Sampling parts of random integer partitions: a probabilistic and asymptotic analysis

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Abstract. Let λ be a partition of the positive integer n, selected uniformly at random among all such partitions. Corteel et al. (1999) proposed three different procedures of sampling parts of λ at random. They obtained limiting distributions of the multiplicity $\mu_n = \mu_n(\lambda)$ of the randomly-chosen part as $n \to \infty$. The asymptotic behavior of the part size $\sigma_n = \sigma_n(\lambda)$, under these sampling conditions, was found by Fristedt (1993) and Mutafchiev (2014). All these results motivated us to study the relationship between the size and the multiplicity of a randomly-selected part of a random partition. We describe it obtaining the joint limiting distributions of (μ_n, σ_n), as $n \to \infty$, for all these three sampling procedures. It turns out that different sampling plans lead to different limiting distributions for (μ_n, σ_n). Our results generalize those obtained earlier and confirm the known expressions for the marginal limiting distributions of μ_n and σ_n.

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1 Introduction

Partitioning integers into summands (parts) is a subject of intensive research in combinatorics, number theory and statistical physics. If n is a positive integer, then by a partition, λ, of n, we mean a representation

$$\lambda: \quad n = \sum_{j=1}^{n} jm_j, \tag{1}$$

in which m_j, called multiplicities of parts $j, j = 1, 2, ..., n$, are non-negative integers. We use $\Lambda(n)$ to denote the set of all partitions of n and let $p(n) = | \Lambda(n) |$. The number $p(n)$ is determined asymptotically by the famous partition formula of Hardy and Ramanujan [9]:

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right), \quad n \to \infty. \tag{2}$$

A precise asymptotic expansion for $p(n)$ was found later by Rademacher [14] (more details may be also found in [2]). For instance, Rademacher's result implies that

$$p(n) = \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right) - \frac{1}{4\pi\sqrt{2}n^{3/2}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$
$$+ O\left(\exp\left(\frac{\pi}{2}\sqrt{\frac{2n}{3}}\right)\right), \quad n \to \infty. \tag{3}$$

Further on, we assume that, for fixed integer $n \geq 1$, a partition $\lambda \in \Lambda(n)$ is selected uniformly at random (uar), i.e. with probability $1/p(n)$. In this way, each numerical characteristic of λ can be regarded as a random variable defined on the space $\Lambda(n)$.

Corteel et al. [3] proposed and studied three procedures of sampling parts of a random partition $\lambda \in \Lambda(n)$. Basic statistics of a randomly selected part are the part size and its multiplicity. Corteel et al. [3] focused on the multiplicity $\mu_{n,j} = \mu_{n,j}(\lambda)$ ($j = 1, 2, 3,$) of the randomly-selected part and found limiting distributions for $\mu_{n,j}$, as $n \to \infty$, in these three cases of sampling (here the subscript j specifies the concrete sampling procedure that is followed; the definitions of these three sampling procedures will be given in the next section). In the same way, let $\sigma_{n,j} = \sigma_{n,j}(\lambda)$ ($j = 1, 2, 3$) be the size of the randomly-selected part. Limit theorems for $\sigma_{n,j}$ were obtained in [6] and [13]. All these results motivated us to study the relationship between the size and the multiplicity of a randomly-selected part of a random integer partition. We describe it obtaining the joint limiting distributions of $\mu_{n,j}$ and $\sigma_{n,j}$ (j=1,2,3) as $n \to \infty$. Our results generalize those obtained earlier in [6, 3, 13] and confirm the known expressions for the marginal limiting distributions of $\mu_{n,j}$ and $\sigma_{n,j}$.

We organize our paper as follows. In Section 2 we describe the sampling procedures proposed by Corteel et al. [3]. The main results of this paper are stated in Section 3. The method of proof is also briefly described there. Section 4 contains some auxiliary facts on generating functions and some asymptotics that we need further. We present the proofs of our limit theorems in Sections 5-7.

2   Basic random variables and definitions of the sampling procedures

For any $\lambda \in \Lambda(n)$ selected uar, we define the random variables

$$\alpha_j^{(n)} = \alpha_j^{(n)}(\lambda) = \text{the number of parts of size } j \text{ in } \lambda.$$

By I_A we denote the indicator of an event A and, for any two real numbers $d, s \geq 1$ and integer $m \geq 1$, we set

$$Z_{d,s}^{(n)} = \sum_{1 \leq j \leq s} \alpha_j^{(n)} I_{\{\alpha_j^{(n)} \leq d\}}, \tag{4}$$

$$Y_{m,s}^{(n)} = \sum_{1 \leq j \leq s} I_{\{\alpha_j^{(n)} = m\}}. \tag{5}$$

($Z_{d,s}^{(n)}$ counts the number of parts of size not grater than s and multiplicity not greater than d in a randomly-chosen partition λ, while $Y_{m,s}^{(n)}$ is the number of distinct parts with multiplicity m and size

not greater than s). Obviously,

$$Z_n = \sum_{j=1}^{n} \alpha_j^{(n)} \tag{6}$$

equals the total number of parts and

$$Y_n = \sum_{j=1}^{n} I_{\{\alpha_j^{(n)} > 0\}} \tag{7}$$

- the number of distinct parts in $\lambda \in \Lambda(n)$.

To describe the sampling procedures introduced by Corteel et al. [3] we notice that they are two-step procedures that combine the outcomes of two experiments. Therefore, they lead to three different product probability spaces. Since in each procedure we first sample uar a partition $\lambda \in \Lambda(n)$, the probability space on $\Lambda(n)$, equipped with the uniform probability measure $Pr(\lambda \in \Lambda(n)) = 1/p(n)$, is included in each product space. The second steps of sampling are, however, different and therefore, for each different procedure we obtain a different product space and different product probability measure. In what follows next, we adopt the common notation $\mathbb{P}(.)$ for the product probability measure of each sampling procedure and follow the concept of a product space developed in [8], Chapter 1.6. By $\mathbb{E}(X)$ we denote the expected value of the random variable X defined on the integer partition space $\Lambda(n)$.

Procedure 1. Given a partition $\lambda \in \Lambda(n)$ chosen uar (step 1), we select a part uar among all Z_n parts of λ (without any bias, step 2). By the product measure formula [8], Chapter 1.6, (4) and (6)

$$\mathbb{P}(\{\lambda \in \Lambda(n)\} \times \{\mu_{n,1} \le d, \sigma_{n,1} \le s\})$$

$$= Pr(\lambda \in \Lambda(n))\mathbb{P}(\mu_{n,1} \le d, \sigma_{n,1} \le s) = \left(\frac{1}{p(n)}\right)\left(\frac{Z_{d,s}^{(n)}}{Z_n}\right).$$

Summation over all $\lambda \in \Lambda(n)$ yields

$$\mathbb{P}(\mu_{n,1} \le d, \sigma_{n,1} \le s) = \mathbb{E}\left(\frac{Z_{d,s}^{(n)}}{Z_n}\right). \tag{8}$$

Procedure 2. Given a partition $\lambda \in \Lambda(n)$ chosen uar (step 1), we select a part among all Y_n different parts (step 2). Recalling definitions (5) and (7) of the random variables $Y_{m,s}^{(n)}$ and Y_n, respectively, we obtain in a similar way that

$$\mathbb{P}(\{\lambda \in \Lambda(n)\} \times \{\mu_{n,2} = m, \sigma_{n,2} \le s\}) = \left(\frac{1}{p(n)}\right)\left(\frac{Y_{m,s}^{(n)}}{Y_n}\right)$$

and

$$\mathbb{P}(\mu_{n,2} = m, \sigma_{n,2} \le s) = \mathbb{E}\left(\frac{Y_{m,s}^{(n)}}{Y_n}\right). \tag{9}$$

Procedure 3. Given a partition $\lambda \in \Lambda(n)$ chosen uar (step 1), we select a part of λ with the probability proportional to its size and multiplicity (step 2). Thus we set

$$\mathbb{P}(\{\lambda \in \Lambda(n)\} \times \{\mu_{n,3} = m, \sigma_{n,3} \le s\}) = \left(\frac{1}{p(n)}\right)\left(\frac{m}{n}\right)\sum_{1 \le j \le s} jI_{\{\alpha_j^{(n)} = m\}}, \tag{10}$$

which in turn implies that

$$\mathbb{P}(\mu_{n,3} = m, \sigma_{n,3} \leq s) = \frac{m}{n} \sum_{1 \leq j \leq s} j Pr(\alpha_j^{(n)} = m). \tag{11}$$

REMARK 2.1 Sampling procedure 3 can be interpreted in terms of Ferrers diagrams - the graphical representations of the integer partitions $\lambda \in \Lambda(n)$ [2]. It is obtained as follows. We use the notation λ_k to denote the kth largest part of λ for k a positive integer; if the number of parts Z_n of λ is $< k$, then $\lambda_k = 0$. The Ferrers diagram illustrates (1) by a two-dimensional array of dots, composed by λ_1 dots in the first (most left) row, λ_2 dots in the second row, ..., λ_{Z_n} dots in the last Z_nth row. Therefore, a Ferrers diagram may be considered as a union of disjoint blocks (rectangles) of dots with base j and height $\alpha_j^{(n)}$ (the multiplicity of part j). So, (10) and (11) imply that the sampling probability in Procedure 3 is proportional to the area of the block to which the chosen part belongs.

3   Statement of the main results and brief description of the method of proof

For sampling procedures 1 - 3, we have proved the following limit theorems.

THEOREM 3.1 *For the reals u and v, we let*

$$F(u,v) = \begin{cases} 0 & if \quad \min\{u,v\} \leq 0 \\ 0 & if \quad \min\{u,v\} > 0 \quad but \quad u+v \leq 1, \\ u+v-1 & if \quad 0 < u \leq 1, 0 < v \leq 1 \quad and \quad u+v > 1, \\ \min\{1,v\} & if \quad u > 1 \quad and \quad 0 < v \leq 1, \\ \min\{1,u\} & if \quad v > 1 \quad and \quad 0 < u \leq 1, \\ 1 & if \quad u > 1 \quad and \quad v > 1. \end{cases}$$

Then, we have

$$\lim_{n \to \infty} \mathbb{P}\left(\frac{2\log\mu_{n,1}}{\log n} \leq u, \frac{2\log\sigma_{n,1}}{\log n} \leq v\right) = F(u,v).$$

THEOREM 3.2 *Let $0 < t < \infty$. Then, for any positive integer m, we have*

$$\lim_{n \to \infty} \mathbb{P}\left(\mu_{n,2} = m, \frac{\pi\sigma_{n,2}}{\sqrt{6n}} \leq t\right) = \int_0^t e^{-my}(1 - e^{-y})dy.$$

THEOREM 3.3 *Let $0 < t < \infty$. Then, for any positive integer m, we have*

$$\lim_{n \to \infty} \mathbb{P}\left(\mu_{n,3} = m, \frac{\pi\sigma_{n,3}}{\sqrt{6n}} \leq t\right) = \frac{6m}{\pi^2} \int_0^t y(1 - e^{-y})e^{-my}dy.$$

REMARK 3.4 Since the inequalities $\frac{2\log\mu_{n,1}}{\log n} \leq u, \frac{2\log\sigma_{n,1}}{\log n} \leq v$ are equivalent to $\mu_{n,1} \leq n^{u/2}, \sigma_{n,1} \leq n^{v/2}$, respectively, Theorem 3.1 implies that the proportion of parts of size $\leq n^{v/2}$ and multiplicity $\leq n^{u/2}$, $0 < u, v < 1$, is approximately equal to $u + v - 1$ if $u + v > 1$; if $u + v \leq 1$ this proportion approaches zero as $n \to \infty$. For the other two sampling procedures, Theorems 3.2 and 3.3 show that typically chosen part sizes are of order $const\sqrt{n}$, while their multiplicities are finite - both converge weakly to discrete random variables whose support is the set $\{1, 2, ...\}$.

REMARK 3.5 For the sake of completeness, we present here a list of the known marginal limiting distributions for the size and multiplicity of the randomly-chosen part. They can be obtained as corollaries of Theorems 3.1-3.3. Proper references are also given.

$$\lim_{n \to \infty} \mathbb{P}\left(\frac{2 \log \mu_{n,1}}{\log n} \leq t\right) = t, \quad 0 < t < 1$$

[3], p. 195;

$$\lim_{n \to \infty} \mathbb{P}\left(\frac{2 \log \sigma_{n,1}}{\log n} \leq t\right) = t, \quad 0 < t < 1$$

[6], p. 712;

$$\lim_{n \to \infty} \mathbb{P}(\mu_{n,2} = m) = \frac{1}{m(m+1)}, \quad m = 1, 2, ..$$

[3], p. 192;

$$\lim_{n \to \infty} \mathbb{P}\left(\frac{\pi \sigma_{n,2}}{\sqrt{6n}} \leq t\right) = 1 - e^{-t} \quad 0 < t < \infty$$

[13], Theorem 2;

$$\lim_{n \to \infty} \mathbb{P}(\mu_{n,2} = m) = \frac{6(2m+1)}{\pi^2 m(m+1)^2}, \quad m = 1, 2, ...$$

[3], p. 195;

$$\lim_{n \to \infty} \mathbb{P}\left(\frac{\pi \sigma_{n,3}}{\sqrt{6n}} \leq t\right) = \frac{6}{\pi^2} \int_0^t \frac{y}{e^y - 1} dy, \quad 0 < t < \infty$$

[13], Theorem 3.

We conclude this section with a description of our method of proof. It combines probabilistic with analytical tools. We employ Fristedt's conditioning device [6], which allows to transfer probability distributions of linear combinations of the multiplicities $\alpha_j^{(n)}$ into conditional distributions of the corresponding linear combinations of independent and geometrically distributed random variables. Using this method, we show that, as $n \to \infty$, the expected values in (8) and (9) are close to the ratios of the expectations of the random variables that are involved there. The asymptotic behavior of the expectations of Y_n and Z_n, defined by (7) and (6), respectively, is well known:

$$\mathbb{E}(Y_n) \sim \sqrt{6n}/\pi, \tag{12}$$

$$\mathbb{E}(Z_n) \sim (\sqrt{6n}/2\pi) \log n \tag{13}$$

(see [16] and [3], respectively). We use combinatorial enumeration identities for generating functions, Cauchy coefficient formula and the saddle-point method in terms of Hayman admissibility theory [10] (see also [5], Chapter VIII.5) to obtain the asymptotic behavior of $\mathbb{E}(Z_{d,s}^{(n)})$ (see (8)). Finally, (9)) and (11) are analyzed using an approach developed by Corteel et al. [3] and based on Euler-MacLaurin sum formula.

4   Generating functions and the analytical background of the proofs

We start with the notation $g(x)$ for the generating function of the sequence $\{p(n)\}_{n\geq 1}$. For $\mid x \mid < 1$, $g(x)$ admits the well known representation

$$g(x) = 1 + \sum_{n=1}^{\infty} p(n)x^n = \prod_{k=1}^{\infty}(1 - x^k)^{-1} \tag{14}$$

(see e.g. [2], Theorem 1.1). Our first lemma is related to the probability generating function and the expectation of the random variable $Z_{d,s}^{(n)}$, defined by (4).

LEMMA 4.1 *For any reals $d, s \geq 1$ and complex variables x and z, satisfying $\mid x \mid < 1$ and $\mid z \mid < 1$, we have*

$$1 + \sum_{n=1}^{\infty} p(n)x^n \mathbb{E}(z^{Z_{d,s}^{(n)}}) = g(x) \prod_{1 \leq j \leq s} \frac{(1 - (zx^j)^{d+1})(1 - x^j)}{1 - zx^j}. \tag{15}$$

Moreover

$$1 + \sum_{n=1}^{\infty} p(n)x^n \mathbb{E}(Z_{d,s}^{(n)}) \tag{16}$$

$$= g(x) \left(\sum_{1 \leq j \leq s} \frac{x^j}{1 - x^j} - (d+1) \sum_{1 \leq j \leq s} \frac{x^{j(d+1)}}{1 - x^{j(d+1)}} \right) \prod_{1 \leq j \leq s}(1 - x^{j(d+1)}).$$

Proof.   The generating function identity (15) follows from a more general argument developed in [15], Chapter V.5. To state it we need some preliminary notations. We let $B \subset \{1, 2, ...\}$ and let $\Omega_j \subset \mathcal{N}_0 = \{0, 1, ...\}, j \geq 1$, be a sequence of sets. By $\widetilde{\sum}$ we denote a sum over all $j \in B$, satisfying (1) with $m_j \in \Omega_j, j \geq 1$. Then, we have

$$\prod_{j \in B} \sum_{m_j \in \Omega_j} (z_j x^j)^{m_j} = 1 + \sum_{n \geq 1} x^n \widetilde{\sum} z_1^{m_1} z_2^{m_2} ... z_n^{m_n}, \tag{17}$$

where $x, z_1, z_2, ...$ are formal variables. In (17) we set $B = \{1, 2, ..., [s]\}$,

$$\Omega_j = \begin{cases} \{0, 1, ..., [d]\} & \text{if} \quad j \leq s, \\ \mathcal{N}_0 & \text{if} \quad j > s, \end{cases}$$

and

$$x_j = \begin{cases} x & \text{if} \quad j \leq s, \\ 1 & \text{if} \quad j > s. \end{cases}$$

(Here $[s]$ and $[d]$ denote the integer parts of s and d, respectively.) The required identity (15) now follows from (4) and (14). A differentiation with respect to z in (15) leads to the the expectations of $Z_{d,s}^{(n)}$ and identity (16).   □

   The next lemma establishes a similar generating function identity for the random variable $Y_{m,s}^{(n)}$ defined by (5). It can be proved repeating the argument from [3], Theorem 1.

LEMMA 4.2 *For any real number $s \geq 1$, positive integer m and complex variables x and z, satisfying $\mid x \mid < 1$ and $\mid z \mid < 1$, we have*

$$1 + \sum_{n \geq 1, j \geq 0} x^n \mathbb{E}(z^{Y_{m,s}^{(n)}}) = g(x) \prod_{1 \leq k \leq s} (1 + (z-1)x^{mk}(1-x^k)).$$

This in turn implies that

$$\mathbb{E}(Y_{m,s}^{(n)}) = \sum_{1 \leq k \leq s} (p(n-mk) - p(n-(m+1)k)). \qquad (18)$$

Further on, for the sake of simplicity, we let

$$c = \frac{\pi}{\sqrt{6}}. \qquad (19)$$

We notice that Hardy-Ramanujan-Rademacher's formula in its form (3) implies that

$$p(n) = \frac{e^{2cn^{1/2}}}{4\sqrt{3}n}(1 + O(n^{-1/2})), \quad n \to \infty.$$

Using this expression, Corteel et al. [3] have obtained the following asymptotic estimates.

LEMMA 4.3 *For enough large n, we have*

$$\frac{p(n-mj)}{p(n)} = \left(1 + O\left(\frac{mj}{n^{3/2}}\right) + O((n-mj)^{-1/2})\right) e^{-cmj/n^{1/2}}$$

$$= \begin{cases} (1 + O(n^{-1/2}))e^{-cmj/n^{1/2}} & if \quad mj \leq n/2, \\ O(e^{-cn^{1/2}/2}) & if \quad mj > n/2. \end{cases} \qquad (20)$$

Lemma 3 enables us to interpret the sum in (18) as a Riemann integral sum.

Our next preliminary fact is related to Hardy-Ramanujan formula (2). We shall present it into a slightly different form, which will be used further to find the asymptotic of $\mathbb{E}(Z_{d,s}^{(n)})$. To introduce the reader into the subject, we notice that Hardy-Ramanujan formula has been subsequently generalized in various directions most notably by Meinardus [11] (see also [2], Chapter 6). Meinardus obtained the asymptotic of the Taylor coefficients of infinite products of the form

$$\prod_{k=1}^{\infty}(1-x^k)^{-b_k} \qquad (21)$$

under certain general assumptions on the sequence of non-negative numbers $\{b_k\}_{k \geq 1}$. Meinardus approach is based on considering the Dirichlet generating series

$$D(z) = \sum_{k=1}^{\infty} b_k k^{-z}, \quad z = u + iv. \qquad (22)$$

Since we shall use this result, below we briefly describe Meinardus assumptions avoiding their precise statements as well as some extra notations and concepts. The first Meinardus assumption (M_1) specifies the domain $\mathcal{H} = \{z : \Re(z) = u \geq -C_0\}, 0 < C_0 < 1$, in the complex plane, in which $D(z)$ has an analytic continuation. The second one (M_2) is related to the asymptotic behavior of $D(z)$, whenever $\mid \Im(z) \mid = \mid v \mid \to \infty$. A function of the complex variable z which is bounded by $O(\mid \Im(z) \mid^{C_1}), 0 < C_1 < \infty$, in certain domain of the complex plane is called function of finite order. Meinardus second condition (M_2) requires that $D(z)$ is of finite order in the whole domain $\mathcal{H}$. Finally, the Meinardus third condition (M_3) implies a bound on the ordinary generating function of the sequence $\{b_k\}_{k \geq 1}$. It can be stated in a way simpler than the Meinardus original expression by the inequality

$$\sum_{k=1}^{\infty} b_k e^{-k\omega} \sin^2(\pi k u) \geq C_2 \omega^{-\epsilon_1}, \quad 0 < \frac{\omega}{2\pi} < \mid u \mid < \frac{1}{2},$$

for sufficiently small ω and some constants $C_2, \epsilon_1 > 0$ $(C_2 = C_2(\epsilon_1))$ (see [7], p. 310).

It is known that Euler partition generating function $g(x)$ (which is obviously of the form (21)) satisfies the Meinardus scheme of conditions (M_1)-(M_3) (see e.g. [2], Theorem 6.3).

The proof of our Theorem 3.1 will be based on an asymptotic analysis of a Cauchy integral stemming from (16). We shall apply there the saddle-point method in the sense of Hayman [10] (see also [5], Chapter VIII.5). In [10] Hayman studied a wide class of power series satisfying a set of relatively mild conditions and established general formulas for the asymptotic order of their coefficients. In the proof of Theorem 3.1 we shall essentially use that the generating function $g(x)$ is admissible in the sense of Hayman. To present Hayman's idea and show how it can be applied, we need to introduce some auxiliary notations.

We consider here a function $G(x) = \sum_{n=1}^{\infty} G_n x^n$ that is analytic for $\mid x \mid < \rho, 0 < \rho < \infty$. For $0 < r < \rho$, we let

$$a(r) = r \frac{G'(r)}{G(r)}, \tag{23}$$

$$b(r) = r \frac{G'(r)}{G(r)} + r^2 \frac{G''(r)}{G(r)} - r^2 \left(\frac{G'(r)}{G(r)} \right). \tag{24}$$

In the statement of Hayman's result we use the terminology given in [5], Chapter VIII.5. We assume that $G(x) > 0$ for $x \in (R_0, \rho) \subset (0, \rho)$ and satisfies the following three conditions.

Capture condition. $\lim_{r \to \rho} a(r) = \infty$ and $\lim_{r \to \rho} b(r) = \infty$.

Locality condition. For some function $\delta = \delta(r)$ defined over (R_0, ρ) and satisfying $0 < \delta < \pi$, one has

$$G(re^{i\theta}) \sim G(r)e^{i\theta a(r) - \theta^2 b(r)/2}$$

as $r \to \rho$, uniformly for $\mid \theta \mid \leq \delta(r)$.

Decay condition.

$$G(re^{i\theta}) = o\left(\frac{G(r)}{\sqrt{b(r)}} \right)$$

as $r \to \rho$, uniformly for $\delta(r) \leq \theta < \pi$.

Hayman Theorem. Let $G(x)$ be Hayman admissible function and $r = r_n$ be the unique solution in the interval (R_0, ρ) of the equation

$$a(r) = n. \tag{25}$$

Then the Taylor coefficients of $G(x)$ satisfy, as $n \to \infty$,

$$G_n \sim \frac{G(r_n)}{r_n^n \sqrt{2\pi b(r_n)}} \tag{26}$$

with $b(r_n)$ given by (24).

The next lemma presents an alternative formula for the partition function $p(n)$.

LEMMA 4.4 *If $r = r_n$ satisfies (25) for sufficiently large n, then*

$$p(n) \sim \frac{r_n^n g(r_n)}{\sqrt{2\pi b(r_n)}}, \quad n \to \infty,$$

where $a(r_n)$ and $b(r_n)$ are given by (23)and (24) with $G(x) \equiv g(x)$.

Proof. Since in (14) we have $b_k = 1, k \geq 1$, the Dirichlet generating series (22) is $D(z) = \zeta(z)$, where ζ denotes the Riemann zeta function. We set in (23) and (24) $r = r_n = e^{-h_n}, h_n > 0$, where h_n is the unique solution of the equation

$$a(e^{-h_n}) = n. \tag{27}$$

((27) is an obvious modification of (25).) Granovsky et al. [7] showed that the first two Meinardus conditions imply that the unique solution of (27) has the following asymptotic expansion:

$$h_n = \sqrt{\zeta(2)/n} + \frac{\zeta(0)}{2n} + O(n^{-1-\beta}) = \frac{\pi}{\sqrt{6n}} - \frac{1}{4n} + O(n^{-1-\beta}), \tag{28}$$

where $\beta > 0$ is fixed constant (here we have also used that $\zeta(0) = -1/2$; see [1]). We also notice that (24) and (28) impliy that

$$b(e^{-h_n}) = 2\zeta(2)h_n^{-3} + O(h_n^{-2}) \sim \frac{\pi^2}{3} h_n^{-3} \sim \frac{2\sqrt{6}}{\pi} n^{3/2} \tag{29}$$

(see [12], Lemma 2.2, with $D(z) = \zeta(z)$). Hence, by (27) and (29), $a(e^{-h_n}) \to \infty$ and $b(e^{-h_n}) \to \infty$ as $n \to \infty$, that is, Hayman's "capture" condition is satisfied with $r = r_n = e^{-h_n}$. To show next that Hayman's "decay" condition is satisfied by $g(x)$ we set

$$\delta_n = \frac{h_n^{4/3}}{\Omega(n)} = \frac{\pi^{4/3}}{(6n)^{2/3}\Omega(n)} \left(1 + O\left(\frac{1}{\sqrt{n}} \right) \right) \tag{30}$$

with h_n given by (28), where $\Omega(n) \to \infty$ as $n \to \infty$ arbitrarily slowly. We can apply now an estimate for $\mid g(e^{-h_n+i\theta}) \mid$ established in a general form in [12], Lemma 2.4, using all three Meinardus conditions. It states that there are two positive constants c_0 and ϵ_0, such that, for sufficiently large n,

$$\mid g(e^{-h_n+i\theta}) \mid \leq g(e^{-h_n}) e^{-c_0 h_n^{-\epsilon_0}} \tag{31}$$

uniformly for $\delta_n \leq \mid \theta \mid < \pi$.

This, in combination with (29), implies that $\mid g(e^{-h_n+i\theta}) \mid = o(g(e^{-h_n})/\sqrt{b(e^{-h_n})})$ uniformly in the same range for θ, which is just Hayman's "decay" condition. Finally, by Lemma 2.3 of [12], established

using Meinardus conditions (M_1) and (M_2), Hayman's "locality" condition is also satisfied by $g(x)$. In fact, this lemma implies in the particular case $D(z) = \zeta(z)$ that

$$e^{-i\theta n}\frac{g(e^{-h_n+i\theta})}{g(e^{-h_n})} = e^{-\theta^2 b(e^{-h_n})/2}(1 + O(1/\Omega^3(n))) \tag{32}$$

uniformly for $\mid \theta \mid \leq \delta_n$, where $b(e^{-h_n})$ and δ_n are determined by (29) and (30), respectively. Hence all conditions of Hayman's theorem hold and we can apply it with $G_n = p(n), G(x) = g(x), r_n = e^{-h_n}$ and $\rho = 1$ to find that

$$p(n) \sim \frac{e^{nh_n}g(e^{-h_n})}{\sqrt{2\pi b(e^{-h_n})}}, \quad n \to \infty, \tag{33}$$

which completes the proof. $\square$

REMARK 4.5 To show that formula (33) yields (2), one has to replace (28) and (29) in the right hand side of (33). The asymptotic of $g(e^{-h_n})$ is determined by a general lemma due to Meinardus [11] (see also [2], Lemma 6.1). Since $\zeta(0) = -1/2$ and $\zeta'(0) = -\frac{1}{2}\log(2\pi)$ (see [1]), in the particular case of $g(e^{-h_n})$ this lemma implies that

$$g(e^{-h_n}) = \exp\left(\zeta(2)h_n^{-1} - \zeta(0)\log h_n + \zeta'(0) + O(h_n^{c_1})\right)$$
$$= \exp\left(\frac{\pi^2}{6h_n} + \frac{1}{2}\log h_n - \frac{1}{2}\log(2\pi) + O(h_n^{c_1})\right), \quad n \to \infty,$$

where $0 < c_1 < 1$. The rest of the computation leading to (2) is based on simple algebraic manipulations and cancellations.

5   Proof of Theorem 3.1

We base our proof on the definition of Sampling Procedure 1 and eq. (8). We want to replace the expected value in its right-hand side by the ratio $\mathbb{E}(Z_{d,s}^{(n)})/\mathbb{E}(Z_n)$. So, we notice first that Erdös and Lehner [4] proved that, in probability, the total number of parts Z_n is asymptotic to $\mathbb{E}(Z_n)$ as $n \to \infty$. Hence, for any $\epsilon > 0$, the probability of the event

$$A_n = \left\{\lambda \in \Lambda(n) :\mid \frac{Z_n}{\mathbb{E}(Z_n)} - 1 \mid > \epsilon\right\}$$

tends to 0 as $n \to \infty$. Further, we rewrite (8) in the following way:

$$\mathbb{P}(\mu_{n,1} \leq d, \sigma_{n,1} \leq s) = \mathbb{E}\left(\frac{Z_{d,s}^{(n)}}{Z_n}I_{A_n^c}\right) + \mathbb{E}\left(\frac{Z_{d,s}^{(n)}}{Z_n}I_{A_n}\right). \tag{34}$$

For $\lambda \in A_n^c$ and $0 < \epsilon < 1$, we have $(1-\epsilon)\mathbb{E}(Z_n) \leq Z_n \leq (1+\epsilon)\mathbb{E}(Z_n)$ and therefore,

$$\frac{\mathbb{E}(Z_{d,s}^{(n)})}{(1+\epsilon)\mathbb{E}(Z_n)} \leq \mathbb{E}\left(\frac{Z_{d,s}^{(n)}}{Z_n}I_{A_n^c}\right) \leq \frac{\mathbb{E}(Z_{d,s}^{(n)})}{(1-\epsilon)\mathbb{E}(Z_n)}. \tag{35}$$

Since $Z_{d,s}^{(n)} \leq Z_n$, the second summand in (34) is not greater than $\mathbb{P}(A_n)$. Hence, combining (34) and (35), we obtain

$$\mathbb{P}(\mu_{n,1} \leq d, \sigma_{n,1} \leq s) = (1 + O(\epsilon)) \frac{\mathbb{E}(Z_{d,s}^{(n)})}{\mathbb{E}(Z_n)} + \mathbb{P}(A_n).$$

Letting $n \to \infty$ and then $\epsilon \to 0$ and replacing $\mathbb{E}(Z_n)$ by the right-hand side of (13), uniformly for $d, s \geq 1$, we finally get

$$\mathbb{P}(\mu_{n,1} \leq d, \sigma_{n,1} \leq s) \sim \frac{2\pi \mathbb{E}(Z_{d,s}^{(n)})}{\sqrt{6n} \log n} = \frac{2c\mathbb{E}(Z_{d,s}^{(n)})}{\sqrt{n} \log n}, \tag{36}$$

where c is the constant from (19).

Our proof continues with an application of Cauchy coefficient formula to (16). We use the circle $x = e^{-h_n + i\theta}, -\pi < \theta \leq \pi$, as a contour of integration and the notation

$$\varphi_{d,s}(x) = \left(\sum_{1 \leq j \leq s} \frac{x^j}{1 - x^j} - (d+1) \sum_{1 \leq j \leq s} \frac{x^{j(d+1)}}{1 - x^{j(d+1)}} \right) \prod_{1 \leq j \leq s} (1 - x^{j(d+1)}) \tag{37}$$

to obtain

$$p(n)\mathbb{E}(Z_{d,s}^{(n)}) = \frac{e^{nh_n}}{2\pi} \int_{-\pi}^{\pi} g(e^{-h_n + i\theta}) \varphi_{d,s}(e^{-h_n + i\theta}) e^{-i\theta n} d\theta.$$

Then, we break up the range of integration as follows:

$$p(n)\mathbb{E}(Z_{d,s}^{(n)}) = J_1(d, s, n) + J_2(d, s, n), \tag{38}$$

where

$$J_1(d, s, n) = \frac{e^{nh_n}}{2\pi} \int_{-\delta_n}^{\delta_n} g(e^{-h_n + i\theta}) \varphi_{d,s}(e^{-h_n + i\theta}) d\theta, \tag{39}$$

$$J_2(d, s, n) = \frac{e^{nh_n}}{2\pi} \int_{\delta_n < |\theta| \leq \pi} g(e^{-h_n + i\theta}) \varphi_{d,s}(e^{-h_n + i\theta}) d\theta \tag{40}$$

and δ_n is defined by (30).

In our next step we set

$$d = n^{u/2}, \quad s = n^{v/2}, \quad 0 \leq u, v \leq 1 \tag{41}$$

and obtain estimates for the sums:

$$S_1 = \sum_{1 \leq j \leq s} \frac{e^{-jh_n}}{1 - e^{-jh_n}}, \tag{42}$$

$$S_2 = \sum_{1 \leq j \leq s} \frac{e^{-j(d+1)h_n}}{1 - e^{-j(d+1)h_n}}. \tag{43}$$

Here the sequence $\{h_n\}_{n \geq 1}$ is defined by (28).

Using the approximation of a Riemann sum by an integral, (28), (41) and (19), for S_1 we get

$$S_1 = \left(1 + O\left(\frac{1}{n}\right)\right) \sqrt{n} \sum_{1 \leq j \leq n^{v/2}} \frac{e^{-cj/\sqrt{n}}}{1 - e^{-cj/\sqrt{n}}} \frac{1}{\sqrt{n}}$$

$$\sim \sqrt{n} \int_{1/\sqrt{n}}^{n^{\frac{v-1}{2}}} \frac{e^{-cz}}{1 - e^{-cz}} dz = \frac{\sqrt{n}}{c} \int_{c/\sqrt{n}}^{n^{\frac{v-1}{2}}} \frac{e^{-z}}{1 - e^{-z}} dz$$

$$= \frac{\sqrt{n}}{c} \log\left(\frac{1 - e^{-cn^{\frac{v-1}{2}}}}{1 - e^{-c/\sqrt{n}}}\right) = \frac{v\sqrt{n}}{2c} \log n + O(n^{v/2}). \tag{44}$$

In the same way one can show that

$$S_2 = \begin{cases} \frac{1-u}{2c} n^{\frac{1-u}{2}} \log n + O(n^{\frac{1-u}{2}}) & \text{if} \quad v + u \geq 1, \\ \frac{v}{2c} n^{\frac{1-u}{2}} \log n + O(n^{\frac{1-u}{2}}) & \text{if} \quad v + u < 1. \end{cases} \tag{45}$$

We are now ready to find an estimate for the second integral in (38) (see (40)). First, we have

$$| \prod_{1 \leq j \leq s} (1 - e^{-h_n j(d+1) + ij\theta(d+1)}) |$$

$$\leq \prod_{1 \leq j \leq s} (1 - e^{-h_n j(d+1)}) + \prod_{1 \leq j \leq s} e^{-h_n j(d+1)} | 1 - e^{ij\theta(d+1)} |$$

$$\leq 1 + e^{-h_n(d+1)s}(1 + \prod_{1 \leq j \leq s} | e^{ij\theta(d+1)} |) = 1 + 2e^{-h_n(d+1)s} \leq 3. \tag{46}$$

Hence, in terms of notations (37), (42) and (43), by (41), (44) and (45),

$$| \varphi_{d,s}(e^{-h_n+i\theta}) | = O((S_1 + (d+1)S_2) = O(\sqrt{n}\log n), \quad -\pi \leq \theta \leq \pi.$$

Replacing this estimate and applying inequality (31) to the integrand of (40), we obtain

$$| J_2(d, s, n) | = O(e^{nh_n} g(e^{-h_n}) \sqrt{n}(\log n) e^{-c_0 h_n^{-\epsilon_0}}).$$

The required estimate now follows from (29) and (33) in the following way:

$$| J_2(d, s, n) | = O\left(\frac{e^{nh_n} g(-e^{-h_n})}{\sqrt{b(e^{-h_n})}} n^{1/2+3/4}(\log n) e^{-c_0 h_n^{-\epsilon_0}}\right)$$

$$= O(p(n)n^{5/4}(\log n) e^{-c_0 h_n^{-\epsilon_0}}) = O(p(n)n^{5/4}(\log n) e^{-c_0' n^{\epsilon_0/2}})$$

$$= o(\sqrt{n}(\log n)p(n)), \tag{47}$$

where $c_0' > 0$.

The estimate for $J_1(d, s, n)$ follows from Hayman's "locality" condition (32). First, we need to expand $\varphi_{d,s}$ by Taylor formula. We have

$$\varphi_{d,s}(e^{-h_n+i\theta}) = \varphi_{d,s}(e^{-h_n}) + O\left(| \theta | \frac{d}{dx} \varphi_{d,s}(x) |_{x=e^{-h_n}}\right)$$

$$= \varphi_{d,s}(e^{-h_n}) + O\left(\delta_n \frac{d}{dx} \varphi_{d,s}(x) |_{x=e^{-h_n}}\right). \tag{48}$$

To find the asymptotic of $\varphi_{d,s}(e^{-h_n})$, in addition to (44) and (45), we also need the limit of $\prod_{1 \le j \le s}(1 - e^{-j(d+1)h_n})$ as $n \to \infty$, whenever d and s satisfy (41) (see (37)). Using approximations by Riemann integrals as in the analysis of S_1 and S_2, it is easy to show that

$$\prod_{1 \le j \le s}(1 - e^{-j(d+1)h_n}) = \exp\left(\sum_{1 \le j \le s} \log\left(1 - e^{-j(d+1)h_n}\right)\right) \to \begin{cases} 1 & \text{if} & v + u \ge 1, \\ 0 & \text{if} & v + u < 1. \end{cases}$$

Hence, from (41)-(45) it follows that

$$\varphi_{d,s}(e^{-h_n}) \begin{cases} \sim \frac{u+v-1}{2c} \sqrt{n} \log n & \text{if} & v + u \ge 1, \\ = o(\sqrt{n}) & \text{if} & v + u < 1. \end{cases} \tag{49}$$

The estimate of the error term in (48) is tedious and follows the same line of reasoning. We have

$$\frac{d}{dx} \varphi_{d,s}(x) \mid_{x=e^{-h_n}}$$

$$= \left(\sum_{1 \le j \le s} \frac{jx^{j-1}}{(1 - x^j)^2} - (d+1) \sum_{1 \le j \le s} \frac{jx^{j(d+1)}}{(1 - x^{j(d+1)})^2}\right)$$

$$\times \left(\prod_{1 \le j \le s}(1 - x^{j(d+1)})\right) \mid_{x=e^{-h_n}} + \left(\prod_{1 \le j \le s}(1 - x^{j(d+1)})\right)$$

$$\times \exp\left(-(d+1) \sum_{1 \le j \le s} \frac{jx^{j(d+1)-1}}{1 - x^{j(d+1)}}\right) \mid_{x=e^{-h_n}}. \tag{50}$$

It can be seen that the first two sums in the right-hand side of (50) are of order $O(n \log n)$, while the first product factor is estimated by (46). Hence, the first summand in (50) is of order $O(n \log n)$. For the sum in the exponent of the second summand of the right-hand side of (50), one can show that there exists a constant $C > 0$ such that

$$(d+1) \sum_{1 \le j \le s} \frac{jx^{j(d+1)-1}}{1 - x^{j(d+1)}} \mid_{x=e^{-h_n}} \ge C\sqrt{n} \log n.$$

Therefore the second summand in (50) is $O(e^{-C\sqrt{n}\log n})$. Hence

$$\frac{d}{dx} \varphi_{d,s}(x) \mid_{x=e^{-h_n}} = O(n \log n)$$

and by (30) and (49), the expansion in (48) becomes

$$\varphi_{d,s}(e^{-h_n+i\theta}) = \varphi_{d,s}(e^{-h_n}) + O\left(\frac{n^{1/3} \log n}{\Omega(n)}\right), \tag{51}$$

where $\Omega(n) \to \infty$ as $n \to \infty$ arbitrarily slowly. Inserting this estimate and (32) into (39) and applying the asymptotic for the partition function $p(n)$ from (33), we obtain

$$
\begin{aligned}
J_1(d,s,n) &= \frac{e^{nh_n}g(e^{-h_n})}{2\pi}\left(\int_{-\delta_n}^{\delta_n} e^{-\theta^2 b(e^{-d_n})/2}(1+O(1/\Omega^3(n))d\theta\right) \\
&\times \left(\varphi_{d,s}(e^{-h_n}) + O\left(\frac{n^{1/3}\log n}{\Omega(n)}\right)\right) \\
&\sim \frac{e^{nh_n}g(e^{-h_n})}{\sqrt{b(e^{-h_n})}2\pi}\left(\int_{-\delta_n\sqrt{b(e^{-h_n})}}^{\delta_n\sqrt{b(e^{-h_n})}} e^{-y^2/2}dy\right)\left(\varphi_{d,s}(e^{-h_n}) + O\left(\frac{n^{1/3}\log n}{\Omega(n)}\right)\right) \\
&\sim \frac{e^{nh_n}g(e^{-h_n})}{\sqrt{b(e^{-h_n})}2\pi}\left(\int_{-\infty}^{\infty} e^{-y^2/2}dy\right)\left(\varphi_{d,s}(e^{-h_n}) + O\left(\frac{n^{1/3}\log n}{\Omega(n)}\right)\right) \\
&= \frac{e^{nh_n}g(e^{-h_n})}{\sqrt{2\pi b(e^{-h_n})}}\left(\varphi_{d,s}(e^{-h_n}) + O\left(\frac{n^{1/3}\log n}{\Omega(n)}\right)\right) \\
&\sim p(n)\left(\varphi_{d,s}(e^{-h_n}) + O\left(\frac{n^{1/3}\log n}{\Omega(n)}\right)\right),
\end{aligned}
\tag{52}
$$

where for the second asymptotic equivalence we have used (29) and (30) in order to get

$$
\delta_n\sqrt{b(e^{-h_n})} \sim \frac{\pi^{5/6}\sqrt{2}}{6^{1/6}\Omega(n)}n^{1/12} \to \infty
$$

if $\Omega(n) \to \infty$ as $n \to \infty$ not too fast, so that $\frac{n^{1/12}}{\Omega(n)} \to \infty$. It is now clear that (38)-(40), (47) and (52) yield

$$
p(n)\mathbb{E}(Z_{d,s}^{(n)}) = p(n)\varphi_{d,s}(e^{-h_n}) + o(p(n)\sqrt{n}\log n)
$$

and therefore

$$
\mathbb{E}(Z_{d,s}^{(n)}) = \varphi_{d,s}(e^{-h_n}) + o(\sqrt{n}\log n).
$$

The result of Theorem 3.1 now follows from (36), (13), (19) and (49).

6   Proof of Theorem 3.2

We base our proof on (9), Lemmas 2 and 3 and asymptotic equivalence (12). To replace the expectation in the right hand side of (9) by the ratio $\mathbb{E}(Y_{m,s}^{(n)})/\mathbb{E}(Y_n)$, similarly to what we did in the proof of Theorem 3.1, we shall study how unlikely is the event

$$
B_n = \left\{\lambda \in \Lambda(n) : \left|\frac{cY_n(\lambda)}{\sqrt{n}} - 1\right| > \epsilon\right\}, \quad \epsilon > 0,
$$

where c is constant from (19). Using Fristeft's method [6], Corteel et al. [3] showed that

$$
Pr(B_n) \le e^{-c_2\sqrt{n}}, \quad c_2 = c_2(\epsilon) > 0.
\tag{53}
$$

REMARK 6.1 Fristedt's approach [6] is based on the identity

$$Pr(\alpha_j^{(n)} = m_j, j = 1, ..., n) = Pr\left(\gamma_j = m_j, j = 1, ..., n \mid \sum_{j \geq 1} j\gamma_j = n\right), \tag{54}$$

where $\{\gamma_j\}_{j \geq 1}$ is a sequence of independent geometrically distributed random variables, whose distribution is given by

$$Pr(\gamma_j = k) = (1 - q^j)q^{jk}, \quad k = 0, 1, ...$$

and $\{m_j\}_{j \geq 1}$ are non-negative integers. Eq. (54) holds for every fixed $q \in (0, 1)$. It is natural to take q so that $Pr(\sum_{j \geq 1} j\gamma_j = n)$ is as large as possible. Fristedt's almost optimal choice for q is $q = e^{-c/\sqrt{n}}$. Then, the bound in (53) is easily obtained using this value of q.

Next, we represent the probability in (9) in the following way

$$\mathbb{P}(\mu_{n,2} = m, \sigma_{n,2} \leq s) = \mathbb{E}\left(\frac{Y_{m,s}^{(n)}}{Y_n} I_{B_n^c}\right) + \mathbb{E}\left(\frac{Y_{m,s}^{(n)}}{Y_n} I_{B_n}\right), \tag{55}$$

where I_{B_n} and $I_{B_n^c}$ denote the indicators of events B_n and B_n^c, respectively. Since, for any $\lambda \in B_n^c$,

$$\frac{c}{\sqrt{n}(1 + \epsilon)} < \frac{1}{Y_n} < \frac{c}{\sqrt{n}(1 - \epsilon)}$$

if $0 < \epsilon < 1$, the first summand in (55) is estimated by

$$\mathbb{E}\left(\frac{Y_{m,s}^{(n)}}{Y_n} I_{B_n^c}\right) = \frac{c}{\sqrt{n}}(1 + O(\epsilon))\mathbb{E}(Y_{m,s}^{(n)} I_{B_n^c})$$

$$= \frac{c}{\sqrt{n}}(1 + O(\epsilon))(\mathbb{E}(Y_{m,s}^{(n)}) - \mathbb{E}(Y_{m,s}^{(n)} I_{B_n})). \tag{56}$$

Clearly, with probability 1, $Y_{m,s}^{(n)} \leq n$. Hence, using (53), we obtain

$$\mathbb{E}(Y_{m,s}^{(n)} I_{B_n})) = O(nPr(B_n)) = O(ne^{-c_2\sqrt{n}})$$

and (56) becomes

$$\mathbb{E}\left(\frac{Y_{m,s}^{(n)}}{Y_n} I_{B_n^c}\right) = \frac{c}{\sqrt{n}}(1 + O(\epsilon))\mathbb{E}(Y_{m,s}^{(n)}) + O(ne^{-c_2\sqrt{n}}).$$

The second term in the right hand side of (55) is easily estimated using (53) since it is not greater than $Pr(B_n)$. Consequently,

$$\mathbb{P}(\mu_{n,2} = m, \sigma_{n,2} \leq s) = \frac{c}{\sqrt{n}}(1 + O(\epsilon))\mathbb{E}(Y_{m,s}^{(n)}) + O(ne^{-c_2\sqrt{n}}) \tag{57}$$

uniformly for any fixed integer $m \geq 1$ and real $s \geq 1$. Hence, our next task is to obtain an estimate for $\mathbb{E}(Y_{m,s}^{(n)})$, as $n \to \infty$, whenever $s = t\sqrt{n}/c$, $m \geq 1$ is fixed integer and $t \in (0, \infty)$ is also fixed.

Combining results of (18) and (20) of Lemmas 2 and 3, respectively, and approximating the sum by the corresponding Riemann integral, we get

$$
\begin{aligned}
\mathbb{E}(Y_{m,s}^{(n)}) &= (1 + O(1/\sqrt{n})) \\
&\times \sum_{1 \leq k \leq c^{-1}\sqrt{n}t} (\exp(-cmk/\sqrt{n}) - \exp(-c(m+1)k/\sqrt{n})) \\
&\sim \sqrt{n} \int_0^{c^{-1}t} (e^{-cmy} - e^{-c(m+1)y})dy.
\end{aligned}
$$

Replacing this expression into (57) and letting first $n \to \infty$ and then $\epsilon \to 0$, we obtain

$$
\begin{aligned}
\mathbb{P}\left(\mu_{n,2} = m, \frac{c\sigma_{n,2}}{\sqrt{n}} \leq t\right) &\to c \int_0^{c^{-1}t} (e^{-cmy} - e^{-c(m+1)y})dy \\
&= \int_0^t e^{-my}(1 - e^{-y})dy,
\end{aligned}
$$

which completes the proof of Theorem 3.2.

7   Proof of Theorem 3.3

The proof will be based on an asymptotic analysis of formula (11), setting there $s = c^{-1}\sqrt{n}t$ as $n \to \infty$ (see again (19)) and assuming that m is fixed positive integer. First, we let $\Lambda_k(n)$ to denote the set of partitions of n with no part equal to k. Also, let $P_k(n) =| \Lambda_k(n) |$. In [3] Corteel et al. give a combinatorial proof of the following identity:

$$
Pr(\alpha_j^{(n)} = m) = \frac{P_j(n - mj)}{p(n)} = \frac{p(n - jm) - p(n - j(m+1))}{p(n)}.
$$

Replacing this expression into the right hand side of (11) and applying (20), as in the proof of Theorem 3.2, we obtain

$$
\begin{aligned}
\mathbb{P}\left(\mu_{n,3} = m, \frac{c\sigma_{n,3}}{\sqrt{n}} \leq t\right) &\sim m \sum_{1 \leq j \leq c^{-1}\sqrt{n}t} \frac{j}{\sqrt{n}}(e^{-cmj/\sqrt{n}} - e^{c(m+1)j/\sqrt{n}})\frac{1}{\sqrt{n}} \\
&\to m \int_0^{c^{-1}t} y(e^{-cmy} - e^{-c(m+1)y})dy = \frac{m}{c^2} \int_0^t y(1 - e^{-y})e^{-my}dy.
\end{aligned}
$$

This completes the proof of Theorem 3.3.

References

[1] M. ABRAMOVITZ AND I. A. STEGUN , *Handbook of Mathemathical Functions with Formulas, Graphs and Mathematical Tables*, Dover Publ. Inc., New York, 1965.

[2] G. E. ANDREWS, *The Theory of Partitions*, Encyclopedia of Mathematics and its Applications, 2, Addison-Wesley, Reading, MA, 1976.

[3] S. Corteel, B. Pittel, C. D. Savage and H. S. Wilf, *On the multiplicity of parts in a random partition*, Random Stuctures Algorithms, 14 (1999) 185–197.

[4] P. Erdös and J. Lehner, *The distribution of the number of summands in the partition of a positive integer*, Duke Math. J., 8 (1941) 335–345.

[5] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, Cambridge, 2009.

[6] B. Fristedt , *The structure of random partitions of large integers*, Trans. Amer. Math. Soc., 337 (1993) 703–735.

[7] B. Granovsky, D. Stark and M. Erlihson, *Meinardus theorem on weighted partitions: extensions and a probabilistic proof*, Adv. in Appl. Math., 41 (2008) 307–328.

[8] G. Grimmett and D. Stirzaker, *Probability and Random Processes*, Oxford University Press, Oxford, 2001.

[9] G. H. Hardy and S. Ramanujan, *Asymptotic formulae in combinatory analysis*, Proc. Lond. Math. Soc., 17 (1918) 75–115.

[10] W. K. Hayman, *A generalization of Stirling's formula*, J. Reine Angew. Math., 196 (1956) 67–95.

[11] G. Meinardus, *Asymptotische aussagen über partitionen*, Math. Z., 59 (1954) 388–398.

[12] L. Mutafchiev, *The size of the largest part of random weighted partitions of large integers*, Combin. Probab. Comput., 22 (2013) 433–454.

[13] L. Mutafchiev, *Sampling part sizes of random integer partitions*, Ramanujan J., 37 (2015) 329–343.

[14] H. Rademacher, *On the partition function $p(n)$*, Proc. London Math. Soc., 43 (1937) 241–254.

[15] V. N. Sachkov, *Combinatorial Methods in Discrete Mathematics*, Encyclopedia of Mathematics and its Applications, 55, Cambridge University Press, Cambridge, 1999.

[16] H. Wilf, *Three problems in combinatorial analysis*, J. Combin. Theory Ser. A, 35 (1983) 199–207.