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Recursions for the flag-excedance number in colored permutations groups

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Abstract. The excedance number for S_n is known to have an Eulerian distribution. Nevertheless, the classical proof uses descents rather than excedances. We present a direct proof based on a recursion which uses only excedances and extend it to the flag-excedance parameter defined on the group of colored permutations $G_{r,n} = \mathbb{Z}_r \wr S_n$. We have also computed the distribution of a variant of the flag-excedance number, and show that its enumeration uses the Stirling number of the second kind. Moreover, we show that the generating function of the flag-excedance number defined on $\mathbb{Z}_r \wr S_n$ is symmetric, and its variant is log-concave on $\mathbb{Z}_r \wr S_n$..

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1 Introduction

Let S_n be the symmetric group on n letters. The parameter *excedance*, which is defined on a permutation $\pi \in S_n$ by:

$$\text{exc}(\pi) = |\{i \in \{1, 2, \dots, n\} \mid \pi(i) > i\}|,$$

is well-known (see [16, Vol. I, pp. 135, 186; Vol. II, p. viii], [17]). Another classical parameter defined on permutations of S_n is the *descent number*, defined by:

$$\text{des}(\pi) = |\{i \in \{1, 2, \dots, n-1\} \mid \pi(i) > \pi(i+1)\}|.$$

Both parameters have the same distribution, which can be read from the following recursion:

$$a(n, k) = (k + 1)a(n - 1, k) + (n - k)a(n - 1, k - 1), \tag{1}$$

with the following initial conditions:

$$a(n, 0) = 1, \; a(0, k) = 0 \; \forall k \forall n,$$

where $a(n, k)$ is the number of permutations in S_n with k excedances or k descents. The generating function $\sum\limits_{\pi \in S_n} q^{\text{exc}(\pi)+1}$ is called *the Eulerian polynomial*.

There is a well-known proof for this recursion by enumerating the descents [14], and there is a bijection from S_n onto itself, taking the descents into the excedances [19]. Later, a different proof of this recursion, which uses only excedances, was given independently by Jansonn [15] and by the authors (in this paper).

There are several different definitions of the excedance number for generalizations of the symmetric group. Brenti [8] defined a version for the hyperoctahedral group $B_n = \mathbb{Z}_2 \wr S_n$. Chen, Tang and Zhao [9] used this definition to construct a type-B analogue of the derangement polynomials, having properties such as the Sturm sequence property and their coefficients having the spiral property.

A different generalization of the excedance number for the *colored permutation groups* $G_{r,n} = \mathbb{Z}_r \wr S_n$ was introduced by Steingrimsson [20]. This version of the excedance number equidistributes with his version of the descent number for the colored permutation groups. He supplies some Eulerian-type recursions for these parameters and presents some geometric applications.

In [5], the first two authors defined a different version of the excedance number for the colored permutation groups, called the *flag-excedance number*. This definition was motivated by the view of $\mathbb{Z}_r \wr S_n$ as a subgroup of $\text{Sym}(\Sigma_n)$, where $\Sigma_n = \{i^{[c]} \mid 1 \leq i \leq n, 0 \leq c < r\}$ is the set of n digits colored by r colors:

$$\text{fexc}(\pi) = |\{i \in \Sigma_n \mid \pi(i) > i\}|.$$

One can compute the flag-excedance number in a different way (all the notations will be defined later):

$$\text{fexc}(\pi) = r \cdot \text{exc}_A(\pi) + \text{csum}(\pi).$$

A similar approach was used in the definition of the flag major index in the colored permutation groups, see [1, 3, 11].

An interesting application of the parameter $\mathrm{fexc}(\pi)$ was introduced by Athanasiadis: Consider an $(n-1)$-dimensional simplicial complex Δ and let $f_i(\Delta)$ be the number of $i-$dimensional faces of Δ. The $h-$polynomial of Δ is defined as:

$$h(\delta, x) = \sum_{i=0}^{n} f_{i-1}(\Delta) x^i (1-x)^{n-i}.$$

Now, let V be an n-element set and let Γ be a finite geometric subdivision of the abstract simplicial complex 2^V. The *local h-polynomial* $\ell_V(\Gamma, x)$ is:

$$\ell_V(\Gamma, x) = \sum_{F \subset 2^V} (-1)^{n-|F|} h(\Gamma_F, x),$$

where Γ_F is the restriction of Γ to the face $F \in 2^V$ and $h(\Delta, x)$ is the $h-$polynomial of the simplicial complex Δ. The $r-$th edgewise subdivision of a simplicial complex is a standard way to subdivide the complex Δ in such a way that each face $F \in \Delta$ is subdivided into $r^{\dim(F)}$ faces of the same dimension.

Denote by $\mathrm{sd}(\Delta)^r$ the $r-$th barycentric subdivision of Δ. A permutation $\pi \in \mathbb{Z}_r \wr S_n$ is called **balanced** if the parameter $\mathrm{csum}(\pi)$ (which is defined to be the sum of the colors of the digits of π) is a multiple of r. The subset of $\mathbb{Z}_r \wr S_n$, consisting of all the permutations without absolute fixed points, is defined as:

$$D_n^r = \{\pi \in \mathbb{Z}_r \wr S_n \mid \forall i \in \{1, 2, \ldots, n\}, \pi(i) \neq i\}.$$

Let $(D_n^r)^b$ denote the set of all balanced permutations in D_n^r. Then, Athansiadis [2] shows the following result:

THEOREM 1.1 (Athanasiadis) *Let V be an n-element set. Then*

$$\ell_V(\mathrm{sd}(2^V)^r, q) = \sum_{\pi \in (D_n^r)^b} q^{\frac{\mathrm{fexc}(\pi)}{r}}.$$

We survey here some other results dealing with the flag-excedance parameter defined on $G_{r,n}$.

In [5], the multi-distributions of the excedance number with some natural parameters were computed. In [4], these definitions and results were generalized to the so-called *multi-colored permutation group* $(\mathbb{Z}_{r_1} \times \cdots \times \mathbb{Z}_{r_k}) \wr S_n$. In [6], the multi-distribution of the excedance number with the number of fixed points on the set of involutions in $G_{r,n}$ was computed. In [18], Mansour and Sun consider similar problems in more general cases.

Recently, Foata and Han [12, 13] have found that this version of the excedance number is equidistributed with some version of the descent number for generalized permutation groups. Moreover, Clark and Ehrenborg [10] mentioned this version of the excedance number as a possible candidate for a generalization for an excedance statistic for all finite Coxeter groups.

We start this paper by presenting a classical way to obtain recursion (1) using only counting of excedances. This argument appears also in [15].

By our definition of excedance, we generalize this recursion for the cases of the hyperoctahedral group $B_n = G_{2,n}$ and the colored permutation groups $G_{r,n}$ (all the notations will be defined in the sequel):

Proposition 1.2 *Define:*

$$f_i^A(r,n,k) = |\{\pi \in G_{r,n} \mid \mathrm{exc}_A(\pi) = k, \mathrm{csum}(\pi) = i\}|.$$

Then:

$$\begin{aligned} f_i^A(r,n,k) &= (n-k)f_i^A(r,n-1,k-1) + (k+1)f_i^A(r,n-1,k) + \\ &\quad + \sum_{j=1}^{r-1} \left[(n-k)f_{i-j}^A(r,n-1,k) + (k+1)f_{i-j}^A(r,n-1,k+1)\right], \end{aligned}$$

with the following initial conditions:

$$f_i^A(r,n,0) = \sum_{\substack{(t_1,\dots,t_j),\ j \leq i \\ 1 \leq t_1 < t_2 < \cdots < t_j \leq n}} j!\binom{i-1}{i-j}(j+1)^{n-t_j}\prod_{u=1}^{j} u^{t_u - t_{u-1} - 1}$$

where $t_0 = 0$*, and*

$$f_i^A(r,0,k) = 0;\ \ f_0^A(r,1,0) = 1;\ \ f_{-1}^A(r,n,k) = 0\ \forall n \forall k.$$

We have also computed the distribution of a variant of the flag-excedance number, denoted by exc_A. The interesting point is that its enumeration uses the Stirling number of the second kind:

Proposition 1.3 *The number of permutations* π *in* $G_{r,n}$ *which satisfy* $\mathrm{exc}_A(\pi) = k$ *is given by:*

$$r\sum_{j=1}^{n}\sum_{i=0}^{j-1}(-1)^{k+j-1-i} r^i j! S_{n,j}\binom{j-1}{i}\binom{n-1-i}{k},$$

where $S_{n,j}$ *is the* (n,j)*-Stirling number of the second kind.*

It is well-known that the generating function $\sum\limits_{\pi \in S_n} q^{\mathrm{exc}(\pi)} = \sum\limits_{i=0}^{d} a_i q^i$ has some symmetry properties. It is symmetric in the sense that $a_i = a_{d-i}$ for $i \in \{1,\dots,\lfloor \frac{d}{2} \rfloor\}$. We prove here the corresponding symmetry property for $G_{r,n}$. We also prove that its variant exc_A is log-concave in $G_{r,n}$.

The paper is organized as follows. In Section 2, we introduce the colored permutation group $G_{r,n} = \mathbb{Z}_r \wr S_n$ and we define some of its parameters and statistics. Section 3 deals with the proof of the recursion for S_n. In Sections 4 and 5, we give the corresponding recursions for B_n and $G_{r,n}$, respectively. Section 6 deals with the distribution of the parameter exc_A, which involves the Stirling number of the second kind. In Section 7, we present the symmetry of the generating function of the excedance number, and in Section 8 we prove the log-concavity property of the parameter exc_A.

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2 The group of colored permutations and its statistics

Definition 2.1 Let r and n be positive integers. *The group of colored permutations of n digits with r colors* is the wreath product

$$G_{r,n} = \mathbb{Z}_r \wr S_n = \mathbb{Z}_r^n \rtimes S_n,$$

consisting of all pairs $(\vec{z}, \tau)$, where $\vec{z}$ is an n-tuple of integers between 0 and $r-1$ and $\tau \in S_n$. The multiplication is defined by the following rule: for $\vec{z} = (z_1, \dots, z_n)$ and $\vec{z'} = (z'_1, \dots, z'_n)$,

$$(\vec{z}, \tau) \cdot (\vec{z'}, \tau') = ((z_1 + z'_{\tau^{-1}(1)}, \dots, z_n + z'_{\tau^{-1}(n)}), \tau \circ \tau') \tag{2}$$

(the operation $+$ is taken modulo r).

Here is an example for the multiplication in $G_{5,3}$:

$$\left((2,1,0), \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\right) \cdot \left((2,2,0), \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\right) = \left((4,3,0), \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}\right).$$

Another way to present $G_{r,n}$ is as follows: Consider the alphabet

$$\Sigma = \{1, \dots, n, \bar{1}, \dots, \bar{n}, \dots, 1^{[r-1]}, \dots, n^{[r-1]}\}$$

as the set $\{1, \dots, n\}$ colored by the colors $0, \dots, r-1$. Then, an element of $G_{r,n}$ is a *colored permutation*, i.e., a bijection $\pi : \Sigma \to \Sigma$ satisfying the following condition: if $\pi\left(i^{[\alpha]}\right) = j^{[\beta]}$, then $\pi\left(i^{[\alpha+1]}\right) = j^{[\beta+1]}$ (the addition in the exponents is taken modulo r). Using this approach, the element $\pi = ((z_1, \dots, z_n), \tau) \in G_{r,n}$ is the permutation of Σ, satisfying $\pi(i) = \pi(i^{[0]}) = \tau(i)^{[z_{\tau(i)}]}$ for each $1 \le i \le n$. For example, the element $\pi = \left((2,1,0,3,0,0), \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \end{pmatrix}\right) \in G_{6,6}$ satisfies:

$$\pi(1) = 2^{[1]}, \pi(2) = 1^{[2]}, \pi(3) = 4^{[3]}, \pi(4) = 3^{[0]}, \pi(5) = 6^{[0]}, \pi(6) = 5^{[0]}.$$

For an element $\pi = (\vec{z}, \tau) \in G_{r,n}$ with $\vec{z} = (z_1, \dots, z_n)$, we write $z_i(\pi) = z_i$, and denote $|\pi| = (\vec{0}, \tau)$. We define also $c_i(\pi) = r - z_i(\pi^{-1})$ and $\vec{c}(\pi) = \vec{c} = (c_1, \dots, c_n)$. Using this notation, the element $\pi = (\vec{z}, \tau) = \left((2,1,0,3,0,0), \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \end{pmatrix}\right)$ satisfies $\vec{c} = (1,2,3,0,0,0)$.

We usually write π in its *window notation* (or *one line notation*): $\pi = \left(a_1^{[c_1]} \cdots a_n^{[c_n]}\right)$, where $a_i = \tau(i)$, so in the above example, we have: $\pi = (2^{[1]}1^{[2]}4^{[3]}3^{[0]}6^{[0]}5^{[0]})$ or just $\left(\bar{2}\bar{\bar{1}}\bar{\bar{\bar{4}}}365\right)$.

Note that z_i is the color of the digit i (i is taken from the window notation), while c_j is the color of the digit $\tau(j)$. Here, j stands for the place, whence i stands for the value.

In particular, $G_{1,n} = \mathbb{Z}_1 \wr S_n$ is the classical symmetric group S_n, while $G_{2,n} = \mathbb{Z}_2 \wr S_n$ is the group of signed permutations B_n, also known as the *hyperoctahedral group*, or the *classical Coxeter group of type B*.

Given any ordered alphabet Σ', we recall the definition of the *excedance set* of a permutation π on Σ' (see [5]):

$$\mathrm{Exc}(\pi) = \{i \in \Sigma' \mid \pi(i) > i\}$$

and the *flag-excedance number* is defined to be $\mathrm{fexc}(\pi) = |\mathrm{Exc}(\pi)|$.

DEFINITION 2.2 The *color order* on Σ is defined to be:

$$1^{[r-1]} < \cdots < n^{[r-1]} < 1^{[r-2]} < 2^{[r-2]} < \cdots < n^{[r-2]} < \cdots < 1 < \cdots < n.$$

EXAMPLE 2.3 Given the color order:

$$\bar{\bar{1}} < \bar{\bar{2}} < \bar{\bar{3}} < \bar{1} < \bar{2} < \bar{3} < 1 < 2 < 3,$$

we write $\sigma = (3\bar{1}\bar{\bar{2}}) \in G_{3,3}$ in an extended form:

$$\left(\begin{array}{ccc|ccc|ccc} \bar{\bar{1}} & \bar{\bar{2}} & \bar{\bar{3}} & \bar{1} & \bar{2} & \bar{3} & 1 & 2 & 3 \\ \bar{\bar{3}} & 1 & \bar{2} & \bar{3} & \bar{\bar{1}} & 2 & 3 & \bar{1} & \bar{\bar{2}} \end{array} \right),$$

and compute: $\mathrm{Exc}(\sigma) = \{\bar{\bar{1}}, \bar{\bar{2}}, \bar{\bar{3}}, \bar{1}, \bar{3}, 1\}$ and $\mathrm{fexc}(\sigma) = 6$.

We present now an alternative way to compute the flag-excedance number. Let $\sigma \in G_{r,n}$. We define:

$$\mathrm{csum}(\sigma) = \sum_{i=1}^{n} c_i(\sigma).$$

Note that in the case $r = 2$ (i.e. the group B_n), the alphabet Σ can be seen as containing the digits $\{\pm 1, \ldots, \pm n\}$ and the parameter $\mathrm{csum}(\pi)$ counts the number of digits $i \in [n]$ such that $\pi(i) < 0$, so it is also called $\mathrm{neg}(\pi)$.

Define now:

$$\mathrm{Exc}_A(\sigma) = \{i \in \{1, 2, \ldots, n-1\} \mid \sigma(i) > i\},$$

where the comparison is with respect to the color order, and denote:

$$\mathrm{exc}_A(\sigma) = |\mathrm{Exc}_A(\sigma)|.$$

EXAMPLE 2.4 Given $\sigma = (\bar{1}\bar{\bar{3}}4\bar{2}) \in G_{3,4}$, we have $\mathrm{csum}(\sigma) = 4$, $\mathrm{Exc_A}(\sigma) = \{3\}$ and hence $\mathrm{exc}_A(\sigma) = 1$.

We have now (see [5]):

LEMMA 2.5

$$\mathrm{fexc}(\sigma) = r \cdot \mathrm{exc}_A(\sigma) + \mathrm{csum}(\sigma).$$

A similar result for the flag major index statistic was achieved by Adin and Roichman [1].

3 The recursion for S_n

We supply a classical proof for the recursion for the Eulerian polynomial using its interpretation as a generating function for the excedance number for S_n (this proof appears independently in [15]). Denote by $a(n, k)$ the number of permutations in S_n with exactly k excedances. Then we have the following recursion:

Proposition 3.1
$$a(n,k) = (k+1)a(n-1,k) + (n-k)a(n-1,k-1),$$
with the following initial conditions:
$$a(n,0) = 1, a(0,k) = 0, \forall n \forall k.$$

Proof. For any n and $0 \leq k \leq n-1$, denote by $S(n,k)$ the set of permutations in S_n with exactly k excedances. Denote also:
$$R = \{\pi \in S(n,k) \mid \pi^{-1}(n) < \pi(n)\}$$
and
$$T = \{\pi \in S(n,k) \mid \pi^{-1}(n) \geq \pi(n)\}.$$

Define $\Phi : S(n,k) \to S(n-1,k) \cup S(n-1,k-1)$ as follows: Let $\pi \in S(n,k)$. Then $\Phi(\pi)$ is the permutation in S_{n-1} obtained from $(n, \pi(n))\pi$ by ignoring the last digit.

Let $\pi \in S(n,k) = R \cup T$. If $\pi \in R$, then $\Phi(\pi) \in S(n-1,k)$. Note that $|\Phi^{-1}(\Phi(\pi))| = k+1$. On the other hand, if $\pi \in T$, then $\Phi(\pi) \in S(n-1,k-1)$ and $|\Phi^{-1}(\Phi(\pi))| = n-1-(k-1) = n-k$. □

We give the following example for illustrating the proof.

Example 3.2 Consider $S(5,2)$, i.e. the set of permutations in S_5 having exactly 2 excedances.

Let
$$R \ni \pi = \begin{pmatrix} ① & ② & 3 & 4 & 5 \\ 5 & 3 & 1 & 2 & 4 \end{pmatrix} \mapsto \begin{pmatrix} ① & ② & 3 & 4 & 5 \\ \mathbf{4} & 3 & 1 & 2 & \mathbf{5} \end{pmatrix} \mapsto \begin{pmatrix} ① & ② & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix} = \Phi(\pi),$$
so:
$$\Phi^{-1}(\Phi(\pi)) = \left\{ \begin{pmatrix} ① & ② & 3 & 4 & 5 \\ 5 & 3 & 1 & 2 & 4 \end{pmatrix}, \begin{pmatrix} ① & ② & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} ① & ② & 3 & 4 & 5 \\ 4 & 3 & 1 & 2 & 5 \end{pmatrix} \right\}.$$

Let
$$T \ni \pi = \begin{pmatrix} ① & 2 & ③ & 4 & 5 \\ 5 & 2 & 4 & 3 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 & ③ & 4 & 5 \\ \mathbf{1} & 2 & 4 & 3 & \mathbf{5} \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 & ③ & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix} = \Phi(\pi).$$

Then:
$$\Phi^{-1}(\Phi(\pi)) = \left\{ \begin{pmatrix} ① & 2 & ③ & 4 & 5 \\ 5 & 2 & 4 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & ② & ③ & 4 & 5 \\ 1 & 5 & 4 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & ③ & ④ & 5 \\ 1 & 2 & 4 & 5 & 3 \end{pmatrix} \right\}.$$

4 The recursion for B_n

In this section, we generalize the above recursion to $B_n = \mathbb{Z}_2 \wr S_n$. We start with some notation.
$$F_i^A(n,k) = \{\pi \in B_n \mid \mathrm{exc}_A(\pi) = k, \mathrm{neg}(\pi) = i\}$$
$$f_i^A(n,k) = \left|F_i^A(n,k)\right|,$$
$$F_i(n,k) = \{\pi \in B_n \mid \mathrm{fexc}(\pi) = k, \mathrm{neg}(\pi) = i\}$$

$$f_i(n,k) = |F_i(n,k)|,$$
$$F(n,k) = \bigcup_{i=0}^{n} F_i(n,k)$$
$$f(n,k) = \sum_{i=0}^{n} f_i(n,k).$$

In B_n, we have $\mathrm{fexc}(\pi) = 2 \cdot \mathrm{exc}_A(\pi) + \mathrm{neg}(\pi)$, hence:

$$f_i(n,k) = f_i^A\left(n, \frac{k-i}{2}\right).$$

In the following proposition, we give a recursion for $f_i^A(n,k)$:

PROPOSITION 4.1

$$\begin{aligned} f_i^A(n,k) &= (n-k)f_i^A(n-1,k-1) + (k+1)f_i^A(n-1,k) + \\ &\quad +(n-k)f_{i-1}^A(n-1,k) + (k+1)f_{i-1}^A(n-1,k+1), \end{aligned}$$

with the following initial conditions:

$$f_i^A(n,0) = \sum_{\substack{(t_1,\dots,t_i) \\ 1 \le t_1 < t_2 < \cdots < t_i \le n}} i!(i+1)^{n-t_i} \prod_{u=1}^{i} u^{t_u - t_{u-1} - 1},$$

where $0 \le i \le n$ *and* $t_0 = 0$, *and*

$$f_i^A(0,k) = 0;\ f_0^A(1,0) = 1;\ f_{-1}^A(n,k) = 0\ \forall n \forall k.$$

Proof. We start with the proof of the recursion. Define:

$$F_{i,0}^A(n,k) = \{\pi \in B_n \mid \mathrm{exc}_A(\pi) = k,\ \mathrm{neg}(\pi) = i,\ \pi^{-1}(n) > 0\}$$
$$f_{i,0}^A(n,k) = \left|F_{i,0}^A(n,k)\right|,$$
$$F_{i,1}^A(n,k) = \{\pi \in B_n \mid \mathrm{exc}_A(\pi) = k,\ \mathrm{neg}(\pi) = i,\ \pi^{-1}(n) < 0\}$$
$$f_{i,1}^A(n,k) = \left|F_{i,1}^A(n,k)\right|.$$

Obviously, $F_i^A(n,k) = F_{i,0}^A(n,k) \uplus F_{i,1}^A(n,k)$, and hence:

$$f_i^A(n,k) = f_{i,0}^A(n,k) + f_{i,1}^A(n,k).$$

Define $\Phi_0 : F_{i,0}^A(n,k) \to F_i^A(n-1,k) \cup F_i^A(n-1,k-1)$ as follows: Let $\pi \in F_{i,0}^A(n,k)$. Then $\Phi_0(\pi)$ is the permutation of B_{n-1} obtained from $(n,\pi(n))\pi$ by ignoring the last digit.

Now, define:

$$R_0 = \{\pi \in F^A_{i,0}(n,k) \mid \pi^{-1}(n) < \pi(n)\}$$

and

$$T_0 = \{\pi \in F^A_{i,0}(n,k) \mid \pi^{-1}(n) \geq \pi(n)\}.$$

Let $\pi \in F^A_{i,0}(n,k) = R_0 \cup T_0$. If $\pi \in R_0$, then $\Phi_0(\pi) \in F^A_i(n-1,k)$. Note that $|\Phi_0^{-1}(\Phi_0(\pi))| = k+1$. On the other hand, if $\pi \in T_0$, then $\Phi_0(\pi) \in F^A_i(n-1,k-1)$ and $|\Phi_0^{-1}(\Phi_0(\pi))| = n-1-(k-1) = n-k$.

Define $\Phi_1 : F^A_{i,1}(n,k) \to F^A_{i-1}(n-1,k) \cup F^A_{i-1}(n-1,k+1)$ similarly.

Now, define:

$$R_1 = \{\pi \in F^A_{i,1}(n,k) \mid |\pi^{-1}(n)| < \pi(n)\}$$

and

$$T_1 = \{\pi \in F^A_{i,1}(n,k) \mid |\pi^{-1}(n)| \geq \pi(n)\}.$$

Let $\pi \in F^A_{i,1}(n,k) = R_1 \cup T_1$. If $\pi \in R_1$, then $\Phi_1(\pi) \in F^A_{i-1}(n-1,k+1)$. Note that $|\Phi_1^{-1}(\Phi_1(\pi))| = k+1$. On the other hand, if $\pi \in T_1$, then $\Phi_1(\pi) \in F^A_{i-1}(n-1,k)$ and $|\Phi_1^{-1}(\Phi_1(\pi))| = n-1-(k-1) = n-k$.

Combining together all the parts, we get the desired recursion for $f^A_i(n,k)$.

Now we prove the initial condition:

$$f^A_i(n,0) = \sum_{\substack{(t_1,\dots,t_i) \\ 1 \leq t_1 < t_2 < \cdots < t_i \leq n}} i!(i+1)^{n-t_i} \prod_{u=1}^{i} u^{t_u - t_{u-1} - 1}$$

where $t_0 = 0$.

Let $\pi \in B_n$ and let $1 \leq t_1 < \cdots < t_i \leq n$ be such that $\pi(t_j) < 0$ for all $1 \leq j \leq i$.

In order to insure that $\mathrm{exc}_A(\pi) = 0$, we have to require that for each $\ell \notin \{t_1,\dots,t_i\}$, $\pi(\ell) \leq \ell$. For each $1 \leq \ell < t_1$ (if there are any), we have only one possibility: $\pi(\ell) = \ell$. For $t_1 < \ell < t_2$ (if there are any), we have exactly two possibilities, and so on: for $t_m < \ell < t_{m+1}$, $1 \leq m \leq i-1$ (if there are any), we have exactly $m+1$ possibilities. Finally, for $t_i < \ell \leq n$, we have exactly $i+1$ possibilities.

After fixing $\pi(\ell)$ for each $\ell \notin \{t_1,\dots,t_i\}$, we have exactly $i!$ possibilities for locating $\pi(t_j)$, for $1 \leq j \leq i$. This gives us the desired initial condition. □

The following example should clarify the proof for the initial condition. Let $\pi \in B_9$ and assume that $t_1 = 3, t_2 = 6, t_3 = 8$. Then in order to get $\mathrm{exc}_A(\pi) = 0$, we must have $\pi(1) = 1, \pi(2) = 2$. $\pi(4)$ can be 3 or 4. $\pi(5) \in \{3,4,5\}$ but once $\pi(4)$ has been chosen we have only 2 possibilities for it. $\pi(7) \in \{3,4,5,6,7\}$ which yields 3 possibilities and for $\pi(9)$ we have 4 possibilities. The values corresponding to $\{\pi(3), \pi(6), \pi(8)\}$ are already fixed, so we just have to order them.

5 The corresponding recursion for $G_{r,n}$

The recursion for B_n can be generalized to $G_{r,n} = \mathbb{Z}_r \wr S_n$ very easily. We continue with similar notations.

$$F^A_i(r,n,k) = \{\pi \in G_{r,n} \mid \mathrm{exc}_A(\pi) = k, \mathrm{csum}(\pi) = i\}$$

$$f_i^A(r,n,k) = \left|F_i^A(r,n,k)\right|,$$
$$F_i(r,n,k) = \{\pi \in G_{r,n} \mid \mathrm{fexc}(\pi) = k, \mathrm{csum}(\pi) = i\}$$
$$f_i(r,n,k) = |F_i(r,n,k)|,$$
$$F(r,n,k) = \bigcup_{i=0}^{(r-1)n} F_i(r,n,k)$$
$$f(r,n,k) = \sum_{i=0}^{(r-1)n} f_i(r,n,k).$$

In $G_{r,n}$, we have $\mathrm{fexc}(\pi) = r \cdot \mathrm{exc}_A(\pi) + \mathrm{csum}(\pi)$, hence:

$$f_i(r,n,k) = f_i^A\left(r, n, \frac{k-i}{r}\right).$$

In the following proposition, we give a recurrence for $f_i^A(r,n,k)$:

Proposition 5.1

$$\begin{aligned} f_i^A(r,n,k) &= (n-k)f_i^A(r,n-1,k-1) + (k+1)f_i^A(r,n-1,k) + \\ &\quad + \sum_{j=1}^{r-1}\left[(n-k)f_{i-j}^A(r,n-1,k) + (k+1)f_{i-j}^A(r,n-1,k+1)\right], \end{aligned}$$

with the following initial conditions:

$$f_i^A(r,n,0) = \sum_{\substack{(t_1,\dots,t_j),\ j \le i \\ 1 \le t_1 < t_2 < \dots < t_j \le n}} j!\binom{i-1}{i-j}(j+1)^{n-t_j}\prod_{u=1}^{j} u^{t_u - t_{u-1} - 1},$$

where $0 \le i \le n$ *and* $t_0 = 0$*, and*

$$f_i^A(r,0,k) = 0;\ f_0^A(r,1,0) = 1;\ f_{-1}^A(r,n,k) = 0\ \forall n \forall k.$$

For completeness, we present here the proof for the general case.

Proof. We start with the proof of the recursion. Define for all j such that $0 \le j \le r-1$:

$$F_{i,j}^A(r,n,k) = \{\pi \in G_{r,n} \mid \mathrm{exc}_A(\pi) = k, \mathrm{csum}(\pi) = i, c_n(\pi) = j\},$$
$$f_{i,j}^A(r,n,k) = \left|F_{i,j}^A(r,n,k)\right|.$$

Obviously, $F_i^A(r,n,k) = \biguplus_{j=0}^{r-1} F_{i,j}^A(r,n,k)$, and hence:

$$f_i^A(r,n,k) = \sum_{j=0}^{r-1} f_{i,j}^A(r,n,k).$$

Define $\Phi_0 : F^A_{i,0}(r,n,k) \to F^A_i(r,n-1,k) \cup F^A_i(r,n-1,k-1)$ as follows: Let $\pi \in F^A_{i,0}(r,n,k)$. Then $\Phi_0(\pi)$ is the permutation of $G_{r,n-1}$ obtained from $(n,\pi(n))\pi$ by ignoring the last digit.

Now, define:

$$R_0 = \{\pi \in F^A_{i,0}(r,n,k) \mid \pi^{-1}(n) < \pi(n)\}$$

and

$$T_0 = \{\pi \in F^A_{i,0}(r,n,k) \mid \pi^{-1}(n) \geq \pi(n)\}.$$

Let $\pi \in F^A_{i,0}(r,n,k) = R_0 \cup T_0$. If $\pi \in R_0$, then $\Phi_0(\pi) \in F^A_i(r,n-1,k)$. Note that $|\Phi_0^{-1}(\Phi_0(\pi))| = k+1$. On the other hand, if $\pi \in T_0$, then $\Phi_0(\pi) \in F^A_i(r,n-1,k-1)$ and $|\Phi_0^{-1}(\Phi_0(\pi))| = n-1-(k-1) = n-k$.

For all j such that $1 \leq j \leq r-1$, define:

$$\Phi_j : F^A_{i,j}(r,n,k) \to F^A_{i-j}(r,n-1,k) \cup F^A_{i-j}(r,n-1,k+1)$$

similarly.

Now, define:

$$R_j = \{\pi \in F^A_{i,j}(r,n,k) \mid |\pi^{-1}(n)| < \pi(n)\}$$

and

$$T_j = \{\pi \in F^A_{i,j}(r,n,k) \mid |\pi^{-1}(n)| \geq \pi(n)\}.$$

Let $\pi \in F^A_{i,j}(r,n,k) = R_j \cup T_j$. If $\pi \in R_j$, then $\Phi_j(\pi) \in F^A_{i-j}(r,n-1,k+1)$. Note that $|\Phi_j^{-1}(\Phi_j(\pi))| = k+1$. On the other hand, if $\pi \in T_j$, then $\Phi_j(\pi) \in F^A_{i-j}(r,n-1,k)$ and $|\Phi_j^{-1}(\Phi_j(\pi))| = n-1-(k-1) = n-k$.

Combining together all the parts, we get the desired recursion for $f^A_i(r,n,k)$.

Now we prove the initial condition:

$$f^A_i(r,n,0) = \sum_{\substack{(t_1,\dots,t_j),\ j \leq i \\ 1 \leq t_1 < t_2 < \cdots < t_j \leq n}} j!\binom{i-1}{i-j}(j+1)^{n-t_j}\prod_{u=1}^{j} u^{t_u - t_{u-1} - 1}$$

where $t_0 = 0$.

Let $\pi \in G_{r,n}$. Let $j \leq i$ and let $(t_1,\cdots,t_j)$ be such that $1 \leq t_1 < \cdots < t_j \leq n$ and $c_{t_k}(\pi) > 0$ (for $1 \leq k \leq j$).

In order to insure that $\mathrm{exc}_A(\pi) = 0$, we require that for each $\ell \notin \{t_1,\dots,t_j\}$, $\pi(\ell) \leq \ell$. For each $1 \leq \ell < t_1$ (if there are any), we have only one possibility: $\pi(\ell) = \ell$. For $t_1 < \ell < t_2$ (if there are any), we have exactly two possibilities, and so on: for $t_m < \ell < t_{m+1}$, $1 \leq m \leq j-1$ (if there are any), we have exactly $m+1$ possibilities. Finally, for $t_j < \ell \leq n$, we have exactly $j+1$ possibilities.

After fixing $\pi(\ell)$ for each $\ell \notin \{t_1,\dots,t_j\}$, we have exactly $j!\binom{i-1}{i-j}$ possibilities to locate $\pi(t_k)$, $1 \leq k \leq j$, and to color them in such a way that $\mathrm{csum}(\pi) = i$. This gives us the desired initial condition for $G_{r,n}$. □

6 The distribution of the parameter exc_A

Recall that:

$$f_i^A(r,n,k) = |\{\pi \in G_{r,n} \mid \mathrm{exc}_A(\pi) = k, \mathrm{csum}(\pi) = i\}|.$$

Let $d(r,n,k) = \sum\limits_{i=0}^{(r-1)n} f_i^A(r,n,k)$ be the number of permutations $\pi \in G_{r,n}$ having $\mathrm{exc}_A(\pi) = k$.

Proposition 5.1 gives that:

$$\begin{aligned} d(r,n,k) &= (n-k)d(r,n-1,k-1) + (k+1)d(r,n-1,k) \\ &\quad +(n-k)(r-1)d(r,n-1,k) + (k+1)(r-1)d(r,n-1,k+1), \end{aligned}$$

which is equivalent to:

$$\begin{aligned} d(r,n,k) &= (n-k)d(r,n-1,k-1) \\ &\quad +[(k+1)+(r-1)(n-k)]d(r,n-1,k) \\ &\quad +(k+1)(r-1)d(r,n-1,k+1). \end{aligned} \tag{3}$$

In order to solve this recurrence, we define the following generating polynomial:

$$D_{r,n}(t) = \sum_{k=0}^{n} d(r,n,k)t^k.$$

Rewriting Equation (3) in terms of the polynomial $D_{r,n}(t)$ yields:

$$\begin{aligned} D_{r,n}(t) &= ntD_{r,n-1}(t) - t\frac{\partial}{\partial t}(tD_{r,n-1}(t)) \\ &\quad +(1+(r-1)n)D_{r,n-1}(t) - (r-2)t\frac{\partial}{\partial t}D_{r,n-1}(t) \\ &\quad +(r-1)\frac{\partial}{\partial t}D_{r,n-1}(t), \end{aligned}$$

which implies that:

$$D_{r,n}(t) = [rn + (n-1)(t-1)]D_{r,n-1}(t) - (t-1)(t+r-1)\frac{\partial}{\partial t}D_{r,n-1}(t). \tag{4}$$

Now, in order to simplify this recurrence, assume that $D_{r,n}(t)$ can be written as $D_{r,n}(t) = P_{r,n}(t)E_{r,n}(t)$. We will give later the condition which $P_{r,n}(t)$ has to satisfy. Therefore, Equation (4) can be written in terms of $P_{r,n}(t)$ and $E_{r,n}(t)$ as

$$\begin{aligned} E_{r,n}(t) &= \frac{[rn+(n-1)(t-1)]P_{r,n-1}(t) - (t-1)(t+r-1)\frac{\partial}{\partial t}P_{r,n-1}(t)}{P_{r,n}(t)}E_{r,n-1}(t) \\ &\quad -\frac{(t-1)(t+r-1)P_{r,n-1}(t)}{P_{r,n}(t)}\frac{\partial}{\partial t}E_{r,n-1}(t). \end{aligned}$$

Let us assume that

$$[rn+(n-1)(t-1)]P_{r,n-1}(t)=(t-1)(t+r-1)\frac{\partial}{\partial t}P_{r,n-1}(t).$$

One solution of the above differential equation is:

$$P_{r,n}(t)=\frac{(t-1)^{n+1}}{t+r-1}.$$

Note that $\frac{P_{r,n-1}(t)}{P_{r,n}(t)}=\frac{1}{t-1}$. Therefore, for all $n\geq 1$,

$$E_{r,n}(t)=-(t+r-1)\frac{\partial}{\partial t}E_{r,n-1}(t). \tag{5}$$

Checking the recurrence for $n=1$, we have by a direct computation that:

$$E_{r,1}(t)=\frac{D_{r,1}(t)}{P_{r,1}(t)}=\frac{r}{\left(\frac{(t-1)^2}{t+r-1}\right)}=-(t+r-1)\frac{\partial}{\partial t}E_{r,0}(t),$$

which gives that:

$$\frac{\partial}{\partial t}E_{r,0}(t)=-\frac{r}{(t-1)^2}.$$

Hence, we define $E_{r,0}(t)=\frac{r}{t-1}$.

Proposition 6.1 *For all $n\geq 1$,*

$$E_{r,n}(t)=(-1)^{n-1}r\sum_{j=1}^{n}j!S_{n,j}\frac{(t+r-1)^j}{(1-t)^{j+1}},$$

where $S_{n,j}$ is the (n,j)-Stirling number of the second kind.

Proof. By induction on n, the recurrence relation (5) gives that:

$$E_{r,n}(t)=(-1)^n\sum_{j=1}^{n}S_{n,j}(t+r-1)^j\frac{\partial^j}{\partial t^j}E_{r,0}(t).$$

Using the initial condition of this recurrence, namely $E_{r,0}(t)=\frac{r}{t-1}$, we obtain that:

$$E_{r,n}(t)=(-1)^n\sum_{j=1}^{n}S_{n,j}(t+r-1)^j\frac{(-1)^j j!r}{(t-1)^{j+1}}.$$

This is equivalent to:

$$E_{r,n}(t)=(-1)^{n-1}r\sum_{j=1}^{n}j!S_{n,j}\frac{(t+r-1)^j}{(1-t)^{j+1}},$$

which completes the proof. □

Now we are ready to give an explicit formula for the polynomial $D_{r,n}(t)$.

Corollary 6.2 *For all $n \geq 1$,*

$$D_{r,n}(t) = r\sum_{j=1}^{n} j!S_{n,j}(t+r-1)^{j-1}(1-t)^{n-j}.$$

Proof. From the definitions, we have that:

$$D_{r,n}(t) = P_{r,n}(t)E_{r,n}(t) = \frac{(t-1)^{n+1}}{t+r-1}E_{r,n}(t).$$

Hence, by Proposition 6.1, we get the desired result. □

Finding the coefficient of t^k in the polynomial $D_{r,n}(t)$, we obtain an explicit formula for $d(r,n,k)$, as follows:

Theorem 6.3 *The number of permutations $\pi \in G_{r,n}$ which satisfy $\mathrm{exc}_A(\pi) = k$ is given by:*

$$d(r,n,k) = r\sum_{j=1}^{n}\sum_{i=0}^{j-1}(-1)^{k+j-1-i}r^i j!S_{n,j}\binom{j-1}{i}\binom{n-1-i}{k},$$

where $S_{n,j}$ is the (n,j)-Stirling number of the second kind.

7 Symmetry of the flag-excedance number

In this section, we present the symmetry property of the flag-excedance number:

Theorem 7.1 *The generating polynomial*

$$\sum_{\pi \in G_{r,n}} q^{\mathrm{fexc}(\pi)} = \sum_{i=0}^{rn-1} a_i q^i$$

satisfies: $a_i = a_{rn-1-i}$ for $i \in \{1,\dots,\lfloor \frac{rn-1}{2} \rfloor\}$.

Define a bijection of $G_{r,n}$: $\pi \mapsto \pi'$ in the following way: For $1 \leq i \leq n-1$, if $\pi(i) = j^{[\beta]}$, then $\pi'(n-i) = (n+1-j)^{[r-\beta]}$ and if $\pi(n) = j^{[\beta]}$, then $\pi'(n) = (n+1-j)^{[r-1-\beta]}$.

Instead of burdening the reader with the subtle though standard proof, we choose to give an example of the bijection:

$$\pi = \left(\begin{array}{cccc|cccc|cccc} \bar{\bar{1}} & \bar{\bar{2}} & \bar{\bar{3}} & \bar{\bar{4}} & \bar{1} & \bar{2} & \bar{3} & \bar{4} & 1 & 2 & 3 & 4 \\ 2 & \bar{1} & 4 & \bar{\bar{3}} & \bar{\bar{2}} & 1 & \bar{\bar{4}} & \bar{3} & \bar{2} & \bar{\bar{1}} & \bar{4} & 3 \end{array}\right)$$

$$\pi' = \left(\begin{array}{cccc|cccc|cccc} \bar{\bar{1}} & \bar{\bar{2}} & \bar{\bar{3}} & \bar{\bar{4}} & \bar{1} & \bar{2} & \bar{3} & \bar{4} & 1 & 2 & 3 & 4 \\ \bar{1} & 4 & \bar{3} & 2 & 1 & \bar{\bar{4}} & 3 & 2 & \bar{\bar{1}} & \bar{4} & \bar{\bar{3}} & \bar{\bar{2}} \end{array}\right).$$

8 Log-concavity of the parameter exc_A

In this section, we show that some variant of the excedance parameter is log-concave. We start by proving that exc_A on B_n is log-concave. The corresponding proof for $G_{r,n}$ is similar.

THEOREM 8.1 *The parameter* exc_A *on* B_n *is log-concave.*

Proof. Recall from Section 4 the following definitions:

$$F_i^A(n,k) = \{\pi \in B_n \mid \mathrm{exc}_A(\pi) = k, \mathrm{neg}(\pi) = i\},$$

$$f_i^A(n,k) = |F_i^A(n,k)|.$$

Define:

$$X_{n,k} = \sum_{i=0}^{n} f_i^A(n,k).$$

Note that $X_{n,k}$ is the number of permutations $\pi \in B_n$ having $\mathrm{exc}_A(\pi) = k$.

By the recursion given in Proposition 4.1:

$$\begin{aligned} f_i^A(n,k) &= (n-k)f_i^A(n-1,k-1) + (k+1)f_i^A(n-1,k) + \\ &\quad +(n-k)f_{i-1}^A(n-1,k) + (k+1)f_{i-1}^A(n-1,k+1), \end{aligned}$$

we have:

$$\begin{aligned} X_{n,k} &= (n-k)X_{n-1,k-1} + (k+1)X_{n-1,k} + (n-k)X_{n-1,k} + (k+1)X_{n-1,k+1} = \\ &= (n-k)X_{n-1,k-1} + (n+1)X_{n-1,k} + (k+1)X_{n-1,k+1}. \end{aligned}$$

We prove the log-concavity by induction. For $n = 3$, the claim can be easily verified. Now we assume it for $n-1$, and we have to show that:

$$X_{n,k}^2 \geq X_{n,k-1}X_{n,k+1}.$$

Along the following computation, we abbreviate $X_{n-1,j}$ to X_j. We compute:

$$\begin{aligned} X_{n,k}^2 - X_{n,k-1}X_{n,k+1} &= [(n-k)X_{k-1} + (n+1)X_k + (k+1)X_{k+1}]^2 - \\ &\quad - [(n-k+1)X_{k-2} + (n+1)X_{k-1} + kX_k] \cdot \\ &\qquad \cdot [(n-k-1)X_k + (n+1)X_{k+1} + (k+2)X_{k+2}] = \\ &= \big[(n-k)^2X_{k-1}^2 + (n+1)^2X_k^2 + (k+1)^2X_{k+1}^2 + \\ &\qquad +2(n-k)(n+1)X_{k-1}X_k + 2(n-k)(k+1)X_{k-1}X_{k+1} + \\ &\qquad +2(n+1)(k+1)X_kX_{k+1}] - \\ &\quad - [(n-k+1)(n-k-1)X_{k-2}X_k + (n-k+1)(n+1)X_{k-2}X_{k+1} + \\ &\qquad + (n-k+1)(k+2)X_{k-2}X_{k+2} + (n+1)(n-k-1)X_{k-1}X_k + \\ &\qquad + (n+1)^2X_{k-1}X_{k+1} + (n+1)(k+2)X_{k-1}X_{k+2} + \\ &\qquad + k(n-k-1)X_k^2 + k(n+1)X_kX_{k+1} + k(k+2)X_kX_{k+2}\big] = \end{aligned}$$

$$\begin{aligned}
= \;& \big[(n-k)^2X_{k-1}^2-(n-k+1)(n-k-1)X_{k-2}X_k\big]+\\
&+\big[(k+1)^2X_{k+1}^2-k(k+2)X_kX_{k+2}\big]+\\
&+[2(n-k)(n+1)X_{k-1}X_k-(n-k+1)(n+1)X_{k-2}X_{k+1}-\\
&\quad -(n+1)(n-k-1)X_{k-1}X_k]+\\
&+[2(n+1)(k+1)X_kX_{k+1}-(n+1)(k+2)X_{k-1}X_{k+2}-\\
&\quad -k(n+1)X_kX_{k+1}]+\\
&+\big[(n+1)^2X_k^2+2(n-k)(k+1)X_{k-1}X_{k+1}-\\
&\qquad -(n-k+1)(k+2)X_{k-2}X_{k+2}-(n+1)^2X_{k-1}X_{k+1}-\\
&\qquad -k(n-k-1)X_k^2\big].
\end{aligned}$$

We treat each one of the five brackets in the last expression separately.

By the induction hypothesis, the first bracket is greater (or equal) than X_{k-1}^2. Similarly, the second bracket is greater (or equal) than X_{k+1}^2.

Since $X_{k-2}X_{k+1}\le X_{k-1}X_k$ by the log-concavity assumption, we have:

$$2(n-k)(n+1)X_{k-1}X_k-(n-k+1)(n+1)X_{k-2}X_{k+1}-(n+1)(n-k-1)X_{k-1}X_k\ge$$
$$\ge 2(n-k)(n+1)X_{k-1}X_k-(n-k+1)(n+1)X_{k-1}X_k-(n+1)(n-k-1)X_{k-1}X_k=0X_{k-1}X_k=0,$$

and hence the third bracket is non-negative. Similarly, the fourth bracket is non-negative too.

From the first two brackets, we have two positive elements: X_{k-1}^2 and X_{k+1}^2. Their sum can be written as $(X_{k-1}-X_{k+1})^2+2X_{k-1}X_{k+1}$. Adding $2X_{k-1}X_{k+1}$ to the fifth bracket, we have:

$$(n+1)^2X_k^2+2((n-k)(k+1)+1)X_{k-1}X_{k+1}-(n-k+1)(k+2)X_{k-2}X_{k+2}-$$
$$-(n+1)^2X_{k-1}X_{k+1}-k(n-k-1)X_k^2.$$

Since $(n+1)^2=((n-k)+(k+1))^2=(n-k)^2+(k+1)^2+2(n-k)(k-1)$, we can simplify the bracket into:

$$(n+1)^2X_k^2-(n-k+1)(k+2)X_{k-2}X_{k+2}$$
$$-((k+1)^2-1)X_{k-1}X_{k+1}-((n-k)^2-1)X_{k-1}X_{k+1}-k(n-k-1)X_k^2.$$

By the log-concavity assumption, this sum is greater (or equal) than:

$$(n+1)^2X_k^2-(n-k+1)(k+2)X_k^2-((k+1)^2-1)X_k^2-((n-k)^2-1)X_k^2-k(n-k-1)X_k^2,$$

which is equal to 0. So, we have that the sum of all five brackets is non-negative and hence we are done. □

References

[1] R. M. Adin and Y. Roichman, *The flag major index and group actions on polynomial rings*, European J. Combin., 22 (2001) 431–446.

[2] C. A. Athanasiadis, *Edgewise subdivisions, local h-polynomials and excedances in the wreath product $\mathbb{Z}_r \wr S_n$*, SIAM J. Discrete Math., 28 (2014) 1479–1492.

[3] E. Bagno and R. Biagioli, *Colored-descent representations of complex reflection groups $G(r,p,n)$*, Israel J. Math., 160 (2007) 317–347.

[4] E. Bagno, A. Butman and D. Garber, *Statistics on the multi-colored permutation groups*, Electron. J. Combin., 14 (2007) #R24.

[5] E. Bagno and D. Garber, *On the excedance number of colored permutation groups*, Sém. Lothar. Combin., B53f (2006) 17 pp.

[6] E. Bagno, D. Garber and T. Mansour, *Excedance number for involutions in complex reflection groups*, Sém. Lothar. Combin., B56d (2007) 11 pp.

[7] M. Bona, *Combinatorics of permutations*, Chapman and Hall/CRC, Second Edition, 2012.

[8] F. Brenti, *q-Eulerian polynomials arising from Coxeter groups*, European J. Combin., 15 (1994) 417–441.

[9] W. Y. C. Chen, R. L. Tang and A. F. Y. Zhao, *Derangement polynomials and excedances of type B*, Electron. J. Combin., 16 (2009) #R15.

[10] E. Clark and R. Ehrenborg, *Excedances of affine permutations*, Adv. in Appl. Math., 46 (2011) 175–191.

[11] M. Fire, *Statistics on wreath products*, (2005) at http://arxiv.org/abs/math/0409421.

[12] D. Foata and G. Han, *The decrease value theorem with an application to permutation statistics*, Adv. in Appl. Math., 46 (2011) 296–311.

[13] D. Foata and G. Han, *The Flag-Descent and -Excedance numbers*, (2009) preprint. (Electronic version: http://www-irma.u-strasbg.fr/∼foata/paper/pub116b.html)

[14] D. Foata and M.-P. Schützenberger, *Théorie géométrique des polynômes eulériens*, Lecture Notes in Math., Vol. 138, Springer-Verlag, Berlin-New York, 1970.

[15] F. Jansson, *Variations on the excedance statistic in permutations*, (2006) preprint. (Electronic version: http://www.math.chalmers.se/∼einar/exjobb/fredrik-jansson.pdf)

[16] P. A. MacMahon, *Combinatory Analysis, Vol. 1 and 2*, Cambridge Univ. Press, Cambridge, 1915 (reprinted by Chelsea, New York, 1955).

[17] P. A. MacMahon, *The indices of permutations and the derivation therefrom of functions of a single varaible asscoiated with the permutations of any assemblage of objects*, Amer. J. Math., 35 (1913) 281–322.

[18] T. Mansour and Y. Sun, *Excedance numbers for permutations in complex reflection groups*, Sém. Lothar. Combin., B58b (2008) 7 pp.

[19] R. P. Stanley, *Enumerative Combinatorics, Volume I*, Cambridge University Press, 2002.

[20] E. Steingrímsson, *Permutation statistics of indexed permutations*, European J. Combin., 15 (1994) 187–205.