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On permutations avoiding 1324, 2143, and another 4-letter pattern

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Abstract. We enumerate permutations avoiding 1324, 2143, and a third 4-letter pattern τ, a step toward the goal of enumerating avoiders for all triples of 4-letter patterns. The enumeration is already known for all but five patterns τ, which are treated in this paper.

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1 Introduction

Here, we are concerned with counting permutations that avoid three 4-letter patterns in the classical sense. There are a total of $\binom{24}{3} = 2024$ triples of 4-letter patterns, which are partitioned into 317 symmetry classes under the usual operations of reversing, complementing and inverting permutations, and all triples in a symmetry class trivially have the same counting sequence for their avoiders. The symmetry classes with a given counting sequence form a so-called Wilf class. As shown in [3, 4], the 317 symmetry classes split into 242 Wilf classes, hence 242 counting sequences, and [3, 4] establishes the counting sequence for all the big Wilf classes (those containing more than one symmetry class) but, while it determines the small Wilf classes, it leaves open their enumeration.

The aim of this paper is to count the set $S_n(1324, 2143, \tau)$ of permutations of $[n] = \{1, 2, \ldots, n\}$ avoiding 1324, 2143 and τ for each of the 22 patterns $\tau \in S_4\backslash\{1324, 2143\}$. As tabulated in Table 1 below, the 22 triples $\{1324, 2143, \tau\}$ lie in precisely 10 Wilf classes, of which 3 are big (see [3, 4]) and 2 are covered by INSENC algorithm (see [7]). (The case numbers in Table 1 are taken from Table 2 in the Appendix to [3], and INSENC refers to regular insertion encodings—the INSENC algorithm [7]). Thus it remains to treat a convenient triple in each of the other five (small) Wilf classes, and in each case we use the first listed τ in Table 1 (by the symmetry). These five cases are treated, by

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similar methods, in Section 3. We note in passing that Case 30 is the only one of them for which the representative triple in Table 2 in [3], namely $\{1324, 3412, 4231\}$, does not include both 1324 and 2143; however, its reversal does.

τ	$F_\tau(x)$	Reference	Case
4321	$\frac{2x^6-6x^5+21x^4-22x^3+16x^2-6x+1}{(1-x)^7}$	INSENC	24
3421, 4312	$\frac{1-8x+27x^2-48x^3+50x^4-30x^5+6x^6}{(1-x)^5(1-2x)^2}$	Theorem 3.3	29
4231	$\frac{1-6x+14x^2-14x^3+8x^4-2x^6}{(1-x)^3(1-2x)^2}$	Theorem 3.6	30
3412	$\frac{1-9x+33x^2-62x^3+64x^4-38x^5+10x^6}{(1-3x+x^2)(1-2x)^2(1-x)^3}$	Theorem 3.9	35
2341, 4123	$\frac{1-6x+12x^2-8x^3+3x^4-x^5}{(1-x)(1-3x+x^2)^2}$	[3]	55
2431, 3241, 4132, 4213	$\frac{(2-10x+16x^2-8x^3+x^4)C(x)-1+4x-5x^2+x^3}{(1-x)^2(1-3x+x^2)}$	Theorem 3.12	172
1234	$\frac{1-3x-2x^3}{x^4-2x^3+2x^2-4x+1}$	INSENC	181
1342, 1423, 2314, 3124 2413, 3142	$1+\frac{1-2x}{2(1-x)}\left(\frac{1}{\sqrt{1-4x}}-1\right)$	[1]	221
1432, 3214	$\frac{1-6x+12x^2-12x^3+6x^4-x^5-x^2(1-x+x^2)^2C(x)}{1-7x+16x^2-19x^3+11x^4-2x^5-x^6}$	Theorem 3.15	227
1243, 2134	$\frac{2(1-4x)}{2-9x+4x^2-x\sqrt{1-4x}}$	[2]	233

Table 1: Triples of 4 letter patterns containing 1324 and 2143, divided into symmetry classes, and the generating function for their avoiders.

2 Preliminaries

Every nonempty permutation π can be expressed uniquely as $\pi = i_1\pi^{(1)}i_2\pi^{(2)}\cdots i_m\pi^{(m)}$ $(m \geq 1)$ where $i_1, i_2, \ldots, i_m$ are the *left-right maxima*, that is, $i_1 < i_2 < \cdots < i_m$ and $i_j > \max(\pi^{(j)})$ for $1 \leq j \leq m$. In each case below, $P_m(x)$ (depending on T) denotes the generating function for the number of permutations in $S_n(T)$ for which $\pi^{(1)} = \pi^{(2)} = \cdots = \pi^{(m-1)} = \emptyset$ and $i_{m-1} \leq n-2$, in other words, for T-avoiders in which the first m letters increase up to n and $n-1$ occurs after n. Similarly, $Q_m(x)$ denotes the generating function for permutations in $S_n(T)$ with $i_j = n+j-m$, $1 \leq j \leq m$, in other words, for T-avoiders in which the m largest letters are the left-right maxima.

Given nonempty sets of numbers S and T, we will write $S < T$ to mean $\max(S) < \min(T)$ (with the inequality holding vacuously if S or T is empty). In this context, we will often denote singleton sets simply by the element in question. Throughout, $C(x) = \frac{1-\sqrt{1-4x}}{2x}$ denotes the generating function for the Catalan numbers $C_n := \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n-1}$; $C(x)$ is well known to count permutations avoiding τ, where τ is any one of the six 3-letter patterns. The identity $xC(x)^2 = C(x) - 1$ is used to simplify results.

3 Proofs

3.1 Case 29: $T = \{1324, 2143, 3421\}$.

The first two lemmas find P_m and Q_m.

LEMMA 3.1 *For $m \geq 2$,*

$$P_m(x) = x^{m-2}P_2(x) + \frac{(m-2)x^{m+2}}{(1-x)^2(1-2x)}.$$

Proof. To treat the case $m = 2$, we refine $P_2(x)$ to $P_{2,i}(x)$, which counts permutations $in\pi' \in S_n(T)$ with $i \neq n-1$. Clearly, $P_{2,1}(x) = x^2(F_{213,3421}(x) - 1) = \frac{x^3(1-3x+3x^2)}{(1-x)(1-2x)^2}$ [6, Seq. A001519]. For $i = 2$, π has the form $2n\beta 1 j_1 j_2 \cdots j_d$ with $2 < j_1 < j_2 < \cdots < j_d$ (to avoid 2143) giving contributions of $x^3(1-x)/(1-2x)$ and $x^{3+d}(1-x)^2/(1-2x)^2$ according as $d = 0$ (no j's) or not. Hence,

$$P_{2,2}(x) = x^3 \frac{1-x}{1-2x} + \sum_{d \geq 1} x^{3+d} \frac{(1-x)^2}{(1-2x)^2} = \frac{x^3(1-x)^2}{(1-2x)^2}.$$

For $i \geq 3$, similarly to the case $m \geq 3$ below, we find that

$$P_{2,i}(x) = x^{i-2}P_{2,2}(x) + \frac{(i-2)x^{i+2}}{(1-x)(1-2x)}.$$

Summing contributions, we get

$$P_2(x) = P_{2,1}(x) - \frac{x^2}{1-x} + \frac{1}{1-x}P_{2,2}(x) + \sum_{i \geq 2} \frac{(i-2)x^{i+2}}{(1-x)(1-2x)},$$

where the term $\frac{x^2}{1-x}$ is subtracted to omit $(n-1)n\pi' \in S_n(T)$. Substituting the expressions for $P_{2,1}(x)$ and $P_{2,2}(x)$ completes the proof.

Now suppose $m \geq 3$. If $i_1 i_2 \cdots i_{m-1}$ form a block of consecutive integers, the contribution is $x^{m-2}P_2(x)$. Otherwise, there is only one gap in the block (to avoid 1324), say $i_j > i_{j-1} + 1$ with $j \in \{2, 3, \ldots, m-1\}$. So the contribution is $(m-2)x^{m-3}Q(x)$, where $Q(x)$ counts permutations $\pi = i_1 i_2 n \pi^{(3)} \in S_n(T)$ with $i_2 > i_1 + 1$. A technique similar to finding the formula for $P_2(x)$ leads to $Q(x) = \frac{x^5}{(1-x)^2(1-2x)}$. Hence,

$$P_m(x) = x^{m-2}P_2(x) + \frac{(m-2)x^{m+2}}{(1-x)^2(1-2x)},$$

and the result follows. □

By similar methods, one can find Q_m.

LEMMA 3.2 *For $m \geq 2$,*

$$Q_m(x) = x^{m-2}Q_2(x) + \frac{(m-2)x^{m+1}}{(1-x)(1-2x)},$$

where $Q_2(x) = P_2(x) + \frac{x^2}{1-x}$.

THEOREM 3.3 *Let* $T = \{1324, 2143, 3421\}$. *Then*

$$F_T(x) = \frac{1 - 8x + 27x^2 - 48x^3 + 50x^4 - 30x^5 + 6x^6}{(1-x)^5(1-2x)^2}.$$

Proof. Let $G_m(x)$ be the generating function for T-avoiders with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xF_T(x)$. Now suppose $m \geq 2$ and let $\pi = i_1\pi^{(1)} \cdots i_m\pi^{(m)} \in S_n(T)$ with exactly m left-right maxima. Since π avoids 1324, we have $\pi^{(s)} > i_1$ for all $s = 2, 3, \ldots, m-1$. If $\pi^{(m)} < i_1$, then we have a contribution of $Q_m(x)$ by Lemma 3.2. If $i_{m-1} \leq n-2$, then we have $\pi^{(s)} = \emptyset$ (π avoids 2143) for all $s = 1, 2, \ldots, m-1$, so we have a contribution of $P_m(x)$ by Lemma 3.1. Thus, we can assume that there exists j such that $\pi^{(m)}$ contains $i_j + 1$. Since π avoids T, we see that $\pi^{(s)} = \emptyset$ for all $s = 1, 2, \ldots, m-1$, and $\pi^{(m)} = \alpha 12 \cdots (i_1 - 1)(i_j + 1)(i_j + 2) \cdots (i_{j+1} - 1)$. Thus, we have a contribution of $\frac{x^{m+1}}{(1-x)^2}$. Hence,

$$G_m(x) = P_m(x) + Q_m(x) + \frac{(m-2)x^{m+1}}{(1-x)^2}.$$

By summing over all $m \geq 2$, we obtain

$$F_T(x) - 1 - xF_T(x) = \sum_{m\geq 2} P_m(x) + \sum_{m\geq 2} Q_m(x) + \sum_{m\geq 2} \frac{(m-2)x^{m+1}}{(1-x)^2}.$$

From Lemmas 3.1 and 3.2, we have

$$\sum_{m\geq 2} P_m(x) = \frac{x^3(1 - 3x + 5x^2 - 5x^3)}{(1-x)^4(1-2x)^2},$$

$$\sum_{m\geq 2} Q_m(x) = \frac{x^2(1 - 5x + 11x^2 - 11x^3 + 3x^4)}{(1-x)^4(1-2x)^2}.$$

Thus, $F_T(x)$ satisfies

$$(1-x)F_T(x) = 1 + \frac{x^2(1 - 4x + 9x^2 - 10x^3 + 2x^4)}{(1-x)^4(1-2x)^2},$$

and the result follows. □

3.2 Case 30: $T = \{1324, 2143, 4231\}$.

Recall that, for P_m, the first m letters increase up to n and $n-1$ occurs after n, and for Q_m, the m largest letters are the left-right maxima.

LEMMA 3.4 $Q_m(x) = xQ_{m-1}(x) + P_m(x)$ *and* $P_m(x) = \frac{x}{1-x}P_{m-1}(x)$, *for all* $m \geq 2$.

Proof. By symmetric operations we see that $P_m(x) = Q'_m(x)$, where $Q'_m(x)$ counts permutations $\pi = (n+1-m)\pi^{(1)}(n+2-m)\pi^{(2)} \cdots n\pi^{(m)} \in S_n(T)$ with exactly m left-right maxima and $\pi^{(1)} \neq \emptyset$. Also, $Q'_m(x) = Q_m(x) - xQ_{m-1}(x)$. Hence, $Q_m(x) = xQ_{m-1}(x) + P_m(x)$.

An avoider counted by $P_m(x)$ has the form $i_1 i_2 \cdots i_m 12 \cdots (i_1 - 1)\pi''$ with $i_1 < i_2 < \cdots < i_{m-1} \leq n-2$ and $i_m = n$. Hence, $P_m(x) = \frac{x}{1-x}P_{m-1}(x)$. □

LEMMA 3.5 *The generating function for the number of permutations* $in\pi' \in S_n(T)$ *with* $i \leq n-2$ *is given by* $P_2(x) = \frac{x^3(1-x+2x^2-x^3)}{(1-2x)(1-x)^3}$. *The generating function for the number of permutations* $n\pi' \in S_n(T)$ *is given by* $Q_1(x) = \frac{x(1-3x+3x^2)}{(1-x)^2(1-2x)}$.

Proof. We have $Q_1(x) = xF_{\{231,1324,2143\}}(x) = \frac{x(1-3x+3x^2)}{(1-x)^2(1-2x)}$ (we omit the proof). To find $P_2(x)$, we consider the decomposition $in\alpha(n-1)\beta$ where $\alpha < \beta$. By considering whether β is empty or not, we obtain the desired result (again we omit the proof). □

THEOREM 3.6 *Let* $T = \{1324, 2143, 4231\}$. *Then*

$$F_T(x) = \frac{1-6x+14x^2-14x^3+8x^4-2x^6}{(1-x)^3(1-2x)^2}.$$

Proof. Let $G_m(x)$ be the generating function for T-avoiders with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = Q_1(x)$ (see Lemma 3.5). So suppose $m \geq 2$ and let $\pi = i_1\pi^{(1)} \cdots i_m\pi^{(m)} \in S_n(T)$ with exactly m left-right maxima. Since π avoids 1324, we have that $\pi^{(s)} > i_1$ for all $s = 2, 3, \ldots, m-1$. If $\pi^{(m)} < i_1$, then we have an avoider of the precise type counted by $Q_m(x)$. Otherwise, there exists j maximal in $[1, m-1]$ such that $\pi^{(m)} < i_{j+1}$ and so $\pi^{(m)}$ has a letter between i_j and i_{j+1}, giving a contribution of $x^{m-1-j}P_{j+1}(x)$. Hence, $G_m(x) = Q_m(x) + P_m(x) + xP_{m-1}(x) + \cdots + x^{m-2}P_2(x)$. By Lemma 3.4, we have

$$\begin{aligned} G_m(x) &= x^{m-1}Q_1(x) + 2(P_m(x) + xP_{m-1}(x) + \cdots + x^{m-2}P_2(x)) \\ &= x^{m-1}Q_1(x) + 2x^{m-2}P_2(x)\sum_{s=0}^{m-2}\frac{1}{(1-x)^s}. \end{aligned}$$

Therefore, by summing over $m \geq 1$ and using Lemma 3.5, we obtain

$$F_T(x) - 1 = \frac{x(1-5x+11x^2-8x^3+4x^4-2x^5)}{(1-x)^3(1-2x)^2},$$

which completes the proof. □

3.3 Case 35: $T = \{1324, 2143, 3412\}$.

We begin with an expression for $P_m(x)$.

LEMMA 3.7 *Let* $m \geq 2$. *Then*

$$P_m(x) = x^{m-2}P_2(x) + \frac{x^m(K(x)-1)}{1-x}\left(\frac{1}{(1-x)^{m-2}} - 1\right),$$

where $K(x) = \frac{1-2x}{1-3x+x^2}$ *and* $P_2(x) = \frac{x^2(K(x)-1)}{1-x} + \frac{x^4}{(1-2x)^2(1-x)}$.

Proof. An avoider counted by $P_m(x)$ has the form $i_1i_2\cdots i_m\pi'$ with $i_1 < i_2 < \cdots < i_{m-1} \leq n-2$ and $i_m = n$. If each letter of π' is either smaller than i_1 or greater than i_{m-1} then we have a contribution of $x^{m-2}P_2(x)$. Otherwise, since π avoids 1324 and 3412, we have that $\pi'=\alpha\beta$ such that α avoids 132 and 3412 (see [6, Sequence A001519]), $\beta<i_{m-1}$ and β is decreasing. Thus, we have a contribution of $\frac{x^m(K(x)-1)}{1-x}\left(\frac{1}{(1-x)^{m-2}}-1\right)$, where $K(x)=\frac{1-2x}{1-3x+x^2}$ the generating function for the number of permutations in $S_n(132,3412)$. Hence,

$$P_m(x)=x^{m-2}P_2(x)+\frac{x^m(K(x)-1)}{1-x}\left(\frac{1}{(1-x)^{m-2}}-1\right),$$

Thus, it remains to find a formula for $P_2(x)$. Let $\pi=in\pi'\in S_n(T)$ (with exactly 2 left-right maxima) such that $i\leq n-2$. Since π avoids 3412 then π' contains the subsequence $(i-1)(i-2)\cdots1$. If $\pi'=\alpha(i-1)(i-2)\cdots1$ then α avoids 132 and 3412 and we have a contribution of $\frac{x^2(K(x)-1)}{1-x}$. Otherwise, $i>1$ and there exists a letter in π' greater than i and on the right-side of $i-1$. The contribution for such case is $\frac{x^4}{(1-2x)^2(1-x)}$ (we leave the proof to the reader). Hence, $P_2(x)=\frac{x^2(K(x)-1)}{1-x}+\frac{x^4}{(1-2x)^2(1-x)}$, as required. □

By Lemma 3.7 and symmetric operations (complement and reversal), we can state the following.

Lemma 3.8 *Let $T=\{1324,2143,2431\}$ and let $m\geq2$. The generating function for the number of permutations $\pi=i_1\pi^{(1)}i_2\pi^{(2)}\cdots i_m\pi^{(m)}\in S_n(T)$ with exactly m left-right maxima such that $i_j=n+j-m$ for all $j=1,2,\ldots,m$ and $\pi^{(1)}\neq\emptyset$ is given by $P_m(x)$.*

Theorem 3.9 *Let $T=\{1324,2143,3412\}$. Then*

$$F_T(x)=\frac{1-9x+33x^2-62x^3+64x^4-38x^5+10x^6}{(1-3x+x^2)(1-2x)^2(1-x)^3}.$$

Proof. Let $G_m(x)$ be the generating function for T-avoiders with m left-right maxima. Clearly, $G_0(x)=1$ and $G_1(x)=xF_T(x)$. Now suppose $m\geq2$ and let $\pi=i_1\pi^{(1)}i_2\pi^{(2)}\cdots i_m\pi^{(m)}\in S_n(T)$ with exactly m left-right maxima. Since π avoids 1324, we see that $\pi^{(j)}<i_1$ for all $j=2,3,\ldots,m-1$. If $\pi^{(1)}\neq\emptyset$ then $\pi^{(m)}<i_1$ (π avoids 2143), and by Lemma 3.7, we have a contribution of $P_m(x)$. Otherwise, since π avoids 2143, we see that $i_{m-1}\leq n-2$ and $\pi^{(j)}=\emptyset$ for all $j=1,2,\ldots,m-1$, and by Lemma 3.7, we have a contribution of $P_m(x)$. If $\pi^{(1)}=\emptyset$ and $\pi^{(m)}<i_{m-1}$, then since π avoids 3412 we see that $\pi^{(2)}\cdots\pi^{(m-1)}$ is decreasing and $\pi^{(m)}$ is decreasing. Since π avoids 2143, we see that if $\pi^{(2)}\cdots\pi^{(m-1)}\neq\emptyset$ then $\pi^{(m)}<i_1$. Thus, by considering the minimal s such that $\pi^{(2)}=\cdots=\pi^{(s-1)}=\emptyset$ and $\pi^{(s)}\neq\emptyset$, we get a contribution of

$$\sum_{j=2}^{m-1}\frac{x^{m+1}}{(1-x)^j}+\frac{x^m}{(1-x)^{m-1}}.$$

Hence,

$$G_m(x)=\sum_{j=2}^{m-1}\frac{x^{m+1}}{(1-x)^j}+\frac{x^m}{(1-x)^{m-1}}+2P_m(x).$$

By summing over $m\geq2$ and using Lemmas 3.7 and 3.8, we obtain

$$F_T(x)-1-xF_T(x)=\frac{x^2(6x^4-14x^3+11x^2-5x+1)}{(1-2x)^2(1-x)^2(1-3x+x^2)}.$$

Solving for $F_T(x)$ completes the proof. □

3.4 Case 172: $T = \{1324, 2143, 2431\}$.

Again, we begin with expressions for P_m and Q_m.

LEMMA 3.10 *For $m \geq 2$,*

$$P_m(x) = \frac{x^{m+1}}{1-x}C(x) + x^m\left(\frac{1-2x}{1-3x+x^2} - \frac{1}{1-x}\right).$$

Proof. Let us write an equation for $P_m(x)$ for $m \geq 2$. Let $\pi = i_1 i_2 \cdots i_m \pi' \in S_n(T)$ with exactly m left-right maxima ($i_1 < i_2 < \cdots < i_m$) such that $i_{m-1} \leq n-2$. If π' contains the subsequence $(i_{m-1}+1)(i_{m-2}+1)\cdots(n-1)$, then π can be written as

$$\pi = i_1 i_2 \cdots i_m \alpha(i_{m-1}+1)(i_{m-2}+1)\cdots(i_m-1)$$

such that α avoids 132, which gives a contribution $\frac{x^{m+1}}{1-x}C(x)$. Otherwise, π' has a descent ba such that $i_m > b > a > i_{m-1}$, then π can be written as $\pi = i_1 i_2 \cdots i_m \beta$ such that β is not increasing and avoids 132 and 4213 (see [6, Sequence A001519]) and $i_m > \beta > i_{m-1}$. Thus, we have a contribution of $x^m\left(\frac{1-2x}{1-3x+x^2} - \frac{1}{1-x}\right)$. Hence,

$$P_m(x) = \frac{x^{m+1}}{1-x}C(x) + x^m\left(\frac{1-2x}{1-3x+x^2} - \frac{1}{1-x}\right),$$

as required. □

LEMMA 3.11 *For $m \geq 2$,*

$$Q_m(x) = x^{m-1}C(x)^{m-2}(F_T(x)-1) + \frac{x^{m-1}(C(x)^{m-2}-1)}{C(x)-1}\big(F_T(x) - 1 - xC(x)F_T(x)\big).$$

Proof. First, let $m \geq 3$ and suppose $\pi = i_1\pi^{(1)}i_2\pi^{(2)}\cdots i_m\pi^{(m)} \in S_n(T)$ with exactly m left-right maxima such that $i_j = n+j-m$ for all $j = 1, 2, \ldots, m$. Since π avoids 1324 we see that $\pi^{(1)} > \pi^{(j)}$ for all $j = 2, 3, \ldots, m-1$. The contribution of the case $\pi^{(1)} > \pi^{(m)}$ is given by $xC(x)Q_{m-1}(x)$, where $C(x)$ counts the permutations $\pi^{(1)}$ that avoid 132. So we can assume that $\pi^{(m)}$ has a letter greater than the smallest letter of $\pi^{(1)}$. Since π avoids 2143, we see that $\pi^{(j)} = \emptyset$ for all $j = 2, 3, \ldots, m-1$, which gives a contribution of $x^{m-2}L(x)$, where $L(x)$ is the generating function for the number of permutations $(n-1)\pi' n\pi'' \in S_n(T)$ such that π'' has a letter greater than the smallest letter in π'. Hence,

$$Q_m(x) = xC(x)Q_{m-1}(x) + x^{m-2}L(x).$$

Note that the generating function for the number of permutations $(n-1)\pi' n\pi'' \in S_n(T)$ is given by $x(F_T(x)-1)$ and the generating function for the number of permutations $(n-1)\pi' n\pi'' \in S_n(T)$ with $\pi' > \pi''$ is given by $x^2C(x)F_T(x)$. Thus, $L(x) = x\big(F_T(X) - 1 - xC(x)F_T(x)\big)$. Hence,

$$Q_m(x) = xC(x)Q_{m-1}(x) + x^{m-1}\big(F_T(X) - 1 - xC(x)F_T(x)\big)$$

with $Q_2(x) = x(F_T(x)-1)$ (easy to see). By induction on m, we complete the proof. □

THEOREM 3.12 *Let* $T = \{1324, 2143, 2431\}$. *Then*

$$F_T(x) = \frac{(2 - 10x + 16x^2 - 8x^3 + x^4)C(x) - 1 + 4x - 5x^2 + x^3}{(1-x)^2(1-3x+x^2)}.$$

Proof. Let $G_m(x)$ be the generating function for T-avoiders with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xF_T(x)$.

Let us write a formula for $G_2(x)$. Let $\pi = i\pi' n\pi'' \in S_n(T)$ with exactly 2 left-right maxima. If $\pi'' < i$ then we have a contribution of $x(F_T(x)-1)$. Otherwise, since π avoids 2143, we have $\pi = in\pi''$. Since π avoids 2431, $\pi = in\alpha\beta$ with $\alpha < i < \beta$. If β is increasing then α avoids 132, which gives a contribution of $\frac{x^3}{1-x}C(x)$. Otherwise, β avoids 213, 2431 (see [6, Sequence A001519]) and has a descent, so $\alpha = \emptyset$ (see Lemma 3.10), which gives a contribution $x^2\big((1-2x)/(1-3x+x^2) - 1\big)$. Hence,

$$G_2(x) = x(F_T(x) - 1) + \frac{x^3}{1-x}C(x) + x^2\left(\frac{1-2x}{1-3x+x^2} - 1\right).$$

Now, let us write an equation for $G_m(x)$ for $m \geq 3$. Let $\pi = i_1\pi^{(1)}i_2\pi^{(2)}\cdots i_m\pi^{(m)} \in S_n(T)$ with exactly m left-right maxima. Since π avoids 1324, we see that $\pi^{(j)} < i_1$ for all $j = 2, 3, \ldots, m-1$. If $\pi^{(1)} = \emptyset$ and $\pi^{(m)} < i_{m-1}$, then we have a contribution of $xG_{m-1}(x)$. If $\pi^{(1)} \neq \emptyset$ then $\pi^{(m)} < i_1$ (π avoids 2143), and by Lemma 3.11, we have a contribution of $Q_m(x) - xQ_{m-1}(x)$. Otherwise, since π avoids 2143, we see that $i_{m-1} \leq n-2$ and $\pi^{(j)} = \emptyset$ for all $j = 1, 2, \ldots, m-1$, from which, by Lemma 3.10, we have a contribution of $P_m(x)$. Hence,

$$G_m(x) = xG_{m-1}(x) + Q_m(x) - xQ_{m-1}(x) + P_m(x),$$

for all $m \geq 3$.

By summing over $m \geq 3$ and using the expressions for $G_0(x), G_1(x), G_2(x)$, we obtain

$$F_T(x) - 1 - xF_T(x) = G_2(x) + x(F_T(x) - 1 - xF_T(x)) + \sum_{m\geq 3}(Q_m(x) - xQ_{m-1}(x) + P_m(x)).$$

Thus, using Lemmas 3.10 and 3.11, we have

$$\begin{aligned} F_T(x) - 1 - xF_T(x) &= \frac{x^3}{1-x}(C(x)-1) + x(F_T(x)-1) + x^2\left(\frac{1-2x}{1-3x+x^2} - 1\right) \\ &\quad + x(F_T(x) - 1 - xF_T(x)) + x^2C(x)^2(F_T(x) - 1 - xF_T(x)) \\ &\quad + \frac{x^4(C(x) - 3xC(x) + x^2C(x) + x)}{(1-x)^2(1-3x+x^2)}. \end{aligned}$$

Solving for $F_T(x)$ and using $C(x) = 1 + xC(x)^2$ completes the proof. □

3.5 Case 227: $T = \{1324, 1432, 2143\}$.

First, we find Q_m.

LEMMA 3.13 *For all $m \geq 1$,*

$$Q_m(x) = -\frac{1-(1-x)F_T(x)}{C(x)-1}\big(xC(x)\big)^{m-1} + \frac{1-\big(1+xC(x)\big)F_T(x)}{C(x)-1}x^{m-1}.$$

Proof. Clearly, $Q_1(x) = xF_T(x)$ and $Q_2(x) = x(F_T(x)-1)$. Now let $m \geq 2$ and suppose π has the form $(n+1-m)\pi^{(1)}(n+2-m)\pi^{(2)}\cdots n\pi^{(m)} \in S_n(T)$. If $\pi^{(2)} = \emptyset$ then we have a contribution of $xQ_{m-1}(x)$. Otherwise, we can assume that $\pi^{(2)}$ is not empty. Since π avoids 1324 we see that $\pi^{(1)} > \pi^{(s)}$ for all $s = 2, 3, \dots, m-1$. Note that $\pi^{(1)}$ avoids 132 and $\pi^{(1)} > \pi^{(m)}$, so we have a contribution of $xC(x)\big(Q_{m-1}(x) - xQ_{m-2}(x)\big)$, where $C(x)$ is the generating function for 132-avoiders. Hence,

$$Q_m(x) = xQ_{m-1}(x) + xC(x)\big(Q_{m-1}(x) - xQ_{m-2}(x)\big).$$

Solving this recurrence completes the proof. □

LEMMA 3.14 *For all $m \geq 2$,*

$$P_m(x) = \frac{x^{m-2}\big(x^2(F_T(x)-1) - x^4C(x)F_T(x)\big)C(x)}{1-x}.$$

Proof. Suppose $\pi = i_1 i_2 \cdots i_m \pi' \in S_n(T)$ is counted by $P_m(x)$. Recall this means that $i_1, i_2, \dots, i_m$ are the left-right maxima of π and π' contains $n-1$. Since π avoids 1324 and 1432, we see that π' does not contain any letter between i_1 and i_{m-1}. Thus, $P_m(x) = x^{m-2}P_2(x)$.

To find a formula for $P_2(x)$, let us refine it by defining $P_2(x; d)$ to be the generating function for the number of permutations $(n-1-d)n\pi' \in S_n(T)$. Since π avoids 1324, we see that π' contains the subsequence $(n-d)(n-d+1)\cdots(n-1)$, so π' can be written as $\alpha^{(1)}(n-d)\alpha^{(2)}(n-d+1)\cdots\alpha^{(d)}(n-1)\alpha^{(d+1)}$. By using the same techniques as in the proof of Lemma 3.13 (either $\alpha^{(2)}$ is empty or not), we obtain the recurrence

$$P_2(x; d) = xP_2(x; d-1) + xC(x)(P_2(x; d-1) - xP_2(x; d-2))$$

with initial condition $P_2(x; 0) = xF_T(x)$ and $P_2(x; 1) = x(F_T(x)-1)$, where $C(x)$ is the generating function for 132-avoiders. Solving this recurrence completes the proof. □

THEOREM 3.15 *Let $T = \{1324, 1432, 2143\}$. Then*

$$F_T(x) = \frac{1-6x+12x^2-12x^3+6x^4-x^5-x^2(1-x+x^2)^2C(x)}{1-7x+16x^2-19x^3+11x^4-2x^5-x^6}.$$

Proof. Let $G_m(x)$ be the generating function for T-avoiders with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xF_T(x)$.

Now let $m \geq 2$ and suppose $\pi = i_1\pi^{(1)}i_2\pi^{(2)}\cdots i_m\pi^{(m)} \in S_n(T)$ has m left-right maxima, $i_1, i_2, \dots, i_m$. Since π avoids 1324, we see that $\pi^{(s)} < i_1$ for all $s = 1, 2, \dots, m-1$. By the definitions, we see that

$$G_m(x) = Q_m(x) + P_m(x) + xP_{m-1}(x) + \cdots + x^{m-2}P_2(x).$$

By Lemmas 3.13 and 3.14, we obtain

$$\begin{aligned}G_m(x) = &-\frac{(xF_T(x) - F_T(x) + 1)\,(xC(x))^{m-1}}{C(x) - 1}\frac{(1 - F_T(x) + xC(x)F_T(x))\,x^{m-1}}{C(x) - 1}\\ &+ \frac{(m-1)\,x^m\left(F_T(x) - 1 - x^2C(x)F_T(x)\right)C(x)}{1 - x}.\end{aligned}$$

Summing over $m \geq 2$ and using the expressions for $G_0(x)$ and $G_1(x)$, we obtain

$$\begin{aligned}F_T(x) - 1 - xF_T(x) = &-\frac{x(1 - F_T(x) + xF_T(x))C(x)}{(1 - xC(x))(C(x) - 1)} + \frac{x(1 - F_T(x) + xC(x)F_T(x))}{(C(x) - 1)(1 - x)}\\ &- \frac{x^2C(x)(1 - F_T(x) + x^2C(x)F_T(x))}{(1 - x)^3}.\end{aligned}$$

Solving for $F_T(x)$ and using $C(x) = 1 + xC(x)^2$ completes the proof. □

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