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Remark on quasi-ideals of ordered semigroups

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Abstract. The aim is to correct part of the Remark 3 of my paper "On regular, intra-regular ordered semigroups" in Pure Math. Appl. (PU.M.A.) 4, no. 4 (1993), 447–461. On this occasion, some further results and the similarity between the *po*-semigroups and the *le*-semigroups are discussed.

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1 Introduction and prerequisites

There was a typing mistake in Remark 3 in [6], corrected by the editor S. Lajos, by hand, in the reprints: On page 458, line –4, the set $X \cup \Big((XS] \cap (SX]\Big)$ should be replaced by $\Big(X \cup \Big((XS] \cap (SX]\Big)\Big]$. In this Remark we have seen that, for the subset $Q := X \cup \Big((XS] \cap (SX]\Big)$ of S, we have

1. $(QS] \cap (SQ] \subseteq Q$ and
2. if T is a quasi-ideal of S such that $T \supseteq X$, then $Q \subseteq T$, thus $(Q] \subseteq (T] = T$.

We also have $\Big((Q]\Big] = (Q]$ and $(Q] \supseteq (X] \supseteq X$. As a consequence, the set $(Q]$ is the quasi-ideal of S generated by X. Indeed:

$$\begin{aligned}\Big((Q]S\Big] \cap \Big(S(Q]\Big] &= \Big((Q](S]\Big] \cap \Big((S](Q]\Big] \subseteq \Big((QS]\Big] \cap \Big((SQ]\Big] \\ &= (QS] \cap (SQ] \subseteq Q \subseteq (Q].\end{aligned}$$

In addition, on p. 459, l. 17 of the same Remark, X^2S should be replaced by $(X^2S]$ and SX^2S by $(SX^2S]$ (this is because $X(XS] \subseteq (X](XS] \subseteq (X^2S]$ and $SX(XS] \subseteq (SX](XS] \subseteq (SX^2S]$), taking it into account in the rest of the proof of the Remark. For the sake of completeness, we give here detailed proofs of our arguments and the necessary definitions we use concerning the Remark. In addition, some further results on *le*-semigroups are given in the present paper, in an attempt to show the similarity between the *po*-semigroups and the *le*-semigroups.

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For an ordered semigroup $(S, \cdot, \leq)$ and a subset H of S, we denote by $(H]$ the subset of S defined by

$$(H] := \{t \in S \mid t \leq h \text{ for some } h \in H\}.$$

We mention the properties we use in the paper. Clearly $S = (S]$, and for subsets A, B of S, we have the following:

- $A \subseteq (A]$;
- if $A \subseteq B$, then $(A] \subseteq (B]$;
- $(A](B] \subseteq (AB]$;
- $\big((A](B]\big] = \big((A]B\big] = \big(A(B]\big] = (AB]$;
- $\big((A]\big] = (A]$.

Let us prove the last one. Since $A \subseteq (A]$, we have $(A] \subseteq \big((A]\big]$. Let now $t \in \big((A]\big]$. Then $t \leq x$ for some $x \in (A]$ and $x \leq a$ for some $a \in A$. Since $t \in S$ and $t \leq a$ where $a \in A$, we have $t \in (A]$.

As one can easily see, the following are equivalent:

1. if $a \in A$ and $S \ni b \leq a$, then $b \in A$;
2. $(A] \subseteq A$;
3. $(A] = A$.

A nonempty subset A of S is called a quasi-ideal of S if

1. $(AS] \cap (SA] \subseteq A$ and
2. if $a \in A$ and $S \ni b \leq a$, then $b \in A$ (equivalently $(A] \subseteq A$, which in turn is equivalent to $(A] = A$).

It is called a bi-ideal of S if

1. $ASA \subseteq A$ and
2. if $a \in A$ and $S \ni b \leq a$, then $b \in A$ (or $(A] = A$).

Every quasi-ideal of S is a bi-ideal of S as well. Indeed, if A is a quasi-ideal of S, then $ASA \subseteq (AS] \cap (SA] \subseteq A$. A nonempty subset A of S is called a left (resp. right) ideal of S if

1. $SA \subseteq A$ (resp. $AS \subseteq A$) and
2. if $a \in A$ and $S \ni b \leq a$, then $b \in A$ (or $(A] = A$).

For a nonempty subset A of S, the set $(A \cup AS]$ is the right ideal of S generated by A, and the set $(A \cup SA]$ is the left ideal of S generated by A (cf., for example [6]). For any nonempty subset X of S, the (nonempty set) $(XS]$ is a right ideal, $(SX]$ a left ideal and $(SXS]$ an ideal of S. Indeed $(XS]S = (XS](S] \subseteq (XS^2] \subseteq (XS]$ and we have $\big((XS]\big] = (XS]$ since $\big((A]\big] = (A]$ for any subset A of S. So, $(XS]$ is a right ideal of S. In a similar way we can prove that $(SX]$ is a left ideal and $(SXS]$ is an ideal of S. An ordered semigroup S is called intra-regular if, for every $a \in S$, there exist $x, y \in S$ such that $a \leq xa^2y$. This is equivalent to saying that $a \in (Sa^2S]$ for every $a \in S$ or $A \subseteq (SA^2S]$ for every $A \subseteq S$ (cf., for example [6]).

2 Main results

If S is an ordered semigroup, for a nonempty subset X of S, the (nonempty) subset

$$Q := \Big(X \cup \big((XS] \cap (SX]\big)\Big]$$

of S is the quasi-ideal of S generated by X. It is mentioned without proof in [7]. We think it is interesting to give a detailed proof which is the following.

Proof.

1. $(QS] \cap (SQ] \subseteq Q$. Indeed:

$$\begin{aligned} QS &= \Big(X \cup \big((XS] \cap (SX]\big)\Big]S = \Big(X \cup \big((XS] \cap (SX]\big)\Big](S] \\ &\subseteq \big(X \cup (XS]\big](S] \subseteq \Big(\big(X \cup (XS]\big)S\Big] \\ &= \big(XS \cup (XS]S\big] \subseteq \big(XS \cup (XS]\big] \text{ (since } (XS] \text{ is a right ideal of } S) \\ &= \big((XS]\big] = (XS] \text{ (since } \big((A]\big] = (A] \text{ for any subset } A \text{ of } S). \end{aligned}$$

Then $(QS] \subseteq \big((XS]\big] = (XS]$. Similarly we get $(SQ] \subseteq (SX]$, thus we have

$$\begin{aligned} (QS] \cap (SQ] &\subseteq (XS] \cap (SX] \subseteq X \cup \big((XS] \cap (SX]\big) \\ &\subseteq \Big(X \cup \big((XS] \cap (SX]\big)\Big] = Q. \end{aligned}$$

2. $(Q] = Q$. Indeed, since $Q = (A]$ for some $A \subseteq S$, we have $(Q] = \big((A]\big] = (A] = Q$.

3. If T is a quasi-ideal of S such that $T \supseteq X$, then

$$Q := \Big(X \cup \big((XS] \cap (SX]\big)\Big] \subseteq \Big(T \cup \big((TS] \cap (ST]\big)\Big] = (T] = T. \qquad \square$$

The sufficient condition of Proposition 2 in [6] is the following. Suppose $X \cap B \cap Y \subseteq (YBX]$ for every right ideal X, every left ideal Y and every bi-ideal B of S, then S is intra-regular. Its proof in Proposition 2 in [6] is based on the fact that an ordered semigroup S is intra-regular if and only if for every right ideal X and every left ideal Y of S we have $X \cap Y \subseteq (YX]$. In Remark 3 of the same paper, directly by the definition, using a more technical proof, we proved that the more general condition $X \cap Q \cap Y \subseteq (YQX]$ for every right ideal X, every left ideal Y and every quasi-ideal Q of S also implies the intra-regularity of S. Let us give the correct form of that part of Remark 3 in [6] in the following Lemma.

Lemma 2.1 *Let S be an ordered semigroup. Suppose for every right ideal X every left ideal Y and every quasi-ideal Q of S, we have $X \cap Q \cap Y \subseteq (YQX]$. Then S is intra-regular.*

Proof. Let $X \subseteq S$. If $X = \emptyset$, then clearly $X \subseteq (SX^2S]$. Let $X \neq \emptyset$. Denote by $r(X)$, $l(X)$, $q(X)$ the right ideal, left ideal and the quasi-ideal of S, respectively, generated by X. By hypothesis, we have

$$\begin{aligned} X &\subseteq r(X) \cap q(X) \cap l(X) \subseteq \Big(l(X)q(X)r(X)\Big] \\ &= \Big((X \cup SX]\Big(X \cup \big((XS] \cap (SX]\big)\Big](X \cup XS]\Big] \\ &= \Big((X \cup SX)\Big(X \cup \big((XS] \cap (SX]\big)\Big)(X \cup XS)\Big] \\ &\subseteq \Big((X \cup SX)\big(X \cup (XS]\big)(X \cup XS)\Big] \\ &= \Big(\big(X^2 \cup SX^2 \cup X(XS] \cup SX(XS]\big)(X \cup XS)\Big]. \end{aligned}$$

Since $X(XS] \subseteq (X](XS] \subseteq (X^2S]$ and $SX(XS] \subseteq (SX](XS] \subseteq (SX^2S]$, we get

$$\begin{aligned} X &\subseteq \Big(\big(X^2 \cup SX^2 \cup (X^2S] \cup (SX^2S]\big)(X \cup XS)\Big] \\ &= \Big(X^3 \cup SX^3 \cup (X^2S]X \cup (SX^2S]X \cup X^3S \cup SX^3S \cup (X^2S]XS \cup (SX^2S]XS\Big]. \end{aligned}$$

Since

$$(X^2S]X \subseteq (X^2S](X] \subseteq (X^2SX] \subseteq (X^2S],$$

$$(SX^2S]X \subseteq (SX^2S](X] \subseteq (SX^2SX] \subseteq (SX^2S],$$

$$(X^2S]XS \subseteq (X^2S](XS] \subseteq (X^2SXS] \subseteq (X^2S]$$

and

$$(SX^2S]XS \subseteq (SX^2S](XS] \subseteq (SX^2SXS] \subseteq (SX^2S],$$

we obtain

$$X \subseteq \Big(X^3 \cup SX^2S \cup (X^2S] \cup (SX^2S]\Big] = \Big(X^3 \cup (SX^2S] \cup (X^2S]\Big].$$

Since $X^3 \subseteq X^2S \subseteq (X^2S]$, we have $X \subseteq \Big((SX^2S] \cup (X^2S]\Big]$, hence

$$\begin{aligned} X^2 &\subseteq X\Big((SX^2S] \cup (X^2S]\Big] \subseteq (X]\Big((SX^2S] \cup (X^2S]\Big] \\ &\subseteq \Big(X\Big((SX^2S] \cup (X^2S]\Big)\Big] = \Big(X(SX^2S] \cup X(X^2S]\Big] \\ &\subseteq \Big((X](SX^2S] \cup (X](X^2S]\Big] \subseteq \Big((XSX^2S] \cup (X^3S]\Big] \\ &\subseteq \Big((SX^2S]\Big] = (SX^2S], \end{aligned}$$

$$X^2S \subseteq (SX^2S]S = (SX^2S](S] \subseteq (SX^2S^2] \subseteq (SX^2S]$$

and

$$(X^2S] \subseteq \Big((SX^2S]\Big] = (SX^2S].$$

Thus we have

$$X \subseteq \Big((SX^2S] \cup (X^2S]\Big] = \Big((SX^2S]\Big] = (SX^2S],$$

and S is intra-regular. □

Combining this result with Proposition 2 in [6], we get the following theorem.

THEOREM 2.2 *Let S be an ordered semigroup. The following are equivalent:*

1. *S is intra-regular;*
2. *For every right ideal X, every left ideal Y and every bi-ideal B of S, we have $X \cap B \cap Y \subseteq (YBX]$;*
3. *For every right ideal X, every left ideal Y and every quasi-ideal Q of S, we have $X \cap Q \cap Y \subseteq (YQX]$.*

As we already know, the theory of ordered semigroups based on ideals and the theory of le-semigroups based on ideal elements are parallel to each other. All the results on le-semigroups based on ideal elements are expressed in ordered semigroups in terms of ideals, and conversely. It is surprising that, for the results on ordered semigroups based on ideals, points do not play any essential role but the sets [6], as we have also seen in the results above. In this respect, Theorem 2.2, in case of le-semigroups, is the following.

THEOREM 2.3 *(cf. also [5]) Let S be an le-semigroup. The following are equivalent:*

1. *S is intra-regular;*
2. *For every right ideal element x, every left ideal element y and every bi-ideal element b of S, we have $x \wedge b \wedge y \leq ybx$;*

3. *For every right ideal element x, every left ideal element y and every quasi-ideal element q of S, we have $x \wedge q \wedge y \leq yqx$.*

Let us first give the necessary definitions, and then we will prove the theorem. A *poe*-semigroup is an ordered semigroup (*po*-semigroup) S having a greatest element usually denoted by "e" (that is, $e \geq a$ for all $a \in S$) [4]. An *le*-semigroup is a *poe*-semigroup which is at the same time a lattice (under the operations $\vee$ and $\wedge$) such that $a(b \vee c) = ab \vee ac$ and $(a \vee b)c = ac \vee bc$ for all $a, b, c \in S$ [1, 2]. An element a of an ordered semigroup S is called a right (resp. left) ideal element if $ax \leq a$ (resp. $xa \leq a$) for all $x \in S$ [1]. If S is a *poe*-semigroup, then a is a right (resp. left) ideal element of S if and only if $ae \leq a$ (resp. $ea \leq a$). An element a of a *poe*-semigroup S is called a bi-ideal element of S if $aea \leq a$, and it is called a quasi-ideal element of S if the element $ae \wedge ea$ exists (in S) and $ae \wedge ea \leq a$ [3]. For an *le*-semigroup, we denote by $r(a)$, $l(a)$, $q(a)$ the right ideal element, the left ideal element and the quasi-ideal element of S, respectively, generated by a. We have $r(a) = a \vee ae$ and $l(a) = a \vee ea$. A *poe*-semigroup S is intra-regular if and only if $a \leq ea^2e$ for all $a \in S$. If S is an *le*-semigroup, then the element $a \vee (ae \wedge ea)$ is the quasi-ideal element of S generated by a ($a \in S$). In fact:

$$\Big(a \vee (ae \wedge ea)\Big)e \wedge e\Big(a \vee (ae \wedge ea)\Big) = \Big(ae \vee (ae \wedge ea)e\Big) \wedge \Big(ea \vee e(ae \wedge ea)\Big).$$

Since $ae \wedge ea \leq ae$, we have $(ae \wedge ea)e \leq ae^2 \leq ae$. Since $ae \wedge ea \leq ea$, we have $e(ae \wedge ea) \leq e^2a \leq ea$. Thus we have

$$\Big(a \vee (ae \wedge ea)\Big)e \wedge e\Big(a \vee (ae \wedge ea)\Big) = ae \wedge ea \leq a \vee (ae \wedge ea).$$

Clearly, $a \vee (ae \wedge ea) \geq a$. If t is a quasi-ideal element of S such that $t \geq a$, then $a \vee (ae \wedge ea) \leq t \vee (te \wedge et) \leq t$. If S is a distributive *le*-semigroup, then we have $q(a) = a \vee (ae \wedge ea) = (a \vee ae) \wedge (a \vee ea) = r(a) \wedge l(a)$.

Proof. (of Theorem 2.3).

(1) $\Longrightarrow$ (2). If S is intra-regular, then for any element a of S, we have

$$a \leq ea^2e = eaae \leq e(ea^2e)(ea^2e)e \leq ea^2ea^2e.$$

Let now x be a right ideal element, y a left ideal element and b a bi-ideal element of S. Then, for the element $x \wedge b \wedge y$ of S, we have

$$\begin{aligned} x \wedge b \wedge y &\leq e(x \wedge b \wedge y)(x \wedge b \wedge y)e(x \wedge b \wedge y)(x \wedge b \wedge y)e \\ &\leq (ey)(beb)(xe) \leq ybx. \end{aligned}$$

(2) $\Longrightarrow$ (3). This is because the quasi-ideal elements of S are bi-ideal elements of S as well. Indeed, if q is a quasi-ideal element of S, then $qeq \leq qe \wedge eq \leq q$.

(3) $\Longrightarrow$ (1). Let $a \in S$. Then $a \leq ea^2e$. In fact: By hypothesis, we have

$$\begin{aligned} a &\leq r(a) \wedge q(a) \wedge l(a) \leq l(a)q(a)r(a) \\ &= (a \vee ea)\Big(a \vee (ae \wedge ea)\Big)(a \vee ae) \\ &\leq (a \vee ea)(a \vee ea)(a \vee ae) \\ &= (a^2 \vee ea^2 \vee aea \vee eaea)(a \vee ae) \\ &= a^3 \vee ea^3 \vee aea^2 \vee eaea^2 \vee a^3e \vee ea^3e \vee aea^2e \vee eaea^2e \\ &\leq a^3 \vee ea^2e \vee ea^2. \end{aligned}$$

Then

$$a^3 \leq (a^3 \vee ea^2e \vee ea^2)a^2 = a^5 \vee ea^2ea^2 \vee ea^4 \leq ea^2e.$$

Thus we have

$$a \leq ea^2e \vee ea^2,\ a^2 \leq ea^2ea \vee ea^3 \leq ea^2e,\ ea^2 \leq e(ea^2e) \leq ea^2e.$$

Thus we get $a \leq ea^2e$, and S is intra-regular. □

Remark. The implication (1) ⇒ (2) of Theorem 2.3 holds in a *poe*-semigroup S in general for which for any right ideal element x, every left ideal element y and every bi-ideal element b of S, the element $x \wedge b \wedge y$ exists. The implication (2) ⇒ (3) of Theorem 2.3 holds in *poe*-semigroups in general.

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References

[1] G. Birkhoff, *Lattice Theory,* Amer. Math. Soc., Coll. Publ., Vol. XXV, Providence, Rhode Island, 1967.

[2] L. Fuchs, *Partially Ordered Algebraic Systems,* Pergamon Press, 1963.

[3] N. Kehayopulu, *A characterization of some regular duo le-semigroups,* In: Contributions to General Algebra, Proceedings of the Klagenfurt Conference, May 25–28, 1978, Verlag Johannes Heyn, Klagenfurt (1979), pp. 137–143.

[4] N. Kehayopulu, *On intra-regular $\vee$e-semigroups,* Semigroup Forum, 19 (1980) 111–121.

[5] N. Kehayopulu, *On regular and intra-regular poe-semigroups,* Semigroup Forum, 29 (1984) 255–257.

[6] N. Kehayopulu, *On regular, intra-regular ordered semigroups,* Pure Math. Appl. (PU.M.A.), 4 (1993) 447–461.

[7] J. Tang, X. Y. Xie, Y. Luo, *A note on "fuzzy quasi-ideals of ordered semigroups",* preprint.