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On permutations avoiding 1243, 2134, and another 4-letter pattern

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Abstract. We enumerate permutations avoiding 1243, 2134, and a third 4-letter pattern τ, a step toward the goal of enumerating avoiders for all triples of 4-letter patterns. The enumeration is already known for all but three patterns τ, which are treated in this paper.

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1 Introduction

This paper is a companion to [2] which enumerates the permutations avoiding 1324, 2143, and a third 4-letter pattern τ, part of a project to enumerate avoiders for *all* triples of 4-letter patterns. As usual, S_n denotes the set of permutations of $[n] = \{1, 2, \ldots, n\}$, considered as lists (words) of distinct letters. For a permutation π to avoid a pattern $\tau \in S_k$ means that π contains no k-letter subsequence whose standardization (replace smallest letter by 1, second smallest by 2, and so on) is τ. For patterns $\tau_1, \ldots, \tau_r$, $S_n(\tau_1, \ldots, \tau_r)$ denotes the set of permutations of $[n]$ that avoid each of $\tau_1, \ldots, \tau_r$. Here, we count the set $S_n(1243, 2134, \tau)$ for all 22 permutations $\tau \in S_4 \backslash \{1243, 2134\}$ (for counting the set $S_n(T)$ with $T \subseteq S_4$, see [1, 3, 4, 5, 6, 7, 8]). The three involutions reverse, complement, invert on permutations generate a dihedral group that divides pattern sets into so-called symmetry classes. All pattern sets in a symmetry class have the same counting sequence for their avoiders. The pattern sets with a given counting sequence form a Wilf class, by definition. We say a Wilf class is *big* if it contains more than one symmetry class. All 242 big Wilf classes of triples of 4-letter patterns are enumerated in [6]. Some small Wilf classes have been enumerated [2].

Table 1 below lists the generating function $F_{1243,2134,\tau}(x)$ to count $\{1243, 2134, \tau\}$-avoiders for each of the 22 permutations τ. The 22 triples $\{1243, 2134, \tau\}$ lie in precisely 11 Wilf classes, of which 3 are

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big, hence covered by [6], 5 are small but can be counted by the INSENC algorithm (INSENC refers to regular insertion encodings, see [9]), and 3 are small and not treatable by the INSENC algorithm. These 3 are the triples with $\tau = 3412$, $\tau = 2341$ and $\tau = 1423$, which are treated in turn in Section 2. Our method is to consider the left-right maxima of an avoider when $\tau = 3412$ and to focus on the initial letters in the other two cases. We use the usual left-right maxima decomposition of a nonempty permutation π: $\pi = i_1\pi^{(1)}i_2\pi^{(2)}\cdots i_m\pi^{(m)}$ where $i_1,\dots,i_m$ are letters, $\pi^{(1)},\dots,\pi^{(m)}$ are words, $i_1 < i_2 < \cdots < i_m$ and $i_j > \max(\pi^{(j)})$ for $1 \le j \le m$. Then $i_1, i_2, \dots, i_m$ are the left-right maxima of π.

Throughout, $C(x) = \frac{1-\sqrt{1-4x}}{2x}$ denotes the generating function for the Catalan numbers $C_n := \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n-1}$. The identity $xC(x)^2 = C(x) - 1$ is used to simplify results.

τ	$F_{1243,2134,\tau}(x)$	Reference	Wilf class
4321	$\frac{-9x^7+24x^6+23x^5+8x^4+2x^3+2x^2-2x+1}{(1-x)^3}$	INSENC	7
3421, 4312	$\frac{-3x^7-5x^6+3x^5+10x^4-11x^3+11x^2-5x+1}{(1-x)^6}$	INSENC	9
4231	$\frac{4x^9-11x^8+10x^7+2x^6-7x^5+21x^4-22x^3+16x^2-6x+1}{(1-x)^7}$	INSENC	14
3412	$\frac{1-8x+28x^2-54x^3+65x^4-49x^5+18x^6-7x^7+2x^8}{(1-x)^7(1-2x)}$	Theorem 2.4	15
2431, 3241, 4132, 4213	$\frac{x^{10}-4x^9-6x^8+68x^7-186x^6+291x^5-283x^4+170x^3-61x^2+12x-1}{(2x-1)^2(x^2-3x+1)^2(x-1)^3}$	INSENC	53
1432, 3214	$\frac{1-4x+4x^2-3x^3+x^4}{(1-x+x^2)(1-4x+2x^2)}$	[6]	112
2341, 4123	$\frac{(1-x)(1-2x+2x^2)(1-2x+x^3+x^5)C(x)-x(1-2x+x^3+x^4-2x^5+2x^6)}{(1-x)^3(1-2x)(1-x-x^2)}$	Theorem 2.11	134
2413, 3142	$\frac{(1-3x+x^2)(1-2x-x^2)}{(1-x)(1-5x+5x^2+2x^3-x^4)}$	INSENC	138
1342, 1423, 2314, 3124	$\frac{1-x(1-x)C(x)}{(1-x)(2-C(x))+x^2}$	Theorem 2.15	207
2143	$\frac{1-4x+2x^2}{(1-x)(1-4x+x^2)}$	[6]	215
1234, 1324	$\frac{2(1-4x)}{2-9x+4x^2-x\sqrt{1-4x}}$	[6]	233

Table 1: Triples of 4-letter patterns containing 1243,2134, divided into Wilf classes.

2 Proofs

2.1 Case 15: $T = \{1243, 2134, 3412\}$.

We count T-avoiders by number of left-right maxima. Let $G_m(x)$ denote the generating function for T-avoiders with exactly m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xF_T(x)$.

LEMMA 2.1 *For* $m \geq 4$, $G_m(x) = \frac{x^m(1+x)}{(1-x)^3}$.

Proof. Suppose $\pi = i_1\pi^{(1)} \cdots i_m\pi^{(m)} \in S_n(T)$ with $m \geq 4$ left-right maxima. Since π avoids T, we have that $\pi^{(s)} = \emptyset$ for all $s = 1, 2, \ldots, m-2$ and $\pi^{(m-1)}, \pi^{(m)} > i_2$. Moreover, $\pi^{(m-1)}\pi^{(m)}$ is decreasing. Thus, by considering whether $\pi^{(m)}$ has a letter between i_1 and i_2 or not, we obtain that $G_m(x) = \frac{x^m}{(1-x)^2}\left(1 + \frac{2x}{1-x}\right)$. □

LEMMA 2.2 *We have* $G_3(x) = \frac{x^3(1-x+x^2+x^3)}{(1-x)^5}$.

Proof. Suppose $\pi = i_1\pi^{(1)}i_2\pi^{(2)}i_3\pi^{(3)} \in S_n(T)$ with exactly 3 left-right maxima. Since π avoids T, we have that $\pi^{(1)} = \emptyset$ and $\pi^{(2)} > i_2$. We consider four cases:

- $\pi^{(2)} = \emptyset$ and $\pi^{(3)}$ has no letter between i_1 and i_2: Since π avoids 3412, we can express π as $\pi = i_1i_2\pi^{(2)}i_3(i_1-1)\cdots 1$ where $\pi^{(2)}$ avoids $\{132, 213, 3412\}$. By a simple decomposition, we see that $K(x) = F_{\{132,213,3412\}}(x) = \frac{1}{1-x} + \frac{x^2}{(1-x)^3}$. Thus, we have a contribution of $\frac{x^3}{1-x}K(x)$.

- $\pi^{(2)} = \emptyset$ and $\pi^{(3)}$ has a letter between i_1 and i_2: Again, in this subcase, π can be expressed as $\pi = i_1i_2\pi^{(2)}i_3i'(i'-1)\cdots(i_1+1)(i_1-1)\cdots 21$ where $i_2 > \pi^{(2)} > i'$ and $\pi^{(2)}$ avoids $\{132, 213, 3412\}$. Thus, we have a contribution of $\frac{x^4}{(1-x)^2}K(x)$.

- $\pi^{(2)} \neq \emptyset$ and $\pi^{(3)}$ has no letter between i_1 and i_2: Similarly, in this subcase, π can be written as $\pi = i_1i_2\pi^{(2)}(i_1-1)\cdots(i'+1)i'i_3(i'-1)\cdots 21$ where $\pi^{(2)}$ avoids $\{132, 213, 3412\}$. Thus, we have a contribution of $\frac{x^4}{(1-x)^2}K(x)$.

- $\pi^{(2)} \neq \emptyset$ and $\pi^{(3)}$ has a letter between i_1 and i_2: Since π avoids 3412, we can write π as $i_1i_2(i_2-1)\cdots i_2'(i_1-1)(i_1-2)\cdots i_1'i_3(i_2'-1)\cdots(i_1+1)(i_1'-1)\cdots 21$. Thus, we have a contribution of $\frac{x^5}{(1-x)^4}$.

Hence, $G_3(x) = \frac{x^3}{1-x}K(x) + \frac{2x^4}{(1-x)^2}K(x) + \frac{x^5}{(1-x)^4}$, which simplifies to the stated expression. □

LEMMA 2.3 *We have*

$$G_2(x) = \frac{x^2(1-5x+13x^2-16x^3+8x^4-7x^5+2x^6)}{(1-x)^6(1-2x)}.$$

Proof. Let us write $G_2(x) = H(x) + J(x) + P(x)$, where $H(x)$ (respectively, $J(x)$ and $P(x)$) is the generating function for the number of T-avoiders π with exactly 2 left-right maxima of form $\pi = (n-1)\pi'n\pi''$ (respectively, $\pi = in\pi'$ with $i \leq n-2$, and $\pi = i\pi'n\pi''$ with $i \leq n-2$ and π' is not empty).

First, we find $H(x)$. Let $\pi = (n-1)\pi'n\pi'' \in S_n(T)$ with exactly 2 left-right maxima and suppose $\pi'n$ has exactly $d \geq 1$ left-right maxima. Clearly, for $d = 1$, we have a contribution of $\frac{x^2}{1-x}$. For $d = 2$, we see that π can be written as $\pi = (n-1)j_1\beta'n(n-2)(n-3)\cdots(j_1+1)\beta''$, where β' is decreasing. Thus, by considering the two cases either $j_1 = n-2$ or $j_1 < n-2$, we have a contribution of $xH(x)+\frac{x^4}{(1-x)(1-2x)}$. For $d\geq 3$, by Lemma 2.1, we obtain a contribution of $xG_d(x)=\frac{x^{d+1}(1+x)}{(1-x)^3}$. Hence,

$$H(x)=xH(x)+\frac{x^2}{1-x}+\frac{x^4}{(1-x)(1-2x)}+\sum_{d\geq 3}\frac{x^{d+1}(1+x)}{(1-x)^3},$$

which implies

$$H(x)=\frac{x^2(1-6x+16x^2-22x^3+16x^4-8x^5+x^6)}{(1-x)^6(1-2x)}.$$

For permutations in $S_n(T)$ with n in position 2, we see by considering left-right maxima that their generating function is given by $H(x)$, while $|\{\pi\in S_n(T)\,:\,\pi_1=n-1,\,\pi_2=n\}|=1$ for $n\geq 2$. Thus, $J(x)=H(x)-\frac{x^2}{1-x}$.

Next, write $P(x)=\sum_{d\geq 1}P_d(x)$, where $P_d(x)$ is the generating function for the number of T-avoiders π with exactly 2 left-right maxima and first letter $n-d-1$. Then $\pi=(n-d-1)j_1j_2\cdots j_e n\pi''$ with $j_1>j_2>\cdots j_e$ and $e\geq 1$ (decreasing because π avoids 1243 and $d\geq 1$). Write π'' as $\alpha^{(1)}(n-1)\cdots\alpha^{(d)}(n-d)\alpha^{(d+1)}$. Since π avoids 3412, we see that $\alpha^{(1)}\alpha^{(2)}\cdots\alpha^{(d)}$ is decreasing.

- Case $d\geq 2$. Since π avoids 2134, we see that $\alpha^{(1)}<j_e$. By considering whether $\alpha^{(1)}$ is empty or not, we have $P_d(x)=xP_{d-1}(x)+\frac{x^{d+4}}{(1-x)^{d+2}}$.

- Case $d=1$. First, suppose that $\alpha^{(1)}$ is empty. In this case $\alpha^{(2)}$ is decreasing, so from the structure of π we see that the contribution is given by $x^{e+3}/(1-x)^{e+1}$. Otherwise, $\alpha^{(1)}$ is not empty. So from the fact that $\alpha^{(1)}\alpha^{(2)}$ is decreasing we see that there two options: either $\alpha^{(1)}=\gamma\gamma'$ with $\gamma>j_e>\gamma'>\alpha^{(2)}$ and $\gamma\gamma'\alpha^{(2)}$ is decreasing, or $\alpha^{(2)}=\gamma\gamma'$ with $\alpha^{(1)}>\gamma>j_e>\gamma'$ and $\alpha^{(1)}\gamma\gamma'$ is decreasing. Each option gives a contribution of $x^{e+4}/(1-x)^3$. Thus,

$$P_1(x)=\frac{x^{e+3}}{(1-x)^{e+1}}+2\frac{x^{e+4}}{(1-x)^3},$$

  and, summing over $e\geq 1$, we find that $P_1(x)=\frac{x^4(1-x-x^2-x^3)}{(1-x)^4(1-2x)}$.

Therefore, $P(x)-P_1(x)=xP(x)+\frac{x^6}{(1-x)^3(1-2x)}$, which gives $P(x)=\frac{x^4(1-x-2x^3)}{(1-x)^5(1-2x)}$. Hence, by adding $H(x)$, $J(x)$ and $P(x)$, we complete the proof. □

Since $G_0(x)=1$ and $G_1(x)=xF_T(x)$ and $F_T(x)=\sum_{d\geq 0}G_d(x)$, the preceding three lemmas imply

THEOREM 2.4 *Let $T=\{1243,2134,3412\}$. Then*

$$F_T(x)=\frac{1-8x+28x^2-54x^3+65x^4-49x^5+18x^6-7x^7+2x^8}{(1-x)^7(1-2x)}.$$

2.2 Case 134: $T=\{3421,3214,4312\}$.

Here, T is in the symmetry class of $\{1243,2134,2341\}$. Let $a(n;i_1,i_2,\ldots,i_m)$ be the number of permutations in $\pi=i_1i_2\cdots i_m\pi'\in S_n(T)$ and $a_n=|S_n(T)|$. Thus $|S_n(T)|=\sum_{i=1}^n a(n;i)$.

LEMMA 2.5 *We have*

$$L(x) := \sum_{n\geq 3} a(n;n,2)x^n = x\left(\frac{1-2x+2x^3+x^4}{(1-2x)(1-x-x^2)} - 1\right).$$

Proof. First we find the generating function $A(x) = F_{\{312,3214,3421\}}(x)$. By symmetry, $A(x) = F_{\{132,2341,2134\}}(x)$. For $\pi \in S_n(132,2341,2134)$, by considering the position of n, we obtain

$$A(x) = 1 + xF_{\{132,213,2341\}}(x) + \frac{x}{1-x}(A(x)-1)$$

and

$$F_{\{132,213,2341\}}(x) = 1 + \frac{x}{1-x} + (x+x^2)(F_{\{132,213,2341\}}(x)-1).$$

Thus,

$$F_{\{312,3214,3421\}}(x) = \frac{1-2x+2x^3+x^4}{(1-2x)(1-x-x^2)}, \quad F_{\{132,213,2341\}}(x) = \frac{1-x+x^3}{(1-x)(1-x-x^2)}.$$

Note that $\pi = n2\pi' \in S_n$ avoids T if and only if π' avoids $\{312,3214,3421\}$. Thus, $L(x) = x(A(x)-1)$, which ends the proof. □

LEMMA 2.6 *We have*

$$B(x,v) := \sum_{n\geq 4}\sum_{i=3}^{n-1} a(n;i,n)v^i x^n = \frac{x^3v^3}{1-xv}\left(\frac{1-2x+2x^3+x^4}{(1-2x)(1-x-x^2)} - 1\right).$$

Proof. Let $\pi = in\pi' \in S_n(T)$. Since π avoids 3421, we see that π contains the subsequence $in12\cdots(i-1)$. Since π avoids 4312, there exists π'' such that $\pi = in12\cdots(i-2)\pi'' \in S_n(T)$. Thus,

$$a(n;i,n) = |S_{n-i}(312,3421,3214)| = |S_{n-i}(132,2341,2134)|,$$

which leads to $B(x,v) = \frac{x^3v^3}{1-xv}\sum_{n\geq 1}|S_n(132,2341,2134)|x^n$. Hence, by Lemma 2.5

$$B(x,v) = \frac{x^3v^3}{1-xv}\left(\frac{1-2x+2x^3+x^4}{(1-2x)(1-x-x^2)} - 1\right),$$

as required. □

LEMMA 2.7 *We have*

$$K(x,v) := \sum_{n\geq 3}\sum_{i=3}^{n} a(n;i,2)v^i x^n = \frac{x^2v^3}{1-xv}L(x) + \frac{x^3v^3}{1-xv}L(xv)$$
$$+ \frac{x^3v^3(x^3v^3(1-x)(1-xv) + x^2v^2(1-3x) - xv(2-3x) + 1 - x)}{(1-x)(1-xv)^2(1-2xv)},$$

where $L(x)$ is given in Lemma 2.5.

Proof. Let $K'(x,v) = \sum_{n\geq 5}\sum_{i=3}^{n} a(n;i,2)v^i x^n$. Let $\pi = i2\pi' \in S_n(T)$. Since π avoids 3214, we can write π as $\pi = i2\alpha 1\beta$ such that $2 < \beta < i$. So $a(n;3,2) = |S_{n-3}(4312,231,3214)| = |S_{n-3}(132,2134,2341)|$; for $4 \leq i \leq n-1$, we have that $a(n;i,2) = a(n-1,i-1,2)+1$, and $a(n;n,2)$ is given by Lemma 2.5. So

$$\sum_{n\geq 5}\sum_{i=4}^{n-1} a(n;i,2)v^i x^n = v\sum_{n\geq 5}\sum_{i=4}^{n-1} a(n-1;i-1,2)v^{i-1}x^n + \sum_{n\geq 5}\sum_{i=4}^{n-1} v^i x^n,$$

which implies

$$\begin{aligned} K'(x,v) &- \sum_{n\geq 5} a(n;n,2)v^n x^n - v^3\sum_{n\geq 5} a(n;3,2)x^n \\ &= vx\sum_{n\geq 4}\sum_{i=3}^{n-1} a(n;i,2)v^i x^n + \frac{v^4x^4}{(1-v)(1-x)} - \frac{v^4x^4}{(1-v)(1-vx)}. \end{aligned}$$

By Lemma 2.5, we have $\sum_{n\geq 5} a(n;3,2)v^3x^n = x^2v^3(L(x)-x^2)$ and

$$\sum_{n\geq 5} a(n;n;2)v^n x^n = xv(L(xv) - x^2v^2 - 2x^3v^3),$$

so

$$\begin{aligned} K'(x,v) &= xv(L(xv) - x^2v^2 - 2x^3v^3) + x^2v^3(L(x)-x^2) \\ &\quad + vx(K'(x,v) + v^3x^4 - xv(L(xv) - x^2v^2 - 2x^3v^3) + \frac{v^4x^5}{(1-x)(1-xv)}. \end{aligned}$$

We have $K(x,v) = K'(x,v) + x^3v^3 + x^4(v^3+2v^4)$, and the result follows. □

Lemma 2.8 *We have $\sum_{n\geq 2} a(n;n)x^n = L(x)$, where $L(x)$ is given in Lemma 2.5.*

Proof. Since $n\pi' \in S_n$ avoids T if and only if π' avoids $312, 3421, 3214$, the result follows from Lemma 2.5. □

Lemma 2.9 *Let $3 \leq i \leq n-1$. Then*

$$a(n;i) = a(n;i,1) + a(n;i,2) + a(n;i,n) + \sum_{j=i+1}^{n} a(n;i,j)\,.$$

Proof. Let $\pi = ij\pi' \in S_n(T)$ with $3 \leq j < i \leq n-1$. Since π avoids 4312, we see that π contains the subsequence $ij21$. Since π avoids 3214, we see that π contains the subsequence $ijn21$, and $jn21$ is order isomorphic to 3421. Thus $a(n;i,j) = 0$ for all j with $3 \leq j < i \leq n-1$, and the lemma follows. □

Lemma 2.10 *Let $3 \leq i < j \leq n-1$. Then*

$$a(n;i,j) = a(n-1;i-1,j-1) + a(n-1;j-1) - a(n-1;j-1,1) - a(n-1;j-1,2)$$

Proof. Let $\pi = ij\pi' \in S_n(T)$ with $3 \le i < j \le n-1$. By considering the third letter in π, we see that $a(n;i,j) = a(n-1;i-1,j-1) + a(n-1;j-1,j) + a(n-1;j-1,j+1) + \cdots + a(n-1;j-1,n-1)$. Note that

$$a(n-1;j-1) = \sum_{\ell=1}^{n-1} a(n-1;j-1,\ell) = a(n-1;j-1,1) + a(n-1;j-1,2) + \sum_{\ell=j}^{n-1} a(n-1;j-1,\ell).$$

Therefore,

$$a(n;i,j) = a(n-1;i-1,j-1) + a(n-1;j-1) - a(n-1;j-1,1) - a(n-1;j-1,2),$$

as claimed. □

Theorem 2.11 *Let $T = \{3421, 3214, 4312\}$. Then*

$$F_T(x) = \frac{(1-x)(1-2x+2x^2)(1-2x+x^3+x^5)C(x) - x(1-2x+x^3+x^4-2x^5+2x^6)}{(1-x)^3(1-2x)(1-x-x^2)},$$

Proof. Note that $a(n;k,1) = a(n-1;k-1)$ for $2 \le k \le n$ (a permutation $k1\pi' \in S_n$ avoids T if and only if $k\pi'$ avoids T). This fact will be used repeatedly. Let $3 \le i \le n-1$. Then

$$\begin{aligned}
&a(n;i) - \big(a(n;i,1) + a(n;i,2) + a(n;i,n)\big) = \sum_{j=i+1}^{n-1} a(n;i,j)\\
&= \sum_{j=i+1}^{n-1} \big(a(n-1;i-1,j-1) + a(n-1;j-1) - a(n-1;j-1,1) - a(n-1;j-1,2)\big)\\
&= \sum_{j=i}^{n-2} a(n-1;i-1,j) + \sum_{j=i}^{n-2} a(n-1;j) - \sum_{j=i-1}^{n-3} a(n-2;j) - \sum_{j=i}^{n-2} a(n-1;j,2)\\
&= a(n-1;i-1) - \big(a(n-1;i-1,1) + a(n-1;i-1,2) + a(n-1;i-1,n-1)\big)\\
&\quad + \sum_{j=i}^{n-2} a(n-1;j) - \sum_{j=i-1}^{n-3} a(n-2;j) - \sum_{j=i}^{n-2} a(n-1;j,2),
\end{aligned}$$

the first equality by Lemma 2.9, the second equality by Lemma 2.10, the third equality by reindexing and the fact that $a(n;k,1) = a(n-1;k-1)$, and the last equality by Lemma 2.9 again.

By Lemma 2.6, we see that $a(n;i,n) = a(n-1;i-1,n-1)$ for all $3 \le i \le n-1$. The preceding identities thus simplify to

$$\begin{aligned}
a(n;i) = a(n-1;i-1) &+ \sum_{j=i-1}^{n-2} a(n-1;j) - \sum_{j=i-2}^{n-3} a(n-2;j)\\
&+ a(n;i,2) - a(n-1;i-1,2) - \sum_{j=i}^{n-2} a(n-1;j,2).
\end{aligned}$$

Define $A_n(v)=\sum_{i=1}^{n}a(n;i)v^{i-1}$ Thus $A_n(1)=|S_n(T)|$. Define $B_n(v)=\sum_{i=3}^{n}a(n;i,2)v^i$ and $\ell_n=a(n;n)$. Note that $a(n;1)=a(n;2)=a(n-1)$, where $a(n)=|S_n(T)|$.

Multiplying the recurrence for $a(n;i)$ by v^{i-1} and summing over $i=3,4,\ldots,n-1$, we obtain

$$\begin{aligned}A_n(v)-(1+v)A_{n-1}(1)-v^{n-1}a(n;n)&=v(A_{n-1}(v)-A_{n-2}(1)-a(n-1;n-1)v^{n-2})\\&+\frac{1}{1-v}(v^2A_{n-1}(1)-v^2a(n-1;n-1)-v^2A_{n-1}(v)+a(n-1,n-1)v^n)\\&-\frac{1}{1-v}(v^2A_{n-2}(1)-a(n-2;n-2)v^2-v^3A_{n-2}(v)+a(n-2;n-2)v^n)\\&+\frac{1}{v}(B_n(v)-vB_{n-1}(v)-a(n;n,2)v^n+a(n-1;n-1,2)v^n)\\&-\frac{1}{1-v}(v^2B_{n-1}(1)-a(n-1;n-1)v^2-vB_{n-1}(v)+a(n-1;n-1)v^{n-1})\end{aligned}$$

with $A_0(v)=A_1(v)=1$, $A_2(v)=1+v$ and $A_3(v)=2+2v+2v^2$. Define $A(x,v)=\sum_{n\geq0}A_n(v)x^n$ and $K(x,v)=\sum_{n\geq3}B_n(v)x^n$. Thus $F_T(x)=A(x,1)$. By Lemma 2.8, $\sum_{n\geq2}a(n;n)x^n=L(x)$. By Lemma 2.5, $\sum_{n\geq3}a(n;n,2)x^n=xL(x)$. By Lemma 2.7, $\sum_{n\geq3}B_n(v)x^n=K(x,v)$.

Multiplying the recurrence for $A_n(v)$ by x^n and summing over $n\geq4$, we obtain

$$\begin{aligned}&A(x,v)-p(x,v)-(2+2v+2v^2)x^3\\&=x(1+v)(A(x,1)-p(x,1))+\frac{1}{v}(L(xv)-x^2v^2-2x^3v^3)\\&+xv(A(x,v)-p(x,v)-x(A(x,1)-1-x)-(L(xv)-x^2v^2)/v)\\&+\frac{v^2x}{1-v}(A(x,1)-p(x,1)-L(x)+x^2-A(x,v)+p(x,v)+(L(xv)-x^2v^2)/v)\\&-\frac{x^2v^2}{1-v}(A(x,1)-1-x-L(x)-v(A(x,v)-1-x)+L(xv))+\frac{1-xv}{v}(K(x,v)-xvL(xv))\\&-\frac{x}{1-v}(v^2K(x,1)-K(x,v)-xv^2L(x)+xvL(xv)),\end{aligned}$$

where $p(x,v)=A_0(v)+A_1(v)x+A_2(v)x^2=1+x+(1+v)x^2$. Hence, $A(x,v)$ satisfies

$$\begin{aligned}&\frac{(1-xv)(1-v+xv^2)}{1-v}A(x,v)\\&=-\frac{v^2x(1-2x)}{1-v}L(x)+\frac{(1-2xv)(1-v+xv^2)}{v(1-v)}L(xv)+\frac{v^2x}{1-v}K(x,1)+\frac{1-v+v^2x}{v(1-v)}K(x,v)\\&+\frac{x(1-xv)}{1-v}A(x,1)+(1-xv)(1-xv-vx^2).\end{aligned}$$

This equation for $A(x,v)$ can be solved by the kernel method, taking $v=C(x)$ and using the expressions for $L(x)$ and $K(x,v)$ from Lemmas 2.5 and 2.7. After simplification $A(x,1)$, which coincides with $F_T(x)$, agrees with the stated expression. □

2.3 Case 207: $T=\{1243,2134,1423\}$.

Let $a(n;i_1,i_2,\ldots,i_m)$ be the number of permutations in $\pi=i_1i_2\cdots i_m\pi'\in S_n(T)$ and $a_n=|S_n(T)|$. Thus $|S_n(T)|=\sum_{i=1}^{n}a(n;i)$.

LEMMA 2.12 *We have*

$$a(n;i,j) = \begin{cases} a(n-1;i,j) + \sum_{k=1}^{j-1} a(n-1;j,k), & 1 \le j < i \le n-2, \\ a(n-1,i,j), & 1 \le i < j \le n-2, \\ \sum_{k=1}^{i} a(n-1;k,n-1), & 1 \le i \le j-2 = n-2, \\ a(n-1;i,n-2) + \sum_{k=1}^{i} a(n-1;k,n-1), & 1 \le i \le j-2 = n-3, \end{cases}$$

with $a(n;n) = a(n;n-1) = a_{n-1}$*,* $a(n;i,1) = 1$ *for all* $i = 2,3,\ldots,n-2$*, and* $a(n;n-2,n-1) = a(n;n-2,n) = a_{n-2}$*.*

Proof. It is not hard to check the initial conditions. Let $1 \le j < i \le n-2$, then

$$\begin{aligned} a(n;i,j) &= a(n;i,j,n) + \sum_{k=1}^{j-1} a(n;i,j,k) \\ &= a(n-1;i,j) + \sum_{k=1}^{j-1} a(n-1;j,k) \end{aligned}$$

with $a(n;i,1) = 1$ (by definitions). For $1 \le i < j \le n-2$, we have $a(n;i,j) = a(n;i,j,j+1) = a(n-1,i,j)$.

For all $1 \le i \le j-2 = n-2$, we have

$$\begin{aligned} a(n;i,n) &= a(n;i,n,1) + \cdots + a(n;i,n,i-1) + a(n;i,n,n-1) \\ &= a(n-1;1,n-1) + \cdots + a(n-1,i-1,n-1) + a(n-1;i,n-1). \end{aligned}$$

Similarly, for all $1 \le i \le j-2 = n-3$,

$$a(n;i,n-1) = a(n-1;i,n-2) + a(n-1;i,n-1) + a(n-1;i-1,n-1) + \cdots + a(n-1;1,n-1),$$

which completes the proof. □

COROLLARY 2.13 *Define* $b(n;i) = a(n;i,n)$ *and* $c(n;i) = a(n;i,n-1)$*. Then* $b(n;i) = \sum_{j=1}^{i} b(n-1;j)$ *and* $c(n;i) = c(n-1;i) + b(n;i)$ *with* $b(n;n) = c(n;n-1) = 0$*,* $b(n;n-1) = b(n;n-2) = a_{n-2}$ *and* $c(n;n) = c(n;n-2) = a_{n-2}$*.*

Define $B(n;v) = \sum_{i=1}^{n} a(n;i,n)v^{i-1}$ and $C(n;v) = \sum_{i=1}^{n} a(n;i,n-1)v^{i-1}$. By Corollary 2.13, we obtain

$$\begin{aligned} B(n;v) &= a_{n-2}v^{n-2} + a_{n-2}v^{n-3} + \frac{1}{1-v}(B(n-1;v) - v^{n-3}B(n-1;1)), \\ C(n;v) &= C(n-1;v) + a_{n-2}(v^{n-1} - v^{n-2}) - a_{n-3}v^{n-2} + B(n;v) \end{aligned}$$

with $B(1;v) = C(1;v) = 0$, $B(2;v) = 1$ and $C(2;v) = v$.

Define $B(x,v) = \sum_{n\ge1} B(n;v)x^n$ and $C(x,v) = \sum_{n\ge1} C(n;v)x^n$. Note that $F_T(x) = \sum_{n\ge0} a_n x^n$. So the above recurrences can be formulated as

$$\begin{aligned} \left(1 - \frac{x}{v(1-v)}\right) B(x/v,v) &= \frac{x^2}{v^2}F_T(x) + \frac{x^2}{v^3}(F_T(x)-1) - \frac{x}{v^3(1-v)}B(x;1), \\ C(x;v) &= xC(x;v) + x^2(v-1)F_T(xv) - \frac{x^3}{v}F_T(xv) + B(x;v). \end{aligned}$$

Using the kernel method with $v = \frac{1}{C(x)}$, we obtain

$$B(x;1) = x^2 F_T(x) + x^2 C(x)(F_T(x) - 1). \tag{1}$$

Then, by substituting $v = 1$ in the second equation, we obtain

$$C(x;1) = x^2 F_T(x) + \frac{x^2}{1-x} C(x)(F_T(x) - 1). \tag{2}$$

LEMMA 2.14 *For all* $1 \le j < i \le n-2$, $a(n;i,j) = b(n;j)$.

Proof. Clearly, $b(n;1) = b(n-1;1)$ for all $n \ge 3$. But $b(2;1) = 1$, so $b(n;1) = 1 = a(n;i,1)$ for all $i = 2,3,\ldots,n-2$. Assume by induction that $a(n-1;i,j) = b(n-1;j)$ for all $n-3 \ge i > j \ge 1$. Then by Lemma 2.12,

$$a(n;i,j) = a(n-1;i,j) + \sum_{k=1}^{j-1} a(n-1;j,k) = b(n-1;j) + \sum_{k-1}^{j-1} b(n-1;k) = b(n;j)\,.$$

□

Now, we are ready to find an explicit formula for $F_T(x)$. By Lemmas 2.12 and 2.14, we have

$$\begin{aligned} a(n,i) &= b(n;1) + \cdots + b(n;i) + c(n;i) + a(n-1;i) - b(n-1;1) - \cdots - b(n-1;i) \\ &= a(n-1;i) + c(n+1;i) - b(n,i) \end{aligned}$$

with $a(n;n-2) = b(n;1)+\cdots+b(n;n-2)+c(n;n-2) = c(n+1;n-2)$ and $a(n;n) = a(n;n-1) = a_{n-1}$. Summing over $i = 1,2,\ldots,n-3$, we get that

$$a_n = a_{n-1} + c_{n+1} - b_n$$

with $a_0 = a_1 = 1$. Hence,

$$F_T(x) = 1 - x + xF_T(x) + C(x;1)/x - B(x;1),$$

Solving for $F_T(x)$ and using (1) and (2), we obtain the following result. Recall that $C(x)$ denotes the generating function for the Catalan numbers.

THEOREM 2.15 *Let* $T = \{1243, 1423, 2134\}$. *Then*

$$F_T(x) = \frac{1 - x(1-x)C(x)}{(1-x)\big(2 - C(x)\big) + x^2}\,.$$

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