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A characterization of regular, intra-regular, left quasi-regular and semisimple hypersemigroups in terms of fuzzy sets

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Abstract. We prove that an hypersemigroup H is regular if and only, for any fuzzy subset f of H, we have $f \preceq f \circ 1 \circ f$ and it is intra-regular if and only if, for any fuzzy subset f of H, we have $f \preceq 1 \circ f \circ f \circ 1$. An hypersemigroup H is left (resp. right) quasi-regular if and only if, for any fuzzy subset f of H we have $f \preceq 1 \circ f \circ 1 \circ f$ (resp. $f \preceq f \circ 1 \circ f \circ 1$) and it is semisimple if and only if, for any fuzzy subset f of H we have $f \preceq 1 \circ f \circ 1 \circ f \circ 1$. The characterization of regular and intra-regular hypersemigroups in terms of fuzzy subsets are very useful for applications.

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1 Introduction

In our paper in [5] we gave, among others, a characterization of regular ordered semigroups in terms of fuzzy subsets which is very useful for applications. Using that equivalent definition of regular semigroups many known results on semigroups (without order) or on ordered semigroups can be drastically simplified. In [1], we characterized the left quasi-regular and semisimple ordered semigroups in terms of fuzzy sets. In the present paper we characterize the regular, intra-regular, left quasi-regular and semisimple hypersemigroups using fuzzy sets. According to the equivalent definition of regularity and intra-regularity given in the present paper, many results on hypersemigroups can be drastically simplified. The paper has been inspired by our paper [5], and the aim is to show the way we pass from fuzzy semigroups to fuzzy hypersemigroups. In fact, the results on semigroups or ordered semigroups can be transferred to hypersemigroups in the way indicated in the present paper.

2 Main results

An *hypergroupoid* is a nonempty set H with an hyperoperation

$$\circ : H \times H \to \mathcal{P}^*(H)$$
$$(a, b) \to a \circ b$$

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on H and an operation

$$\begin{aligned} * : & \mathcal{P}^*(H) \times \mathcal{P}^*(H) \to \mathcal{P}^*(H) \\ & (A, B) \to A * B \end{aligned}$$

on $\mathcal{P}^*(H)$ (induced by the operation of H) such that

$$A * B = \bigcup_{(a,b) \in A \times B} (a \circ b)$$

for every $A, B \in \mathcal{P}^*(H)$ ($\mathcal{P}^*(H)$ denotes the set of nonempty subsets of H) (cf. also [2, 3]).

As the operation "$*$" depends on the hyperoperation "$\circ$", an hypergroupoid can be also denoted with $(H, \circ)$ (instead of $(H, \circ, *)$).

If H is an hypergroupoid, then, for any $x, y \in H$, we have $x \circ y = \{x\} * \{y\}$. Indeed,

$$\{x\} * \{y\} = \bigcup_{u \in \{x\}, v \in \{y\}} u \circ v = x \circ y.$$

An hypergroupoid H is called an *hypersemigroup* if

$$(x \circ y) * \{z\} = \{x\} * (y \circ z)$$

for every $x, y, z \in H$. Since $x \circ y = \{x\} * \{y\}$ for any $x, y \in H$, an hypergroupoid H is an hypersemigroup if and only if, for any $x, y, z \in H$, we have

$$\Big(\{x\} * \{y\}\Big) * \{z\} = \{x\} * \Big(\{y\} * \{z\}\Big).$$

If H is an hypersemigroup and $A, B, C \in \mathcal{P}^*(H)$, then we have

$$(A * B) * C = A * (B * C) = \bigcup_{(a,b,c) \in A \times B \times C} \Big(\{a\} * \{b\} * \{c\}\Big).$$

Thus we can write $(A * B) * C = A * (B * C) = A * B * C$ [3, Lemma 2.4]. Using induction, for any finite family $A_1, A_2, ..., A_n$ of elements of $\mathcal{P}^*(H)$, we have

$$A_1 * A_2 * ..... * A_n = \bigcup_{(a_1, a_2 ... a_n) \in A_1 \times A_2 \times ... \times A_n} \Big(\{a_1\} * \{a_2\} * ... * \{a_n\}\Big).$$

Following Zadeh, if $(H, \circ)$ is an hypergroupoid, we say that f is a fuzzy subset of H (or a fuzzy set in H) if f is a mapping of H into the real closed interval $[0, 1]$ of real numbers, that is $f : H \to [0, 1]$. For an element a of H, we denote by A_a the subset of $H \times H$ defined as follows:

$$A_a := \{(y, z) \in H \times H \mid a \in y \circ z\}.$$

For two fuzzy subsets f and g of H, we denote by $f \circ g$ the fuzzy subset of H defined as follows:

$$\begin{aligned} f \circ g : & H \to [0, 1] \\ & a \to \begin{cases} \bigvee\limits_{(y,z) \in A_a} \min\{f(y), g(z)\} & \text{if } A_a \neq \emptyset \\ 0 \quad \text{if } A_a = \emptyset. \end{cases} \end{aligned}$$

We denote the hyperoperation on H and the multiplication between the two fuzzy subsets of H with the same symbol (no confusion is possible). Denote with $F(H)$ the set of all fuzzy subsets of H and with "$\preceq$" the order relation on $F(H)$ defined by:

$$f \preceq g \iff f(x) \leq g(x) \text{ for every } x \in H.$$

We finally denote with 1 the fuzzy subset of H defined by:

$$1 : H \to [0,1] \mid x \to 1(x) := 1.$$

Clearly, the fuzzy subset 1 is the greatest element of the ordered set $(F(H), \preceq)$ (that is, $1 \succeq f$, for every $f \in F(H)$).

For an hypergroupoid H, we denote with f_a the fuzzy subset of H defined by:

$$\begin{aligned} f_a :& H \to [0,1] \\ & x \to f_a(x) := \begin{cases} 1 & \text{if} \quad x = a \\ 0 & \text{if} \quad x \neq a. \end{cases} \end{aligned}$$

The following proposition, though clear, plays an essential role in the theory of hypergroupoids.

PROPOSITION 2.1 *Let $(H, \circ)$ be an hypergroupoid, $x \in H$ and $A, B \in \mathcal{P}^*(H)$. Then we have $x \in A * B$ if and only if $x \in a \circ b$, for some $a \in A$, $b \in B$.*

LEMMA 2.2 *If H is an hypergroupoid and A, B, C nonempty subsets of H, then $A \subseteq B$ implies $A * C \subseteq B * C$ and $C * A \subseteq C * B$.*

LEMMA 2.3 *(cf. also [4, Proposition 9]) If H is an hypersemigroup, then the set of all fuzzy subsets of H is a semigroup.*

According to this lemma, for any fuzzy subsets f, g, h of H, we write $(f \circ g) \circ h = f \circ (g \circ h) := f \circ g \circ h$.

An hypersemigroup H is called *regular* if for every $a \in H$ there exists $x \in H$ such that

$$a \in (a \circ x) * \{a\}.$$

Equivalent definitions are the following:

1. $a \in \{a\} * H * \{a\}$ for every $a \in H$.

2. $A \subseteq A * H * A$ for every $A \in \mathcal{P}^*(H)$.

THEOREM 2.4 *An hypersemigroup H is regular if and only if, for any fuzzy subset f of H, we have*

$$f \preceq f \circ 1 \circ f.$$

Proof.

$\Longrightarrow$) Let f be a fuzzy subset of H and $a \in H$. Since H is regular, there exists $x \in H$ such that $a \in (a \circ x) * \{a\}$. Then, by Proposition 2.1, there exists $u \in a \circ x$ such that $a \in u \circ a$. Since $(u, a) \in A_a$, we have $A_a \neq \emptyset$ and

$$(f \circ 1 \circ f)(a) := \bigvee_{(y,z) \in A_a} \min\{(f \circ 1)(y), f(z)\} \geq \min\{(f \circ 1)(u), f(a)\}.$$

Since $(a, x) \in A_u$, we have $A_u \neq \emptyset$ and

$$(f \circ 1)(u) := \bigvee_{(y,z) \in A_u} \min\{f(y), 1(z)\} \geq \min\{f(a), 1(x)\} = f(a).$$

Thus we have

$$(f \circ 1 \circ f)(a) \geq \min\{(f \circ 1)(u), f(a)\} \geq \min\{f(a), f(a)\} = f(a).$$

$\Longleftarrow$) Let $a \in H$. Since f_a is a fuzzy subset of H, by hypothesis, we have $1 = f_a(a) \leq (f_a \circ 1 \circ f_a)(a)$. Since $f_a \circ 1 \circ f_a$ is a fuzzy subset of H, we have $(f_a \circ 1 \circ f_a)(a) \leq 1$. Thus we have

$$(f_a \circ 1 \circ f_a)(a) = 1.$$

If $A_a = \emptyset$, then $((f_a \circ 1) \circ f_a)(a) = 0$ which is impossible. Thus we have $A_a \neq \emptyset$. Then

$$((f_a \circ 1) \circ f_a)(a) = \bigvee_{(x,y) \in A_a} \min\{(f_a \circ 1)(x), f_a(y)\}.$$

Then there exists $(x, y) \in A_a$ such that $(f_a \circ 1)(x) \neq 0$ and $f_a(y) \neq 0$. Indeed, if $(f_a \circ 1)(x) = 0$ or $f_a(y) = 0$ for every $(x, y) \in A_a$, then $\min\{(f_a \circ 1)(x), f_a(y)\} = 0$ for every $(x, y) \in A_a$, then $((f_a \circ 1) \circ f_a)(a) = 0$ which is impossible.

Since $f_a(y) \neq 0$, we have $y = a$, then $(x, a) \in A_a$. Since $(f_a \circ 1)(x) \neq 0$, we get $A_x \neq \emptyset$. Since $A_x \neq \emptyset$, we have

$$(f_a \circ 1)(x) = \bigvee_{(b,c) \in A_x} \min\{f_a(b), 1(c)\} = \bigvee_{(b,c) \in A_x} f_a(b).$$

If $b \neq a$ for every $(b, c) \in A_x$, then $f_a(b) = 0$ for every $(b, c) \in A_x$, then $(f_a \circ 1)(x) = 0$ which is impossible. Hence there exists $(b, c) \in A_x$ such that $b = a$. Then $(a, c) \in A_x$. We have $(x, a) \in A_a$ and $(a, c) \in A_x$. So we obtain $a \in x \circ a$ and $x \in a \circ c$. Then we have

$$a \in x \circ a = \{x\} * \{a\} \subseteq (a \circ c) * \{a\},$$

where $c \in H$, so the hypersemigroup H is regular. $\square$

An hypersemigroup H is called *intra-regular* if, for every $a \in H$, there exist $x, y \in H$ such that

$$a \in (x \circ a) * (a \circ y).$$

Equivalent definitions are the following:

1. $a \in H * \{a\} * \{a\} * H$, for every $a \in H$.
2. $A \subseteq H * A * A * H$, for every $A \in \mathcal{P}^*(H)$.

THEOREM 2.5 *An hypersemigroup H is intra-regular if and only if, for any fuzzy subset f of H, we have*

$$f \preceq 1 \circ f \circ f \circ 1.$$

Proof.

$\Longrightarrow$) Let f be a fuzzy subset of H and $a \in H$. Since H is intra-regular, there exist $x, y \in H$ such that $a \in (x \circ a) * (a \circ y)$. Then, by Proposition 2.1, there exist $u \in x \circ a$ and $v \in a \circ y$ such that $a \in u \circ v$. Since $(u, v) \in A_a$, we have

$$\begin{aligned}(1 \circ f \circ f \circ 1)(a) &= \bigvee_{(y,z) \in A_a} \min\{(1 \circ f)(y), (f \circ 1)(z)\} \\ &\geq \min\{(1 \circ f)(u), (f \circ 1)(v)\}.\end{aligned}$$

Since $(x, a) \in A_u$, we have

$$(1 \circ f)(u) = \bigvee_{(y,z) \in A_u} \min\{1(y), f(z)\} \geq \min\{1(x), f(a)\} = f(a).$$

Since $(a, y) \in A_v$, we have

$$(f \circ 1)(v) = \bigvee_{(y,z) \in A_v} \min\{f(y), 1(z)\} \geq \min\{f(a), 1(y)\} = f(a).$$

Hence we obtain

$$(1 \circ f \circ f \circ 1)(a) \geq \min\{f(a), f(a)\} = f(a),$$

so $f \preceq 1 \circ f \circ f \circ 1$.

$\Longleftarrow$) Let $a \in H$. Since f_a is a fuzzy subset of H, by hypothesis, we have $1 = f_a(a) \leq (1 \circ f_a \circ 1 \circ f_a)(a) \leq 1$, thus $(1 \circ f_a \circ 1 \circ f_a)(a) = 1$. If $A_a = \emptyset$, then $(1 \circ f_a \circ 1 \circ f_a)(a) = 0$, which is impossible, thus $A_a \neq \emptyset$. Then

$$\Big((1 \circ f_a) \circ (f_a \circ 1)\Big)(a) = \bigvee_{(x,y) \in A_a} \min\{(1 \circ f_a)(x), (f_a \circ 1)(y)\}.$$

Then there exists $(x,y)\in A_a$ such that $(1\circ f_a)(x)\neq 0$ and $(f_a\circ 1)(y)\neq 0$ (otherwise, $(1\circ f_a\circ 1\circ f_a)(a)=0$ which is impossible). If $A_x=\emptyset$, then $(1\circ f_a)(x)=0$ which is not possible, thus $A_x\neq\emptyset$. Then

$$(1\circ f_a)\circ(x)=\bigvee_{(b,c)\in A_x}\min\{1(b),f_a(c)\}=\bigvee_{(b,c)\in A_x}f_a(c).$$

If $c\neq a$ for every $(b,c)\in A_x$, then $f_a(c)=0$ for every $(b,c)\in A_x$, then $(1\circ f_a)(x)=0$ which is impossible. Then there exists $(b,c)\in A_x$ such that $c=a$, so we get $(b,a)\in A_x$. If $A_y=\emptyset$, then $(f_a\circ 1)(y)=0$, which is impossible, thus $A_y\neq\emptyset$, and so

$$(f_a\circ 1)\circ(y)=\bigvee_{(u,d)\in A_y}\min\{f_a(u),1(d)\}=\bigvee_{(u,d)\in A_y}f_a(u).$$

If $u\neq a$ for every $(u,d)\in A_y$, then $f_a(u)=0$ for every $(u,d)\in A_y$, and then $(f_a\circ 1)(y)=0$ which is not possible. Thus there exists $(u,d)\in A_y$ such that $u=a$, then $(a,d)\in A_y$. We have $(x,y)\in A_a$, $(b,a)\in A_x$, $(a,d)\in A_y$, that is,

$$a\in x\circ y,\ x\in b\circ a\ \text{ and }\ y\in a\circ d.$$

Then $a\in x\circ y=\{x\}*\{y\}\subseteq(b\circ a)*(a\circ d)$, where $b,d\in H$, so H is intra-regular. □

An hypersemigroup H is called *left quasi-regular* if for every $a\in H$ there exist $x,y\in H$ such that

$$a\in(x\circ a)*(y\circ a).$$

Equivalent definitions are the following:

1. $a\in H*\{a\}*H*\{a\}$ for every $a\in H$.
2. $A\subseteq H*A*H*A$ for every $A\in\mathcal{P}^*(H)$.

THEOREM 2.6 *An hypersemigroup H is left quasi-regular if and only if, for any fuzzy subset f of H, we have*

$$f\preceq 1\circ f\circ 1\circ f.$$

Proof.

$\Longrightarrow$) Let $a\in H$. Then $f(a)\leq(1\circ f\circ 1\circ f)(a)$. In fact, since H is left quasi-regular, there exist $x,y\in H$ such that $a\in(x\circ a)*(y\circ a)$. Then there exist $u\in x\circ a$ and $v\in y\circ a$ such that $a\in u\circ v$. Since $a\in u\circ v$, we have $(u,v)\in A_a$. Since $(u,v)\in A_a$, A_a is a nonempty set and we have

$$\begin{aligned}(1\circ f\circ 1\circ f)(a)&=\bigvee_{(y,z)\in A_a}\min\{(1\circ f)(y),(1\circ f)(z)\}\\&\geq\min\{(1\circ f)(u),(1\circ f)(v)\}.\end{aligned}$$

Since $u \in x \circ a$, we have $(x,a) \in A_u$. Then A_u is a nonempty set and we have

$$(1 \circ f)(u) = \bigvee_{(s,t)\in A_u} \min\{(1(s), f(t)\} \geq \min\{1(x), f(a)\} = f(a).$$

Since $v \in y \circ a$, we have $(y,a) \in A_v$. Then A_v is a nonempty set and we have

$$(1 \circ f)(v) = \bigvee_{(l,k)\in A_v} \min\{(1(l), f(k)\} \geq \min\{1(y), f(a)\} = f(a).$$

Thus we get

$$(1 \circ f \circ 1 \circ f)(a) \geq \min\{f(a), f(a)\} = f(a),$$

so $f \preceq 1 \circ f \circ 1 \circ f$.

$\Longleftarrow$) Let $a \in H$. By hypothesis, we have $1 = f_a(a) \leq (1 \circ f_a \circ 1 \circ f_a)(a) \leq 1$, so

$$(1 \circ f_a \circ 1 \circ f_a)(a) = 1.$$

Since $(1 \circ f_a \circ 1 \circ f_a)(a) \neq 0$, we have $A_a \neq \emptyset$. Then

$$\Big((1 \circ f_a) \circ (1 \circ f_a)\Big)(a) = \bigvee_{(x,y)\in A_a} \min\{(1 \circ f_a)(x), (1 \circ f_a)(y)\}.$$

Then there exists $(x,y) \in A_a$ such that $(1 \circ f_a)(x) \neq 0$ and $(1 \circ f_a)(y) \neq 0$ (otherwise $(1 \circ f_a \circ 1 \circ f_a)(a) = 0$ which is impossible). Since $(1 \circ f_a)(x) \neq 0$, we have $A_x \neq \emptyset$. Then

$$(1 \circ f_a)(x) = \bigvee_{(b,c)\in A_x} \min\{1(b), f_a(c)\} = \bigvee_{(b,c)\in A_x} f_a(c).$$

If $c \neq a$ for every $(b,c) \in A_x$, then $f_a(c) = 0$ for every $(b,c) \in A_x$, then $(1 \circ f_a)(x) = 0$ which is impossible. Then there exists $(b,c) \in A_x$ such that $c = a$. Then we have $(b,a) \in A_x$. Since $(1 \circ f_a)(y) \neq 0$, we have $A_y \neq \emptyset$, then

$$(1 \circ f_a)(y) = \bigvee_{(c,d)\in A_y} \min\{1(c), f_a(d)\} = \bigvee_{(c,d)\in A_y} f_a(d).$$

If $d \neq a$ for every $(c,d) \in A_y$, then $(1 \circ f_a)(y) = 0$ which is impossible. Thus there exists $(c,d) \in A_y$ such that $d = a$. Thus we get $(c,a) \in A_y$.

We have $(x,y) \in A_a$, $(b,a) \in A_x$, $(c,a) \in A_y$, that is $a \in x \circ y$, $x \in b \circ a$, $y \in c \circ a$. Thus we have

$$a \in x \circ y = \{x\} * \{y\} \subseteq (b \circ a) * (c \circ a),$$

where $b, c \in H$, so H is left quasi-regular. □

An hypersemigroup H is called *right quasi-regular* if for every $a \in H$ there exist $x, y \in H$ such that

$$a \in (a \circ x) * (a \circ y).$$

Equivalent definitions are the following:

1. $a \in \{a\} * H * \{a\} * H$ for every $a \in H$.
2. $A \subseteq A * H * A * H$ for every $A \in \mathcal{P}^*(H)$.

The right analogue of the above theorem also holds, and we have the following theorem.

THEOREM 2.7 *An hypersemigroup H is right quasi-regular if and only if, for any fuzzy subset f of H, we have*

$$f \preceq f \circ 1 \circ f \circ 1.$$

An hypersemigroup H is called *semisimple* if for every $a \in H$ there exist $x, y, z \in H$ such that

$$a \in (x \circ a) * (y \circ a) * \{z\}.$$

Equivalent definitions are the following:

1. $a \in H * \{a\} * H * \{a\} * H$ for every $a \in H$.
2. $A \subseteq H * A * H * A * H$ for every $A \in \mathcal{P}^*(H)$.

THEOREM 2.8 *An hypersemigroup H is semisimple if and only if, for any fuzzy subset f of H, we have*

$$f \preceq 1 \circ f \circ 1 \circ f \circ 1.$$

Proof.

$\Longrightarrow$) Let $a \in H$. Since H is semisimple, there exist $x, y \in H$ such that $a \in (x \circ a) * (y \circ a) * \{z\} = \Big((x \circ a) * \{y\}\Big) * (a \circ z)$. By Proposition 2.1, there exist $u \in (x \circ a) * \{y\}$ and $v \in a \circ z$ such that $a \in u \circ v$. Since $u \in (x \circ a) * \{y\}$, by Proposition 2.1, there exists $w \in x \circ a$ such that $u \in w \circ y$. Thus we have

$$a \in u \circ v,\ u \in w \circ y,\ w \in x \circ a,\ v \in a \circ z.$$

Since $a \in u \circ v$, we have $(u, v) \in A_a$. Then $A_a \neq \emptyset$ and

$$\begin{aligned}(1 \circ f \circ 1 \circ f \circ 1)(a) &= \bigvee_{(c,d) \in A_a} \min\{(1 \circ f \circ 1)(c), (f \circ 1)(d)\} \\ &\geq \min\{(1 \circ f \circ 1)(u), (f \circ 1)(v)\}.\end{aligned}$$

Since $u \in w \circ y$, we have $(w, y) \in A_u$. Then $A_u \neq \emptyset$ and

$$(1 \circ f \circ 1)(u) = \bigvee_{(s,t) \in A_u} \min\{(1 \circ f)(s), 1(t)\} \geq (1 \circ f)(w).$$

Since $w \in x \circ a$, we have $(x, a) \in A_w$. Then $A_w \neq \emptyset$ and

$$(1 \circ f)(w) = \bigvee_{(\xi,\zeta) \in A_w} \min\{(1(\xi), f(\zeta)\} \geq \min\{1(x), f(a)\} = f(a).$$

Since $v \in a \circ z$, we have $(a, z) \in A_v$. Then $A_v \neq \emptyset$ and

$$(f \circ 1)(v) = \bigvee_{(k,h) \in A_v} \min\{f(k), 1(h)\} = \bigvee_{(k,h) \in A_v} f(k) \geq f(a).$$

Thus we have

$$(1 \circ f \circ 1 \circ f \circ 1)(a) \geq \min\{f(a), f(a)\} = f(a),$$

so $f \preceq 1 \circ f \circ 1 \circ f \circ 1$.

$\Longleftarrow$) Let $a \in H$. By hypothesis, we have $(1 \circ f_a \circ 1 \circ f_a \circ 1)(a) = 1$. Then $A_a \neq \emptyset$ and

$$(1 \circ f_a \circ 1 \circ f_a \circ 1)(a) = \bigvee_{(x,y) \in A_a} \min\{(1 \circ f_a \circ 1)(x), (f_a \circ 1)(y)\}.$$

Then there exists $(x, y) \in A_a$ such that $(1 \circ f_a \circ 1)(x) \neq 0$ and $(f_a \circ 1)(y) \neq 0$. Since $(f_a \circ 1)(y) \neq 0$, we have $A_y \neq \emptyset$ and

$$(f_a \circ 1)(y) = \bigvee_{(b,c) \in A_y} \min\{f_a(b), 1(c)\} = \bigvee_{(b,c) \in A_y} f_a(b).$$

If $b \neq a$ for every $(b, c) \in A_y$, then $f_a(b) = 0$ for every $(b, c) \in A_y$, then $(f_a \circ 1)(y) = 0$ which is impossible. Thus there exists $(b, c) \in A_y$ such that $b = a$, then $(a, c) \in A_y$. Since $(1 \circ f_a \circ 1)(x) \neq 0$, we have $A_x \neq \emptyset$ and

$$(1 \circ f_a \circ 1)(x) = \bigvee_{(\rho,\lambda) \in A_x} \min\{1(\rho), (f_a \circ 1)(\lambda)\} = \bigvee_{(\rho,\lambda) \in A_x} (f_a \circ 1)(\lambda).$$

If $(f_a \circ 1)(\lambda) = 0$ for every $(\rho, \lambda) \in A_x$, then $(1 \circ f_a \circ 1)(x) = 0$, which is impossible. Then there exists $(\rho, \lambda) \in A_x$ such that $(f_a \circ 1)(\lambda) \neq 0$. Since $(f_a \circ 1)(\lambda) \neq 0$, we have $A_\lambda \neq \emptyset$ and

$$(f_a \circ 1)(\lambda) = \bigvee_{(k,h) \in A_\lambda} \min\{f_a(k), 1(h)\} = \bigvee_{(k,h) \in A_\lambda} f_a(k).$$

If $a \neq k$ for every $(k, h) \in A_\lambda$, then $(f_a \circ 1)(\lambda) = 0$, which is impossible. Thus there exists $(k, h) \in A_\lambda$ such that $a = k$, then $(a, h) \in A_\lambda$. We have

$$a \in x \circ y,\ y \in a \circ c,\ x \in \rho \circ \lambda,\ \lambda \in a \circ h.$$

Then we have

$$\begin{aligned} a \in \{x\} * \{y\} &\subseteq \{\rho\} * \{\lambda\} * \{a\} * \{c\} \subseteq \{\rho\} * \{a\} * \{h\} * \{a\} * \{c\} \\ &= (\rho \circ a) * (h \circ a) * \{c\}, \end{aligned}$$

where $\rho, h, c \in H$, so H is semisimple. $\square$

Remark. The characterizations of regular and intra-regular hypersemigroups in terms of fuzzy sets given in this paper are very useful for further investigation. Exactly as in semigroups, using these definitions, many proofs on hypersemigroups can be drastically simplified. Let us just give an example to clarify. We begin with the definition of fuzzy right and fuzzy left ideals of hypersemigroups. If H is an hypersemigroup, a fuzzy subset of H is called a *fuzzy right ideal* of H if $f(x \circ y) \geq f(x)$ for every $x, y \in H$, in the sense that if $x, y \in H$ and $u \in x \circ y$, then $f(u) \geq f(x)$. A fuzzy subset of H is called a *fuzzy left ideal* of H if $f(x \circ y) \geq f(y)$ for every $x, y \in H$, that is, if $x, y \in H$ and $u \in x \circ y$, then $f(u) \geq f(y)$. A fuzzy subset f of H is a fuzzy right (resp. fuzzy left) ideal of H if and only if $f \circ 1 \preceq f$ (resp. $1 \circ f \preceq f$) [4]. For two fuzzy subsets f and g of an hypergroupoid H we denote with $f \wedge g$ the fuzzy subset of H defined as follows:

$$f \wedge g : H \to [0,1] \mid x \to (f \wedge g)(x) := \min\{f(x), g(x)\}.$$

(see also [4]). One can easily prove that the fuzzy subset $f \wedge g$ is the infimum of the fuzzy subsets f and g, so we write $f \wedge g = \inf\{f, g\}$. Using the definition of regular hypersemigroups by means of fuzzy subsets given in the present paper, one can immediately see that if H is a regular hypersemigroup then, for every fuzzy right ideal f and every fuzzy left ideal g of H, we have $f \wedge g = f \circ g$. In fact, by Theorem 2.4,

$$\begin{aligned} f \wedge g &\preceq (f \wedge g) \circ 1 \circ (f \wedge g) \preceq (f \circ 1) \circ g \preceq f \circ g \\ &\preceq (f \circ 1) \wedge (1 \circ g) \preceq f \wedge g, \end{aligned}$$

thus $f \wedge g = f \circ g$. One can immediately see that, if H is an intra-regular hypersemigroup, then, for every right ideal f and every left ideal g of H, we have $f \wedge g \preceq g \circ f$. Indeed, by Theorem 2.5, we have

$$f \wedge g \preceq 1 \circ (f \wedge g) \circ (f \wedge g) \circ 1 \preceq (1 \circ g) \circ (f \circ 1) \preceq g \circ f.$$

We can immediately have the following: If H is a regular hypersemigroup and f a fuzzy bi-ideal of H (i.e. $f \circ 1 \circ f \preceq f$ [4]), then there exists a fuzzy right ideal h and a fuzzy left ideal g of H such that $f = h \circ g$. Indeed, by Theorem 2.4, we have

$$f = f \circ 1 \circ f = f \circ 1 \circ (f \circ 1 \circ f) \preceq (f \circ 1) \circ (1 \circ f),$$

where $f \circ 1$ is a fuzzy right ideal and $1 \circ f$ is a fuzzy left ideal of H.

As a conclusion, if we wish to get a result on an hypersemigroup then, exactly as in the case of Gamma semigroups, we have to solve the problem for the semigroup and then transfer the proof to the hypersemigroup. The same holds if we replace the word “hypersemigroup" by “ordered hypersemigroup". Further interesting information concerning this structure will be given in a forthcoming paper.

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