □

We conclude the paper with the following purely algebraic conjecture.

CONJECTURE 3.6 Let $\mathcal{R}$ be a semiprime $*$-ring with a suitable torsion restriction and $D, G : \mathcal{R} \to \mathcal{R}$ be additive mappings such that either the relations

$$D(x^{2n+1}) = D(x^{2n})x^* + x^{2n}G(x),$$

$$G(x^{2n+1}) = G(x^{2n})x^* + x^{2n}D(x)$$

hold for all $x \in \mathcal{R}$, or the relations

$$D(x^{2n+1}) = D(x)(x^*)^{2n} + xG(x^{2n}),$$

$$G(x^{2n+1}) = G(x)(x^*)^{2n} + xD(x^{2n})$$

hold for all $x \in \mathcal{R}$ and some fixed integer $n \geq 1$. Then D and G are Jordan $*$-derivations and $D = G$.

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