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The dominance order for permutations

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Abstract. We define an order relation over S_n considering the Robinson-Schensted bijection and the dominance order over Young tableaux. This order relation makes $S_n(k\ k-1...3\ 2\ 1)$ -the set of permutations of length n that avoid the pattern $k\ k-1...3\ 2\ 1$, $k \leq n$- a principal filter in S_n. We study in detail these order relations on $S_n(321)$ and $S_n(4321)$, finding order-isomorphisms between these sets and sets of lattice paths.

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1 Introduction

The study of particular order relations over sets of combinatorial objects can be a powerful tool to understand the structure of such sets.
The present paper focuses on the set S_n of permutations of length n and his subset $S_n(k\ k-1...3\ 2\ 1)$, namely, the set of permutations that avoid a descending pattern of length k ($k \leq n$).
The order relation that we study over this set is obtained by considering the *dominance order* $\trianglelefteq$ over $Tab(n)$, the set of standard Young tableaux with n boxes, and the product order $\trianglelefteq \times \trianglelefteq$ over the set $SYT(n)$ of pairs of standard Young tableaux of the same shape. This last relation induces, via the Robinson-Schensted bijection, an order relation $\trianglelefteq$ over S_n, called again dominance order, and hence, by restriction, over $S_n(k\ k-1...3\ 2\ 1)$.
The choice of this order $\trianglelefteq$ has two reasons: the first one is the importance of the dominance order over the set of standard Young tableaux in representation theory (see e.g. [5]), the second one is that, as we will see, the posets $(S_n(k\ k-1...3\ 2\ 1), \trianglelefteq)$ have some interesting properties when $k = 3$ or $k = 4$.
In Section 2 we study the order $\trianglelefteq$ over $S_n(k\ k-1...3\ 2\ 1)$. The structure of the partially ordered set $(S_n(k\ k-1...3\ 2\ 1), \trianglelefteq)$ is quite tangled (for $k \geq 4$, it is neither a lattice, nor a graded poset), however, it has some remarkable properties, for example the fact that $S_n(k\ k-1...3\ 2\ 1)$ is a principal filter in $(S_n, \trianglelefteq)$.
In Section 3 we restrict our attention to the case of $k = 3$. In the literature, several bijections between the set $\mathcal{D}_n$ of Dyck paths of semilength n and the subset of S_n of permutations avoiding a pattern of length 3 appear. Most of them are based on the bijections given in [11], [12] and [13]. In this paper, we consider the bijection ϕ given in [8] between $S_n(321)$ and $\mathcal{D}_n$. This bijection is an order isomorphism between $(S_n(321), \trianglelefteq)$ and the distributive lattice $\mathcal{D}_n$ (endowed with the following order:

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a path f is smaller than a path g if and only f lies above g). In particular, $(S_n(321), \trianglelefteq)$ turns out to be a distributive lattice.
We show that the dominance order over $S_n(321)$ can be described in terms of Knuth equivalences and use the map ϕ to find some interesting partitions of the sets $S_n(321)$ and $\mathcal{D}_n$.
In Section 4 we define a bijection ψ between $S_n(4321)$ and a suitable subset of the set of Motzkin paths of length $2n$. This bijection, restricted to the subset $S_n(321)$ of $S_n(4321)$ turns out to be nothing but the map ϕ defined in section 3.

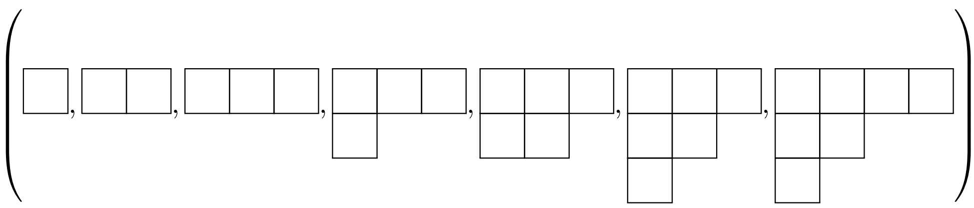
2 An order over the set of permutations avoiding k k-1...321

We begin by recalling some basic notion about standard Young tableaux.
A *partition* of the positive integer n is a non-increasing sequence of non-negative integers $(\lambda_1, \lambda_2, \lambda_3, ...)$ such that $\sum_{i=1}^{k} \lambda_i = n$ (k is the greatest integer such that $\lambda_k > 0$). A partition of n can be identified with a *Ferrers diagram* with n boxes, namely, a left-justified array of n empty boxes such that each row contains at most as many boxes as the preceding one. The Ferrers diagram corresponding to the partition $(\lambda_1, ..., \lambda_k)$ has k rows of length $\lambda_1, \lambda_2, ..., \lambda_k$, respectively. A *standard Young Tableau* P is a completion of a Ferrers diagram with the integers $\{1, 2, ..., n\}$ such that it is strictly increasing along rows and columns; the corresponding partition is called the *shape* of P. For a detailed introduction to the combinatorics of Young tableaux and their applications in representation theory see e.g. [10].
Given a tableau P, for every $1 \leq j \leq n$ let P_j be the tableau induced by the elements $\{1, 2, ..., j\}$ of P and denote by shP_j the shape of the tableau P_j. We can identify the tableau P with the sequence of shapes $(shP_1, shP_2, ..., shP_n)$, with $shP_i \subseteq shP_{i+1}$, and, conversely, every such sequence gives rise to a standard tableau, since we have an order for the insertion of elements.

EXAMPLE 2.1 The standard Young tableau

1	2	3	7
4	5		
6			

can be identified with the sequence



We recall that the *dominance order* between partitions of n is given by $\lambda \trianglelefteq \tau$ if and only if $\lambda_1 + ... + \lambda_j \leq \tau_1 + ... + \tau_j$ for all j (for a detailed study of the lattice of partitions, see [6]). This order relation can be extended to an order over the set $Tab(n)$ of standard Young tableaux with n boxes, called *dominance order over standard tableaux*, by setting

$$P \trianglelefteq Q \tag{1}$$

if and only if $shP_j \trianglelefteq shQ_j$ for all $j \leq n$. This order is used in the representation theory of the symmetric group and of the general linear group, see e.g. [5].

Example 2.2 The tableau

$$\ytableaushort{12,35,4} = \left(\ydiagram{1}, \ydiagram{2}, \ydiagram{2,1}, \ydiagram{2,1,1}, \ydiagram{2,2,1} \right)$$

is greater than the tableau

$$\ytableaushort{13,25,4} = \left(\ydiagram{1}, \ydiagram{1,1}, \ydiagram{2,1}, \ydiagram{2,1,1}, \ydiagram{2,2,1} \right)$$

Note that, in general, the dominance order defined in (1) does not yield a lattice structure over the set of standard Young tableaux. Consider for example the tableaux

$$P = \ytableaushort{1237,456} \quad \text{and} \quad P' = \ytableaushort{12345,6,7}$$

Then two incomparable lower bounds of P and P' are $\ytableaushort{1237,45,6}$ and $\ytableaushort{123,456,7}$. Since P covers both these lower bounds, the infimum of P and P' does not exist.

It will be useful to describe explicitly the covering relation associated with the dominance order.

Proposition 2.3 *If P, P' are two standard tableaux with the same number of boxes, P' covers P in the dominance order (in symbols, $P' \succ P$) if and only if P is obtained from P' by one of the following operations:*

(S) *if $shP = shP'$, choose one box filled with the integer i whose position is right and above the box filled with $i+1$ and swap these two boxes.*

(M) *if $shP \neq shP'$, move one corner-box (i, j) filled with the integer k to the first row below the i-th whose length is smaller than j, with the further condition that between the starting and the ending position of k there are no corner-boxes filled with an integer greater than k.*

Proof. First of all, note that the covering relation in the lattice of integer partitions with the dominance order is the following: the partition σ covers the partition τ if and only if the Ferrers diagram of τ is obtained by the Ferrers diagram of σ by moving one corner-box in position (i,j) to the first row below the i-th whose length is smaller than j (see [6]).
Now we proceed by induction on the number of boxes n. If $n = 1, 2, 3$ the proposition is trivial.
Suppose it true for tableaux with at most $n-1$ boxes and suppose that P and P' are two standard tableaux with n boxes such that $P \prec P'$. Consider P_{n-1} and P'_{n-1}. Since P' covers P, P'_{n-1} covers P_{n-1} and then, by the induction hypothesis, one of the following possibilities is true:

- P_{n-1} is obtained from P'_{n-1} by applying operation **(S)**.
  In this case, the shapes of P and P' must be equal and P and P' are obtained from P_{n-1} and P'_{n-1}, respectively, by adding one box (filled with n) in the same position. So we get P from P' by applying operation **(S)**.

- P_{n-1} is obtained from P'_{n-1} by applying the operation **(M)** to the box filled with the integer j, $j \leq n-1$.
  In this case, P and P' are obtained from P_{n-1} and P'_{n-1}, respectively, by adding the box filled with n. Since P' covers P, the new box must be in a position not between the starting and the ending position of the box filled with j. So we get P from P' by applying the operation **(M)**.

- $P_{n-1} = P'_{n-1}$.
  In this case, the shapes of P and P' are different, so P is obtained from P' by applying operation **(M)** to the corner-box filled with the integer n.

$\square$

Example 2.4 Consider the tableaux

$$P' = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & \multicolumn{2}{c}{} \\ \cline{1-1} 5 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$$

and

$$P = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}.$$

Then P' covers P. In fact, P is obtained from P' by applying the operation **(M)** to the box filled with 4 of P'.
Consider now the tableaux

$$R' = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}$$

and

$$R = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}.$$

Then R' does not cover the tableau R. In fact, the further condition of operation **(M)** is not verified: there is a box filled with 5 between the starting and the ending position of 4. Actually,

$$\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array} \succ \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1} 5\\\cline{1-1}\end{array} \succ \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5\\\cline{1-1}\end{array} \succ \begin{array}{|c|c|}\hline 1&2\\\hline 3&5\\\hline 4\\\cline{1-1}\end{array}.$$

Consider now the set $SYT(n)$ of all pairs of standard Young tableaux of the same shape in the elements $\{1, 2, ..., n\}$.

We can endow the set $SYT(n)$ with the product of the dominance order over standard Young tableaux, namely,

$$(P, Q) \trianglelefteq (P', Q') \text{ if and only if } P \trianglelefteq Q \text{ and } P' \trianglelefteq Q'. \tag{2}$$

We call this order relation *dominance order over $SYT(n)$*.

By restriction, we obtain also an order over $SYT_k(n)$, the subset of $SYT(n)$ of pairs of tableaux with at most k rows, $1 \leq k \leq n$. In this case the poset $SYT_k(n)$ has a minimum, namely the pair $(T_k(n), T_k(n))$, where:

$$T_k(n) = \begin{array}{|c|c|c|l}\hline 1&k+1&2k+1&\ldots\\\hline 2&k+2&\multicolumn{1}{c}{\vdots}\\\cline{1-2} 3&k+3\\\cline{1-2} \multicolumn{1}{c}{\vdots}&\multicolumn{1}{c}{\vdots}\\ \multicolumn{1}{c}{\vdots}&\multicolumn{1}{c}{\vdots}\\\cline{1-2} k&2k\\\cline{1-2}\end{array}$$

For example, if $n = 9$

$$T_4(9) = \begin{array}{|c|c|c|}\hline 1&5&9\\\hline 2&6\\\cline{1-2} 3&7\\\cline{1-2} 4&8\\\cline{1-2}\end{array}.$$

Proposition 2.3 implies that $(T_k(n), T_k(n))$ is in fact the minimum of $SYT_k(n)$.

The well known Robinson-Schensted bijection RS associates with every permutation $\pi \in S_n$ a pair of standard Young tableaux $(P, Q) \in SYT(n)$. P is called the *insertion tableau* of π and Q the *recording tableau* of π (see [10]). One of the fundamental properties of this bijection is expressed by the following theorem (see [14]).

THEOREM 2.5 (Schensted) *Let d and i be the length of the greatest decreasing subsequence and of the greatest increasing subsequence of a permutation π. Then $RS(\pi)$ is a pair of tableaux with d rows and i columns.*

Let τ be a sequence of h integers. The *normalization* $|\tau|$ of τ is the permutation of S_h whose elements have the same relative order as the elements of τ. For istance, $|5683| = 2341 \in S_4$. As usual, we say that the permutation $\pi \in S_n$ *avoids the pattern* $\tau \in S_h$ if there are no subsequences of π whose normalization is τ.

Schensted's theorem implies that the Robinson-Schensted bijection maps the set $S_n(k+1\ k\ ...3\ 2\ 1)$ of permutations of length n that avoid the pattern $k+1\ k\ ...3\ 2\ 1$ to the set $SYT_k(n)$. This fact allows us to define an order relation over the set $S_n(k+1\ k\ ...3\ 2\ 1)$, as follows:

$$\pi \trianglelefteq \sigma \iff RS(\pi) \trianglelefteq RS(\sigma) \tag{3}$$

where the $\trianglelefteq$ on the right-hand side denotes the dominance order over SYT(n). In particular, we have an order over $S_n = S_n(n+1\ n\ ...2\ 1)$ and the order over $S_n(k+1\ k\ ...3\ 2\ 1)$, $1 \leq k \leq n-1$, is simply the restriction of this order over S_n. We call this order *dominance order over permutations.*
Though the structure of the posets $(S_n, \trianglelefteq)$ and $(SYT(n), \trianglelefteq)$ is quite difficult to describe, these posets have some remarkable symmetries given in the next proposition. Here, if τ is a partition of the integer n, τ^T is the conjugate partition (see [10]) and, similarly, if P is a standard tableau, P^T is the conjugate tableau.

PROPOSITION 2.6 *In the poset $SYT_k(n)$, for every k with $1 \leq k \leq n$, the map $(P, Q) \mapsto (Q, P)$ is an order isomorphism. Equivalently, in the poset $S_n(k+1\ k\ k-1...1)$, the map $\pi \mapsto \pi^{-1}$ is an order isomorphism.*
In the poset $SYT(n)$, the map $(P, Q) \mapsto (P^T, Q^T)$ is an order anti-isomorphism.

Proof. It follows directly from the definitions of the order relations given by (2) and (3). ☐

Now consider the permutation

$$\psi_k(n) := \left(\begin{smallmatrix} 1 & 2 & 3 & \cdots & k & k+1 & \cdots & & 2k & \cdots & & & & & & \cdots & n \\ k & k-1 & k-2 & \cdots & 1 & 2k & 2k-1 & \cdots & k+1 & \cdots & tk & tk-1 & \cdots & (t-1)k+1 & n & n-1 \cdots & tk+1 \end{smallmatrix}\right),$$

where $t = \lfloor \frac{n}{k} \rfloor$. For example, if $n = 11$,

$$\psi_4(11) := \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 4 & 3 & 2 & 1 & 8 & 7 & 6 & 5 & 11 & 10 & 9 \end{smallmatrix}\right).$$

We can now state the main result of this section.

THEOREM 2.7 *For every $1 \leq k \leq n$, the set $S_n(k+1\ k...2\ 1)$ is a principal filter in S_n generated by the permutation $\psi_k(n)$, namely,*

$$S_n(k+1\ k...2\ 1) = \{\sigma \in S_n; \sigma \trianglerighteq \psi_k(n)\}.$$

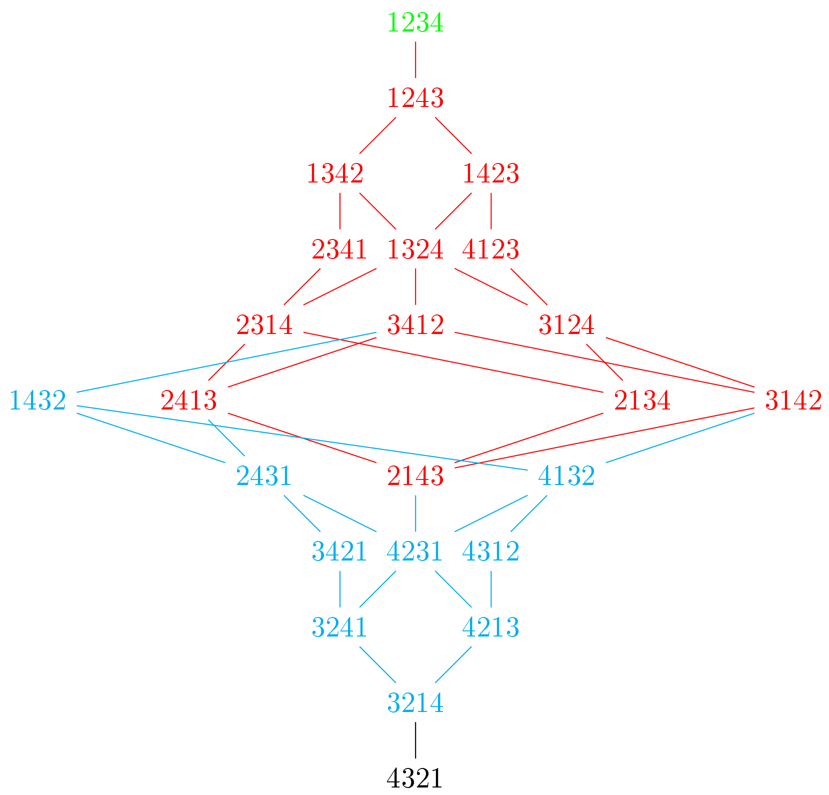
Proof. Recall that the poset $SYT_k(n)$ has a minimum, namely, the pair $(T_k(n), T_k(n))$. The permutation corresponding to this pair via the inverse of the Robinson-Schensted correspondence is $\psi_k(n)$ and hence it is the minimum of $S_n(k+1\ k...3\ 2\ 1)$ in the dominance order defined in 3. ☐

As an immediate consequence we get the following.

COROLLARY 2.8 *Consider the set S_n with the order relation defined above. In this poset there is a chain of principal filters, given by*

$$S_n(21) \subset S_n(321) \subset ... \subset S_n.$$

EXAMPLE 2.9 The Hasse diagram of the chain $S_4(21) \subset S_4(321) \subset S_4(4321) \subset S_4$ is



3 The case of at most two rows

The dominance order defined above reveals to have some remarkable properties when restricted to the set $SYT_2(n)$ of pairs of standard Young tableaux with at most two rows, namely, to permutations avoiding the pattern 321.

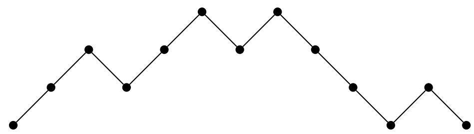
We now describe a bijection between the set $SYT_2(n)$ and the set $\mathcal{D}_n$ of Dyck paths of semilength n, originally stated in [8], essentially due to Knuth [12] and widely used (see e.g. [1]). Recall that a *Dyck path* of semilength n is a lattice path consisting of up steps $U = (1, 1)$ and down steps $D = (1, -1)$, starting at $(0, 0)$, ending at $(0, 2n)$, and never going below the x-axis. Similarly, a *Dyck prefix* of length n is a lattice path consisting of n steps $U = (1, 1)$ and $D = (1, -1)$, starting at $(0, 0)$ and never going below the x-axis.

Given $(P, Q) \in SYT_2(n)$, the corresponding Dyck path is obtained by associating both P and Q with a Dyck prefix in which the up steps correspond to the elements of the first row of the tableau and down steps correspond to the elements of the second row. In this way we obtain a pair of prefixes (p, q) that end at the same height (since P and Q have the same shape). Considering p and q as words in the letters U and D and defining the *inverse* of a word $p = (p_1, p_2, ..., p_n)$ as $p^{-1} = (-p_n, ..., -p_1)$ where $-D := U$ and $-U := D$, the juxtaposition of the words q and p^{-1} is a Dyck path qp^{-1}. Let ϕ_1 be the map $(P, Q) \mapsto qp^{-1}$. ϕ_1 is a bijection between $SYT_2(n)$ and the set $\mathcal{D}_n$ of Dyck paths of semilength n. The composition $\phi := \phi_1 \circ RS$ is the bijection between $S_n(321)$ and $\mathcal{D}_n$ defined in [8].

EXAMPLE 3.1 Consider the permutation $\pi \in S_6(321)$ given by $\pi = 231465$. Then $RS(\pi) = (P, Q)$ with

$$P = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 5 \\ \hline 2 & 6 \\ \cline{1-2}\end{array} \quad \text{and} \quad Q = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 5 \\ \hline 3 & 6 \\ \cline{1-2}\end{array}$$

so $p = UDUUUD$, $q = UUDUUD$, $p^{-1} = UDDDUD$ and $\phi(\pi)$ is the path whose corresponding word is $qp^{-1} = UUDUUDUDDDUD$:



We endow the set $\mathcal{D}_n$ with the following order: $f \leq g$ if the path f lies above the path g. It is a well known fact that $(\mathcal{D}_n, \leq)$ is actually a distributive lattice (see e.g. [9]).

Our next goal is to show that the order relation over $S_n(321)$ induced by the order over $\mathcal{D}_n$ via the bijection ϕ defined above is precisely the dominance relation defined in (3). This fact is an immediate consequence of the following result.

THEOREM 3.2 *Consider the set $SYT_2(n)$ endowed with the dominance order defined in (2). Then the map ϕ_1 is an order isomorphism between $SYT_2(n)$ and $\mathcal{D}_n$.*

Proof. In the lattice $\mathcal{D}_n$ the covering relation can be expressed in terms of valleys and peaks. A *valley* of a Dyck path f is a pair of consecutive steps DU; similarly, a *peak* is a pair of consecutive steps UD. A valley (a peak) is at *position* $j, j+1$ if the first and the second step of the valley (of the peak, respectively) are the j-th and the $j+1$-th step of the path. It follows immediately from the definition of the order relation over $\mathcal{D}_n$ that a path f covers a path g if and only if f is obtained from g by replacing a valley by a peak.
Now, given two Dyck paths f, g of semilength n, suppose that f covers g (in symbols, $f \succ g$). Applying the map ϕ_1^{-1} to f and g, we obtain two pairs of tableaux (P_f, Q_f) and (P_g, Q_g) where $shP_f = shQ_f$ and $shP_g = shQ_g$. Since f covers g, f is obtained from g by the replacement of a valley with a peak. If this valley (and this peak) is at position $i, i+1$, then the following cases are possible:

1. If $i < n$, then $P_f = P_g$, $shQ_g = shQ_g$ and Q_f is obtained from Q_g by swapping the boxes filled with $i+1$ and i. Note that, in Q_g, the box filled with $i+1$ must be right and above the box filled with i.

2. If $i > n$, then $Q_f = Q_g$, $shP_g = shP_g$ and P_f is obtained from P_g by swapping the boxes filled with $i+1$ and i. Note that, in P_g, the box filled with $i+1$ must be right and above the box filled with i.

3. If $i = n$, then $shP_g = shQ_g \neq shP_f = shQ_f$ and P_f and P_g are obtained from Q_f and Q_g by moving the corner box filled with the integer n from the last position of the second row to the end of the first row, respectively.

The previous cases, by proposition 2.3, correspond exactly to the covering relation over the set $SYT_2(n)$. So we have proved:

$$g \prec f \Longleftrightarrow (P_g, Q_g) \prec (P_f, Q_f).$$

□

In this way $(SYT_2(n), \trianglelefteq)$ turns out to be a distributive lattice isomorphic to $\mathcal{D}_n$. The maximum and the minimum elements of this lattice are:

$$\left(\begin{array}{|c|c|c|c|} \hline 1 & 2 & \cdots & n \\ \hline \end{array} , \begin{array}{|c|c|c|c|} \hline 1 & 2 & \cdots & n \\ \hline \end{array} \right)$$

and

$$\begin{cases} \left(\begin{array}{|c|c|c|c|c|} \hline 1 & 3 & \cdots & n-2 & n \\ \hline 2 & 4 & \cdots & n-1 & \\ \hline \end{array} , \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & \cdots & n-2 & n \\ \hline 2 & 4 & \cdots & n-1 & \\ \hline \end{array} \right) & \text{if } n \text{ is odd} \\ \left(\begin{array}{|c|c|c|c|} \hline 1 & 3 & \cdots & n-1 \\ \hline 2 & 4 & \cdots & n \\ \hline \end{array} , \begin{array}{|c|c|c|c|} \hline 1 & 3 & \cdots & n-1 \\ \hline 2 & 4 & \cdots & n \\ \hline \end{array} \right) & \text{if } n \text{ is even} \end{cases}$$

that correspond to the paths U^nD^n and $(UD)^n$ respectively.
We point out that the set $\mathcal{D}_n$ corresponds also to the set $S_n(312)$, via the bijection γ originally stated in [11] and studied in [13]. If the set of Dyck paths is endowed with the order $\leq$, the corresponding order over $S_n(312)$ is the Bruhat order (see [2]). This is not the case for $S_n(321)$, since this set endowed with the Bruhat order is not a lattice. However, if we consider the composition $\gamma \circ \phi : S_n(321) \to S_n(312)$ we obtain an order isomorphism between $(S_n(321), \trianglelefteq)$ and $S_n(312)$ with the Bruhat order. The map $\gamma \circ \phi$ appears also in the classifcation [7] of the bijections between these two sets.
From these considerations, it follows immediately the following.

Corollary 3.3 *The set $S_n(321)$ with the dominance order is a distributive lattice isomorphic to the lattice $S_n(312)$ with the Bruhat order.*

Note in particular that the elements of $SYT_2(n)$ of the form (P, P), i.e., the involutions in $S_n(321)$, correspond to symmetric paths. In particular, the set $Tab_2(n)$ of standard Young tableaux with at most two rows endowed with the dominance order defined in (1) is a distributive lattice isomorphic to the sublattice of $\mathcal{D}_n$ of symmetric paths.

In the following, we will need the next lemma that describes the operations of sup and inf in $SYT_2(n)$ and $Tab_2(n)$.

Lemma 3.4 *In the lattice $Tab_2(n)$ the infimum and the supremum are given by*

$$inf(P, P') = (inf(shP_1, shP'_1), inf(shP_2, shP'_2), ..., inf(shP_n, shP'_n))$$

and

$$sup(P, P') = (sup(shP_1, shP'_1), sup(shP_2, shP'_2), ..., sup(shP_n, shP'_n)).$$

In the lattice $SYT_2(n)$ the infimum and the supremum are given by

$$inf((P,Q),(P',Q')) = (inf(P,P'), inf(Q,Q'))$$

and

$$sup((P,Q),(P',Q')) = (sup(P,P'), sup(Q,Q')).$$

Proof. It follows from the fact that $inf(shP_j, shP'_j) \subseteq inf(shP_{j+1}, shP'_{j+1})$ for all j and that, if $shP = shQ$ and $shP' = shQ'$, $sh(inf(P,P')) = sh(inf(Q,Q'))$, so the infimum operations considered in the lemma are well defined. Similarly, the supremum operations are well defined. □

We now show that the order relation over $S_n(321)$ defined in (3) can be described in terms of Knuth equivalences.
First of all, we recall that, given $\sigma, \tau \in S_n$, with $RS(\tau) = (P_\tau, Q_\tau)$ and $RS(\sigma) = (P_\sigma, Q_\sigma)$, the permutations τ and σ are said to be *Knuth equivalent* ($\tau \sim \sigma$) if $P_\sigma = P_\tau$, and *dual-Knuth equivalent* ($\tau \sim_d \sigma$) if $Q_\sigma = Q_\tau$.
In particular, since $RS(\pi) = (P,Q) \Leftrightarrow RS(\pi^{-1}) = (Q,P)$,

$$\sigma \sim \tau \Leftrightarrow \sigma^{-1} \sim_d \tau^{-1}.$$

It is well known that the relations $\sim$ and $\sim_d$ can be characterized as follows (see e.g. [10]):

$\tau \sim \sigma$ if and only of τ is obtained from σ by a finite sequence of the following transformations :

$$\begin{cases} yzx \mapsto yxz & \text{where } yzx \text{ are three consecutive letters, } x<y<z \\ yxz \mapsto yzx & \text{where } yxz \text{ are three consecutive letters, } x<y<z \\ xzy \mapsto zxy & \text{where } xzy \text{ are three consecutive letters, } x<y<z \\ zxy \mapsto xzy & \text{where } zxy \text{ are three consecutive letters, } x<y<z \end{cases}$$

$\tau \sim_d \sigma$ if and only of τ is obtained from σ by a finite sequence of the following transformations :

$$\begin{cases} cab \mapsto bac & \text{where } cab \text{ are three letters, } b=a+1, c=b+1 \\ bac \mapsto cab & \text{where } bac \text{ are three letters, } b=a+1, c=b+1 \\ acb \mapsto bca & \text{where } acb \text{ are three letters, } b=a+1, c=b+1 \\ bca \mapsto acb & \text{where } bca \text{ are three letters, } b=a+1, c=b+1 \end{cases}$$

Given $\pi = \pi_1\pi_2...\pi_n \in S_n$, define ${}^j\pi$ to be the permutation of S_j obtained by considering the integers $1, 2, ..., j$ in the same order as they appear in π. For example if $\pi = 5143726$ then ${}^4\pi = 1432$. Note that, if P is the insertion tableau of π, then P_j is the insertion tableau of ${}^j\pi$. Moreover, if Q is the recording tableau of π, then Q_j is the recording tableau of $|\pi_1\pi_2...\pi_j|$, the normalization of $\pi_1\pi_2...\pi_j$.
If $\pi \in S_n(321)$, note that π has at most two maximal increasing subsequences. In fact, the greatest elements of such subsequences are in decreasing order. Denote with $l(\pi)$ the length of a maximal increasing subsequence of π and with $M(\pi)$ the maximum between $a_{l(\pi)}$ and $b_{l(\pi)}$, where $a = a_1, a_2, ..., a_{l(\pi)}$ and $b = b_1, b_2, ...b_{l(\pi)}$ are the maximal increasing subsequences in π (b could be empty).
Now we can state the characterization of the covering relation over $S_n(321)$.

THEOREM 3.5 *Consider the dominance order over $S_n(321)$. In this order, a permutation $\sigma = \sigma_1...\sigma_n$ covers a permutation $\pi = \pi_1...\pi_n$ (in symbols, $\sigma \succ \pi$) if and only if one of the following conditions is satisfied:*

P(1) *π and σ are dual-Knuth equivalent and there exists an integer j with $1 \leq j \leq n-2$ such that*

- *${}^j\pi$ and ${}^j\sigma$ are Knuth equivalent*
- *${}^{j+1}\pi$ is obtained from ${}^j\pi$ by inserting $j+1$ in a position to the left of $M({}^j\pi)$ and ${}^{j+1}\sigma$ is obtained from ${}^j\sigma$ by inserting $j+1$ in a position to the right of $M({}^j\sigma)$*
- *${}^{j+2}\pi$ is obtained from ${}^{j+1}\pi$ by inserting $j+2$ in a position to the right of $M({}^{j+1}\pi)$ and ${}^{j+2}\sigma$ is obtained from ${}^{j+1}\sigma$ by inserting $j+2$ in a position to the left of $M({}^{j+1}\sigma)$*
- *${}^k\pi$ and ${}^k\sigma$ with $k \geq j+3$, are obtained from ${}^{j+2}\pi$ and ${}^{j+2}\sigma$, respectively, by inserting $j+3, j+4, ..., k$ in the same positions*

P(2) *π and σ are Knuth equivalent and there exists an integer $1 \leq j \leq n$ such that*

- *$|\pi_1...\pi_{j+1}|$ and $|\sigma_1...\sigma_{j+1}|$ are dual-Knuth equivalent*
- *$l(|\pi_1...\pi_j\pi_{j+1}|) = l(|\pi_1...\pi_j|)$ and $l(|\sigma_1...\sigma_j\sigma_{j+1}|) = l(|\sigma_1...\sigma_j|) + 1$*
- *$l(|\pi_1...\pi_j\pi_{j+1}\pi_{j+2}|) = l(|\pi_1...\pi_j\pi_{j+1}|) + 1$ and $l(|\sigma_1...\sigma_j\sigma_{j+1}\sigma_{j+2}|) = l(|\sigma_1...\sigma_j\sigma_{j+1}|)$*
- *$\forall k \geq j+3$, $l(|\pi_1...\pi_k|) = l(|\sigma_1...\sigma_k|)$*

P(3) *${}^{n-1}\pi$ and ${}^{n-1}\sigma$ are Knuth equivalent, $|\pi_1...\pi_{n-1}|$ and $|\sigma_1...\sigma_{n-1}|$ are dual-Knuth equivalent, $\sigma_n = n$, $l(\pi) = l({}^{n-1}\pi)$ and $l(\sigma) = l({}^{n-1}\sigma) + 1$.*

In particular, the bijection ϕ defined above is a poset isomorphism between the posets $S_n(321)$ and $\mathcal{D}_n$, hence, $S_n(321)$ turns out to be a distributive lattice.

Proof. We want to prove that the dominance order over $S_n(321)$ has the stated covering relation. Suppose that, in this order, $\pi \prec \sigma$ and $RS(\pi) = (P, Q)$, $RS(\sigma) = (P', Q')$.
Recall that in the dominance order defined in (2), $(P, Q) \prec (P', Q')$ if and only if one of the following condition is satisfied

(C1) $shP = shQ = shP' = shQ'$, $Q = Q'$ and $P \prec P'$

(C2) $shP = shQ = shP' = shQ'$, $P = P'$ and $Q \prec Q'$

(C3) $shP = shQ \neq shP' = shQ'$, $P \prec P'$ and $Q \prec Q'$

The three previous cases correspond respectively to the three cases **P(1)**, **P(2)**, **P(3)**.
In fact, in case **(C1)**, since $Q = Q'$, π and σ are dual-Knuth equivalent. By proposition 2.3, P is obtained from P' by choosing one box filled with the integer $j+1$ in the first row, one box filled with the integer $j+2$ in the second row and swapping these boxes. So the tableaux P_j and P'_j are equal, hence ${}^j\pi$ and ${}^j\sigma$ are Knuth equivalent, and the tableaux P_{j+1} and P'_{j+1} have the boxes filled with $j+1$ in the second and first row, respectively.
Considering the definition of the insertion procedure, this implies that ${}^{j+1}\pi$ is obtained from ${}^j\pi$ by adding the element $j+1$ to the left of $M({}^j\pi)$ and that ${}^{j+1}\sigma$ is obtained from ${}^j\sigma$ by adding $j+1$ to the

right of $M({}^j\sigma)$. Similarly, ${}^{j+2}\pi$ and ${}^{j+2}\sigma$ are obtained as described. All the ${}^k\pi$ and ${}^k\sigma$ with $k \geq j+3$ have the elements $j+3, j+4, ..., k$ in the same position because $Q = Q'$ and P and P' differ only in the boxes filled with $j+1$ and $j+2$.
In case **(C2)**, note that, during the insertion procedure, the insertion of the $(j+1)$th element does not modify the tableau Q_j but it only adds the box $j+1$ at the end of the first or of the second row of Q_j. Since, in this case, $P = P'$ and Q is obtained from Q' by swapping the boxes $j+1$ and $j+2$, $|\pi_1...\pi_j|$ and $|\sigma_1...\sigma_j|$ are dual-Knuth equivalent and $l(|\pi_1...\pi_j\pi_{j+1}|) = l(|\pi_1...\pi_j|)$, $l(|\sigma_1...\sigma_j\sigma_{j+1}|) = l(|\sigma_1...\sigma_j|)+1$. Similarly, considering Q_{j+2} and Q'_{j+2} we have $l(|\pi_1...\pi_j\pi_{j+1}\pi_{j+2}|) = l(|\pi_1...\pi_j\pi_{j+1}|)+1$ and $l(|\sigma_1...\sigma_j\sigma_{j+1}\sigma_{j+2}|) = l(|\sigma_1...\sigma_j\sigma_{j+1}|)$ and , considering Q_k and Q'_k, $k \geq j+3$, we have $l(|\pi_1...\pi_k|) = l(|\sigma_1...\sigma_k|)$.
In the same way, in case **(C3)**, the tableaux have the following form:

$$P = \begin{array}{|c|c|c|c|c|}\hline a_1 & a_2 & \dots & \dots & a_l \\ \hline a_{l+1} & \dots & a_{n-1} & n & \\ \cline{1-4}\end{array}, \quad P' = \begin{array}{|c|c|c|c|c|c|}\hline a_1 & a_2 & \dots & \dots & a_l & n \\ \hline a_{l+1} & \dots & a_{n-1} \\ \cline{1-3}\end{array}$$

and

$$Q = \begin{array}{|c|c|c|c|c|}\hline b_1 & b_2 & \dots & \dots & b_l \\ \hline b_{l+1} & \dots & b_{n-1} & n & \\ \cline{1-4}\end{array}, \quad Q' = \begin{array}{|c|c|c|c|c|c|}\hline b_1 & b_2 & \dots & \dots & b_l & n \\ \hline b_{l+1} & \dots & b_{n-1} \\ \cline{1-3}\end{array}.$$

In particular, $P_{n-1} = P'_{n-1}$, $Q_{n-1} = Q'_{n-1}$ and $\sigma_n = n$. So ${}^{n-1}\sigma = \sigma_1\sigma_2...\sigma_{n-1} = |\sigma_1\sigma_2...\sigma_{n-1}|$. This implies that $l({}^{n-1}\sigma) = l(|\sigma_1...\sigma_{n-1}|)$ (so also $l({}^{n-1}\pi) = l(|\pi_1...\pi_{n-1}|)$), that ${}^{n-1}\pi$ and ${}^{n-1}\sigma$ are Knuth equivalent and that $|\pi_1...\pi_{n-1}|$ and $|\sigma_1...\sigma_{n-1}|$ are dual-Knuth equivalent. Since P and Q are obtained from P_{n-1} and Q_{n-1} adding n in the second row and P' and Q' are obtained from P'_{n-1} and Q'_{n-1} adding n in the first row, $l(\pi) = l({}^{n-1}\pi)$ and $l(\sigma) = l({}^{n-1}\sigma)+1$.
Viceversa, it is easy to see with similar arguments that the conditions given in the theorem imply respectively the conditions **(C1)**, **(C2)** and **(C3)** that characterize the covering relation in $SYT_2(n)$. $\square$

The previous theorem allows us to describe the order relation over $S_n(321)$ without using the Robinson-Schensted map.

EXAMPLE 3.6 Consider the permutations $\sigma = 4571236$ and $\pi = 3571246$. Then

$$RS(\sigma) = \left(\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 6 \\ \hline 4 & 5 & 7 \\ \cline{1-3}\end{array}, \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 7 \\ \hline 4 & 5 & 6 \\ \cline{1-3}\end{array} \right)$$

and

$$RS(\pi) = \left(\begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 6 \\ \hline 3 & 5 & 7 \\ \cline{1-3}\end{array}, \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 7 \\ \hline 4 & 5 & 6 \\ \cline{1-3}\end{array} \right)$$

so $\pi \prec \sigma$, π and σ are dual-Knuth equivalent and in fact we are in case **P(1)** of the theorem: $j = 2$, $M({}^2\sigma) = 2$ and $M({}^2\pi) = 2$.

j	${}^j\pi$	${}^j\sigma$
2	12	12
3	312	123
4	3124	4123
5	35124	45123
6	351246	451236
7	3571246	4571236

Consider the permutations $\sigma = 213784596$ and $\pi = 217893456$. Then

$$RS(\sigma) = \left(\begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 4 & 5 & 6 \\ \hline 2 & 7 & 8 & 9 \\ \cline{1-4} \end{array}, \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 4 & 5 & 8 \\ \hline 2 & 6 & 7 & 9 \\ \cline{1-4} \end{array} \right)$$

and

$$RS(\pi) = \left(\begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 4 & 5 & 6 \\ \hline 2 & 7 & 8 & 9 \\ \cline{1-4} \end{array}, \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 4 & 5 & 9 \\ \hline 2 & 6 & 7 & 8 \\ \cline{1-4} \end{array} \right)$$

so $\pi \prec \sigma$, π and σ are Knuth equivalent and we are in case $\mathbf{P(2)}$ of the theorem:
$j = 7$,

j	$\lvert\pi_1 \ldots \pi_j\rvert$	$\lvert\sigma_1 \ldots \sigma_j\rvert$
7	2156734	2136745
8	21678345	21367458
9	217893456	213784596

As an example of case $\mathbf{P(3)}$, consider $\sigma = 1324$ and $\pi = 3412$. Then

$$RS(\sigma) = \left(\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1} \end{array} \right)$$

and

$$RS(\pi) = \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \right)$$

so $\pi \prec \sigma$ and we have: $\sigma_4 = 4$, ${}^3\pi = 312$, ${}^3\sigma = 132$, $|\pi_1\pi_2\pi_3| = 231$ and $|\sigma_1\sigma_2\sigma_3| = 132$.

We conclude this section with an application of the bijection ϕ to the study of the lattice $\mathcal{D}_n$. In the literature, several interesting partitions of the lattice $\mathcal{D}_n$ into sublattices has been considered, in particular in connection with the ECO method (see [9] and [3]). Now we describe other partitions of $\mathcal{D}_n$ obtained by using the map ϕ. Consider the following sets:

$$S_n(321)_k : \text{ the subset of } S_n(321) \text{ of permutations } \pi \text{ with } l(\pi) = k$$

where $l(\pi)$ is the length of a maximal increasing subsequence of π,

$$S_n(321)_{(P,)} : \text{ the subset of } S_n(321) \text{ of permutations } \pi \text{ whose insertion tableau is } P$$

and

$$S_n(321)_{(,Q)} : \text{ the subset of } S_n(321) \text{ of permutations } \pi \text{ whose recording tableau is } Q.$$

Note that the set $S_n(321)_k$ corresponds, via the map RS, to the subset of $SYT_2(n)$ of pairs of tableaux with shape $(k, n-k)$.

THEOREM 3.7 *The family of sets $S_n(321)_k$ with $\lceil \frac{n}{2} \rceil \leq k \leq n$, is a partition of $S_n(321)$ into sublattices. This partition corresponds to the partition of $\mathcal{D}_n$ given by the sublattices $\mathcal{D}_{n,k}$ of paths with exactly k up steps in the first half and k down steps in the second half.*
The family of sets $S_n(321)_{(P,)}$ with $P \in Tab_2(n)$ and $shP = (k, n-k)$, is a partition of $S_n(312)_k$ into sublattices. This partition corresponds to the partition of $\mathcal{D}_{n,k}$ given by the sublattices of paths with a fixed second half.
Similarly, the family of sets $S_n(321)_{(,Q)}$ is a partition and correspond to Dyck paths with a fixed first half.

Proof. Follows directly by a careful analysis of the properties of the bijection ϕ and by lemma 3.4. □

4 The case of at most three rows

Finally we consider the case of standard tableaux with at most three rows. Our goal is to find a suitable set of lattice paths which corresponds bijectively to the set $SYT_3(n)$, and hence to $S_n(4321)$.
We recall that a Motzkin path of length n is a lattice path consisting of steps $U = (1,1)$, $D = (1,-1)$ and $H = (1,0)$, starting at $(0,0)$, ending at $(0,n)$, and never going below the x-axis. Similarly, a Motzkin prefix of length n is a lattice path consisting of n steps $U = (1,1)$, $D = (1,-1)$ and $H = (1,0)$, starting at $(0,0)$, and never going below the x-axis. We recall that a Dyck prefix can be seen as a Motzkin prefix without H steps.
Consider the set $\mathcal{M}_{2n}$ of Motzkin paths of length $2n$. Given $p \in \mathcal{M}_{2n}$ and $i \leq n$, we denote by $(U, p, \leq i)$ the number of up steps U whose position is smaller than or equal to i. Analogously we denote by $(D, p, \leq i)$, $(H, p, \leq i)$ the number of steps D and H whose position is smaller than or equal to i and by $(U, p, \geq i)$, $(D, p, \leq i)$, $(H, p, \leq i)$ the number of steps U, D and H respectively whose position is greater than or equal to i, respectively.
Let $\widehat{M_{2n}}$ be the subset of $\mathcal{M}_{2n}$ of paths p with the following properties:

- $\forall i$ with $1 \leq i \leq n$, $(U, p, \leq i) \geq (D, p, \leq i) \geq (H, p, \leq i)$
- $\forall i$ with $n+1 \leq i \leq 2n$, $(D, p, \geq i) \geq (U, p, \geq i) \geq (H, p, \geq i)$
- $(H, p, \leq n) = (H, p, \geq n+1)$

Note that the set $\mathcal{D}_n$ of Dyck paths of semilength n is a subset of $\widehat{M_{2n}}$.

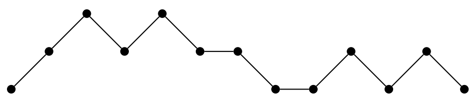
We associate with a given pair $(P, Q) \in SYT_3(n)$ two Motzkin prefixes p, q in the following way:

the prefix p has steps U, D, H in all the positions given by the elements of the first, second and third row of P, respectively, q is built in the same way using the tableau Q. Now we have a Motzkin path qp^{-1} where p^{-1} is the inverse of the word $p = (p_1, ..., p_n)$ defined as $p^{-1} := (-p_n, ..., -p_1)$ where $-U := D$, $-D := U$ and $-H := H$.
Similarly to the case of two rows, define ψ_1 to be the map $(P, Q) \mapsto qp^{-1}$ and $\psi = \psi_1 \circ RS$.
It is immediately seen that ψ_1 is a bijection between $SYT_3(n)$ and $\widehat{M}_{2n}$.

Example 4.1 Consider the permutation $\pi \in S_6(321)$, $\pi = 251643$. Then $RS(\pi) = (P, Q)$ with

$$P = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 4 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array} \quad \text{and} \quad Q = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array}$$

so $p = UDUDHU$, $q = UUDUDH$, $p^{-1} = DHUDUD$ and we obtain the path $\psi(\pi)$ whose word is $qp^{-1} = UUDUDHDHUDUD$:

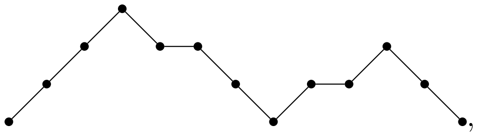


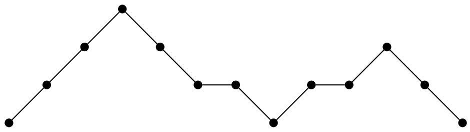
Now, consider the order over $\widehat{M}_{2n}$ whose covering relation is given by: $f \prec g$ if and only if the word $g = (g_1, ..., g_n | g_{n+1}, ..., g_{2n})$ is obtained from the word $f = (f_1, ..., f_n | f_{n+1}, ..., f_{2n})$ by one of the following substitutions

- $...DU... \mapsto ...UD...$ where DU is a pair of consecutive steps of f
- $...HD... \mapsto ...DH...$ where HD is a pair of consecutive steps in the first half of f
- $...HU... \mapsto ...UH...$ where HU is a pair of consecutive steps in the first half of f
- $...DH... \mapsto ...HD...$ where DH is a pair of consecutive steps in the second half of f
- $...UH... \mapsto ...HU...$ where UH is a pair of consecutive steps in the second half of f
- $...H|H... \mapsto ...U|D...$ where $H|H$ is a pair of consecutive steps symmetric with respect to the half of f
- $...D\underbrace{H...H}_{k_1}|\underbrace{H...H}_{k_2}U... \mapsto ...U\underbrace{H...H}_{k_1}|\underbrace{H...H}_{k_2}D...$ where D, U are, respectively, in the first half and in the second half of f and $k_1, k_2 \geq 0$
- $...H\underbrace{U...U}_{k_1}|\underbrace{D...D}_{k_2}H... \mapsto ...D\underbrace{U...U}_{k_1}|\underbrace{D...D}_{k_2}U...$ where H, H are, respectively, in the first half and in the second half of f and $k_1, k_2 \geq 0$

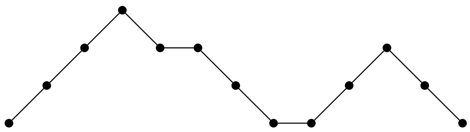
where the bar indicates the half of the path.

EXAMPLE 4.2 Consider the following paths f, g_1 and g_2 in $\widehat{M}_{12}$ given by:

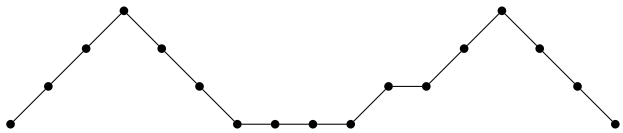
$f :=$ 

$g_1 :=$ 

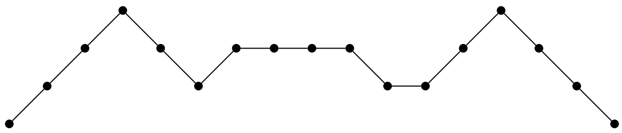
and

$g_2 :=$ 

then $f \prec g_1$ and $f \prec g_2$. In fact, g_1 is obtained from f by the substitution $...HD... \mapsto ...DH...$ where HD is a pair of consecutive steps in the first half of f and g_2 is obtained from f by the substitution $...UH... \mapsto ...HU...$ where UH is a pair of consecutive steps in the second half of f.
Consider the paths f' and g' in $\widehat{M}_{16}$ given by:

$f' :=$ 

and

$g' :=$ 

then $f' \prec g'$. In fact, g' is obtained from f' by the substitution $...DHH|HU... \mapsto ...UHH|HD...$ where the bar $|$ indicates the half of the path.

Note that this covering relation extends the covering relation over the set $\mathcal{D}_n$ of Dyck paths, where $f \prec g$ if and only if g is obtained from f by the substitution of consecutive steps: $...DU... \mapsto ...UD...$
Note also that, if $f \prec g$ and the paths f and g differs in two consecutive positions in the first half or in two consecutive positions in the second half, then the corresponding tableaux P_f, Q_f, P_g and Q_g have the same shape. Otherwise, if $f \prec g$ and f and g differs in two positions, one in the first half and the other in the second half of the path, then $shP_f = shQ_f \neq shP_g = shQ_g$.
We have the following theorem.

THEOREM 4.3 *The map ψ_1 is an order isomorphism between $SYT_3(n)$ and $\widehat{M}_{2n}$ which extends the previous bijection ϕ_1 between $SYT_2(n)$ and $\mathcal{D}_n$. In particular, $\mathcal{D}_n$ is a principal filter in $\widehat{M}_{2n}$ generated by the element $(UD)^n$.*

Proof. The proof is similar to the proof of theorem 3.2. □

As a consequence, by Gessel's formula (see [4]),

$$|\widehat{M_{2n}}| = |S_n(4321)| = \frac{1}{(n+1)^2(n+2)} \sum_{k=0}^{n} \binom{2k}{k} \binom{n+1}{k+1} \binom{n+2}{k+1}.$$

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