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ON STABILITY OF PIN-JOINTED BEAM AFFECTED BY RANDOM PULSATING LOAD

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This paper deals with stability analysis of pin-jointed beams that are affected to random pulsating load. The stability conditions of a pin-jointed beam are analysed using a mathematical model of the beam characterised by longitudinal force with Poisson characteristics and applying the stochastic modification of the second Lyapunov method.

Key words: *beam dynamics, perturbation theory, second Lyapunov method, random harmonic oscillator.*

INTRODUCTION

Many engineering structures consist of elements that can be modelled as a beam. To study the dynamics of this structural component under longitudinal parametrical excitations, the well-known Timoshenko partial differential equation (Timoshenko and Gere, 1961) has been used for a long time:

$$EJ\frac{\partial^4 u}{\partial x^4}+P(t)\frac{\partial^2 u}{\partial x^2}+D\frac{\partial u}{\partial t}+m\frac{\partial^2 u}{\partial t^2}=0, \tag{1}$$

where t is time, x is axial coordinate, E is Young modulus of elasticity, J is axial moment of inertia, $P(t)$ is disturbance longitudinal force, m is mass of unit of beam length, and D is viscous damping coefficient.

The boundary conditions for the above equation depend on the beam fastening. For a simply supported beam with free warping displacement the boundary conditions for (1) are

$$u(t,0)=u(t,L)=0; \tag{2}$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)(t,0)=\left(\frac{\partial^2 u}{\partial x^2}\right)(t,L)=0. \tag{3}$$

The disturbance longitudinal force usually is divided into two terms:

$$P(t)=P_0+P_1(t), \tag{4}$$

where the bounded continuous function $P_1(t)$ satisfies the assumption of zero mean, that is

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t P_1(s)ds=0. \tag{5}$$

The problem of elastic stability of beams may be formulated as the asymptotic stability problem of the trivial solution of equation (1). Substituting the series

$$u(t,x)=\sum_{n=1}^{\infty}T_n(t)\sin\left(\frac{\pi n x}{L}\right) \tag{6}$$

in equation (1) the authors reduce this problem to analyse stability of the second order differential equations of the following type:

$$n\in N:\ \frac{d^2T_n(t)}{dt^2}+\frac{D}{m}\frac{dT_n(t)}{dt}+(\omega_n^2+f_n(t))T_n(t)=0, \tag{7}$$

where $\omega_n^2=\frac{1}{m}\left(\frac{\pi n}{L}\right)^2\left(EJ\left(\frac{\pi n}{L}\right)^2+P_0\right)$ and

$f_n(t)=\frac{1}{m}\left(\frac{\pi n}{L}\right)^2P_1(t)$. The most advanced results are reached for equation (7) with periodic or almost periodic function $f_n(t)$, which is called the Mathieu-Hill equation with damping (Bolotin,1964). This model has been analysed in detail in many classical monographs and textbooks (see, for example, Timoshenko and Gere, 1961; Bolotin, 1964; Leipholz, 1978). The asymptotic stability criterion for these equations can be formulated in the following form: for all free oscillation frequencies ω_n, $n\in$N there exist such positive numbers D_n^{cr} that with unlimited time increment nontrivial solutions of (7) tend to zero for all $D>D_n^{cr}$ and unboundedly increase for all $D<D_n^{cr}$. This means that there exists a critical value of damp-

ing $D_{cr} = \max D_n^{cr}$ which guaranties stability of bridge with dynamics (1^n) for all $D > D_n^{cr}$. Unfortunately, there are no sufficiently efficient methods for analytical calculation of D_n^{cr}, even for periodic continuous functions $f_n(t)$. The most productive analysis of equations (1) and (7) can be done under the assumptions that the perturbation function $P_1(t)$ is sufficiently small. Substituting $P_1(t) = \varepsilon p(t)$ in the formula for $f_n(t)$ we introduce a small positive parameter ε in equation (7) and search for $D_n^{cr}(\varepsilon)$ as an analytical function of ε. In this case we can apply the very productive Krylov–Bogolyubov method (Bogoljubov *et al.*, 1976) of asymptotical analysis and find the critical dissipation $D_n^{cr}(\varepsilon)$ as an infinitesimal of the second order. In reality, there are some random factors that affect beam dynamics. In this case we may not apply the Krylov–Bogolyubov algorithm *per se* and use the proposed stochastic modification for this method (Skorokhod, 1989). This method can by helpful in engineering applications in conditions where the small perturbation function $P_1(t)$ in (1) be modelled as a continuous ergodic Markov process defined by the stochastic Ito differential equation (Pavlovic and Kozic, 2003; Ariaratnam, 1972; Li *et al.*, 2004). In this paper we also propose an algorithm for calculation of the critical damping D_{cr} in (1) under the assumption that perturbation is an impulse type random process given by formula $P_1(t) = \varepsilon h(y(t))$, where $y(t)$ is the compound Poisson process (Dynkin, 1965) with a stationary uniform distribution. To achieve this result we apply the proposed (Katafygiotis and Tsarkov 1996) stochastic averaging procedure for impulse type Markov dynamical systems.

This approach is schematically described in the next chapter of this paper. Applying the proposed diffusion approximation algorithm for a scalar second order differential equation (7) we find in the third chapter the critical damping D_n^{cr} and in the fourth chapter we discuss the dependence of the critical damping $D_{cr} = \max_n D_n^{cr}$ on parameters J, m, L in (1), variance and intensity of perturbations.

STOCHASTIC AVERAGING PROCEDURES FOR DYNAMICAL SYSTEMS WITH IMPULSE TYPE MARKOV SWITCHING

Let $\{y(t), t \geq 0\}$ be the Markov process with values at the segment $\mathbf{Y} := [0,1]$ defined for an arbitrary function $\{v(y), y \in \mathbf{Y}\}$ by the infinitesimal operator (Dynkin, 1965):

$$y \in \mathbf{Y} : (Qv)(y) = \lambda \int_{\mathbf{Y}} [v(z) - v(y)]dz, \tag{8}$$

where $\lambda > 0$. Any realisation of this Markov process (Dynkin, 1965) is a piecewise constant function having jumps at increasing random time moments $\{\tau_j . j \in \mathrm{N}\}$, which may be defined by formulae:

$$\tau_0 = 0, \quad P(\tau_j - \tau_{j-1} > t / y(\tau_{j-1}) = y) = \exp\{-\lambda t\} \tag{9}$$

The jump at any time moment τ_j is the uniform $\mathbf{R}(0,1)$ distributed random variable. We will deal with the impulse type dynamical system on the phase space

$$\mathbf{R} \times \mathbf{S}^1, \mathbf{S}^1 := \{0 \leq \varphi \leq 2\pi / \omega, \varphi(0) = \varphi(2\pi / \omega)\}, \tag{10}$$

defined by the phase coordinates $\{x_\varepsilon(t) \in \mathbf{R}^n, \varphi_\varepsilon(t) \in \mathbf{S}^1\}$. We assume that the random processes $\{x_\varepsilon(t) \in \mathbf{R}^n, \varphi_\varepsilon(t) \in \mathbf{S}^1\}$ satisfy:

- the differential equations

$$\frac{dx_\varepsilon}{dt} = \varepsilon A(y_\varepsilon(\tau_{j-1}), \varphi_\varepsilon, \varepsilon)x_\varepsilon, \tag{11}$$

$$\frac{d\varphi_\varepsilon}{dt} = \varepsilon f(y_\varepsilon(\tau_{j-1}), \varphi_\varepsilon, \varepsilon), \tag{12}$$

for all $j \in N, t \in (\tau_{j-1}, \tau_j)$;

- the jump equations

$$x_\varepsilon(\tau_j) = x_\varepsilon(\tau_j -) + \varepsilon B(y_\varepsilon(\tau_{j-1}), \varphi_\varepsilon(\tau_j -), \varepsilon)x_\varepsilon(\tau_j -), \tag{13}$$

$$\varphi_\varepsilon(\tau_j) = \varphi_\varepsilon(\tau_j -) + \varepsilon g(y_\varepsilon(\tau_{j-1}), \varphi_\varepsilon(\tau_j -), \varepsilon), \tag{14}$$

for all $j \in \mathrm{N}$;

where ε is small positive parameter, $\varepsilon \in (0, \varepsilon_0)$,

$$A(y,\varphi,\varepsilon) = A_1(y,\varphi) + \varepsilon A_2(y,\varphi),$$
$$f(y,\varphi,\varepsilon) = f_1(y,\varphi) + \varepsilon f_2(y,\varphi),$$

$$B(y,\varphi,\varepsilon) = B_1(y,\varphi) + \varepsilon B_2(y,\varphi),$$
$$g(y,\varphi,\varepsilon) = g_1(y,\varphi) + \varepsilon g_2(y,\varphi)$$

and $y_\varepsilon(t) = y(\varepsilon t)$.

Under the above assumption the triple $\{y_\varepsilon(t), x_\varepsilon(t), \varphi_\varepsilon(t), t \geq 0\}$ defines the homogeneous Markov process on the space $\mathbf{Y} \times \mathbf{R} \times \mathbf{S}^1$ (Skorokhod, 1989) with the weak infinitesimal operator

$$\begin{aligned}\mathbf{L}(\varepsilon)v(y,x,\varphi) &= \varepsilon A(y,\varphi,\varepsilon)x \frac{\partial}{\partial x} v(y,x,\varphi) + \\ &+ \varepsilon f(y,\varphi,\varepsilon) \frac{\partial}{\partial \varphi} v(y,x,\varphi) + \frac{1}{\varepsilon} Qv(y,x,\varphi) + \\ &+ \varepsilon G^\varepsilon v(y,x,\varphi),\end{aligned} \tag{15}$$

where

$$G^\varepsilon v(y,x,\varphi) = \frac{a(y)}{\varepsilon} \int_{\mathbf{Y}} [v(z, x + \varepsilon B(y,\varphi,\varepsilon)x, \varphi + \varepsilon g(y,\varphi,\varepsilon)) - v(z,x,\varphi)]dz$$

The stochastic averaging approach is based on the limit theorem (Skorokhod, 1989) for the pair of random processes $\{x_\varepsilon(t), \varphi_\varepsilon(t), t \geq 0\}$ under the condition that $\varepsilon \to 0$. The first step for asymptotic analysis of the Markov dynamical system (11)–(14) is the averaging procedure based on the limit calculation

$$\lim_{\varepsilon\to 0}\mathbf{L}(\varepsilon)(\varepsilon^{-1}v(x,\varphi)+v_1(y,x,\varphi)):=\overline{\mathbf{L}}v(x,\varphi)= \\ =\overline{A}_1(\varphi)x\frac{\partial}{\partial x}v(x,\varphi)+\overline{F}_1(\varphi)\frac{\partial}{\partial x}v(x,\varphi), \quad (16)$$

$$\overline{A}_1(\varphi)=\overline{[A_1(y,\varphi)+a(y)B_1(y,\varphi)]}:= \\ =\int_{\mathbf{Y}}[A_1(\varphi,y)+a(y)B_1(\varphi,y)]dy, \quad (17)$$

$$\overline{F}_1(\varphi)=\overline{[f_1(y,\varphi)+a(y)g_1(y,\varphi)]}:= \\ =\int_{\mathbf{Y}}[f_1(y,\varphi)+a(y)g_1(y,\varphi)]dy, \quad (18)$$

for an arbitrary sufficiently smooth function $v(x,\varphi)$ and specially selected function $v_1(y,x,\varphi)$.

Now we can construct the system of equations for *an average motion*:

$$\frac{d}{dt}\overline{x}(t)=\overline{A}_1(\overline{\varphi})\overline{x}(t), \quad (19)$$

$$\frac{d}{dt}\overline{\varphi}(t)=\overline{F}_1(\overline{\varphi}), \quad (20)$$

and define the averaging principle:

- for any $T>0, C>0$

  $$\lim_{\varepsilon\to 0}\sup_{0\le t\le T}P\left(\{|x_\varepsilon(\varepsilon t)-\overline{x}(t)|+|\varphi_\varepsilon(\varepsilon t)-\overline{\varphi}(t)|\}>C\right)=0;$$

- if the trivial solution of equation (19) is asymptotical stable then there exists such a positive number ε_0 that

  $$\mathrm{P}\left(\lim_{\varepsilon\to 0}\frac{1}{t}\ln|x_\varepsilon(t)|=0\right)=1, \quad (21)$$

  for any $\varepsilon\in(0,\varepsilon_0)$.

If $\overline{A}_1(\varphi)\equiv 0$ we can apply the diffusion approximation (Carkovs and Stoyanov, 2005) for the Markov dynamical system (11)–(14). For that we should look for the limit

$$\lim_{\varepsilon\to 0}\mathbf{L}(\varepsilon)(\varepsilon^{-2}v(x,\varphi)+\varepsilon^{-1}v_1(y,x,\varphi)+v_2(y,x,\varphi))=\overline{\mathbf{L}}v(x,\varphi),$$

where $\overline{\mathbf{L}}$ is a diffusion operator, which is given by equality

$$\overline{\mathbf{L}}v(x,\varphi):=\hat{A}(\varphi)x\frac{\partial}{\partial x}v(x,\varphi)+m(\varphi)\frac{\partial}{\partial\varphi}v(x,\varphi)+ \\ +0.5(x^2\sigma_1^2(\varphi)\frac{\partial^2}{\partial x^2}v(x,\varphi)+\sigma_2^2(\varphi)\frac{\partial^2}{\partial\varphi^2}v(x,\varphi)+ \\ +2x^2\sigma_{12}^2\frac{\partial^2}{\partial x\partial\varphi}v(x,\varphi)) \quad (22)$$

for an arbitrary sufficiently smooth function $v(x,\varphi)$. This operator defines the system of stochastic differential Ito equations (Dynkin, 1965):

$$d\hat{x}(t)=\hat{A}(\hat{\varphi}(t)\hat{x}(t)dt+\sigma_1(\hat{\varphi}(t)\hat{x}(t)dw_1(t)+ \\ +\sigma_{12}(\hat{\varphi}(t)\hat{x}(t)dw_2(t), \quad (23)$$

$$d\hat{\varphi}(t)=m(\hat{\varphi}(t))dt+\sigma_2(\hat{\varphi}(t)dw_1(t)+\sigma_{12}(\hat{\varphi}(t))dw_2(t) \quad (24)$$

where $w_1(t)$ and $w_2(t)$ are the independent standard Wiener processes. The finite dimensional distributions of initial processes $\{x_\varepsilon(t),\varphi_\varepsilon(t)\}$ for sufficiently small $\varepsilon>0$ may be approximated (Tsarkov, 1993) by the corresponding distributions of the processes $\{\hat{x}(t),\hat{\varphi}(t)\}$. For sufficiently small positive ε the asymptotic stability of the trivial solution of equation (4) follows the asymptotic stability of equation (23).

STABILITY ANALYSIS OF THE RANDOM LINEAR OSCILLATOR SUBJECTED TO SMALL RANDOM SWITCHING OF FREQUENCY

As it was mentioned in the Introduction, we assume that $P_1(t)=\varepsilon h(y(t))$ where ε is a small positive parameter and $y(t)$ is defined by the weak infinitesimal operator (8) Poisson process. The substituted $D=2\delta m\varepsilon^2$ will be searching for the critical damping $D_n^{cr}(\varepsilon)$ as an infinitesimal of the second order. After substitution of the decomposition (6) in (1) we have to deal with the second order random differential equation of the following form:

$$\ddot{x}(t)+\omega^2x(t)=-2\delta\varepsilon^2\dot{x}(t)-\varepsilon x(t)p(y(t)). \quad (25)$$

To take advantage of the diffusion approximation method proposed in the previous chapter, we have to rewrite the above equation in polar coordinates. Substituting

$$x(t)=r(t)\cos\frac{\psi(t)}{2};\ \dot{x}(t)=-r(t)\omega\sin\frac{\psi(t)}{2} \quad (26)$$

we may rewrite the second order differential equation (25) as a system of two differential equations:

$$\begin{cases}\dot{\psi}=2\omega+\varepsilon\frac{1}{\omega}[1+\cos\psi]p(y(t))-\varepsilon^2 2\delta\sin\psi, & (27)\\ \dot{r}=-\varepsilon^2 r\delta[1-\cos\psi]+\varepsilon\frac{1}{2\omega}rp(y(t))\sin\psi. & (28)\end{cases}$$

To analyse α-exponential stability of the solution for equation (25) we will apply the second Lyapunov method (Carkovs and Stoyanov, 2005) with Lyapunov function $F(r,\psi,y)=r^\alpha V^\varepsilon(\psi,y)$. By definition

$$(\mathbf{L}_1F)(r,\psi,y)= \\ +r^\alpha\left[-\alpha\varepsilon^2\delta[1-\cos\psi]+\alpha\varepsilon\frac{1}{2\omega}\sin\psi p(y)\right]V^\varepsilon(\psi,y)+ \\ +r^\alpha\left(2\omega+\varepsilon\frac{1}{\omega}[1-\cos\psi]p(y)-\varepsilon^2 2\delta\sin\psi\right)\frac{\partial}{\partial\psi}V^\varepsilon(\psi,y)+ \\ +r^\alpha QV^\varepsilon(\psi,y)=r^\alpha(\mathbf{L}(\varepsilon)V^\varepsilon)(\psi,y), \quad (29)$$

where

$$\mathbf{L}(\varepsilon)=Q_0+\varepsilon Q_1+Q_2\varepsilon^2,$$

$$Q_0:=2\omega\frac{\partial}{\partial\psi}+Q, \quad (30)$$

$$Q_1:=\alpha\frac{1}{2\omega}p(y)\sin\psi+\frac{1}{\omega}[1+\cos\psi]p(y)\frac{\partial}{\partial\psi}, \quad (31)$$

$$Q_2 := -2\alpha\delta[1-\cos\psi] - 2\delta\sin\psi\frac{\partial}{\partial\psi}. \tag{32}$$

If there exists such a function $V^{\varepsilon}(\psi, y)$, which satisfies inequalities

$$-r^{\alpha} \le -\frac{1}{c_2} r^{\alpha} F(\psi, y) = -\frac{1}{c_2} r^{\alpha} F(\psi, y) \tag{33}$$

and

$$\mathbf{L}(\varepsilon) V^{\varepsilon}(\psi, y) = -1, \tag{34}$$

then for any initial condition $r(0) = r_0$ the solution of equation (28) tends to zero with probability one (Carkovs and Stoyanov, 2005). To find a solution of equation (34) we apply the algorithm proposed by Carkovs and Matvejevs (2015). We will look for the solution of this equation as a singular at the point $\varepsilon = 0$ function

$$V^{\varepsilon}(\psi, y) = \varepsilon^{-2} q + \varepsilon^{-1} V_1(\psi, y) + V_2(\psi, y), \tag{35}$$

where q is a constant. Substituting (35) in (34) and equating the coefficients near ε^{-1} we will have an equation for unknown function $V_1(\psi, y)$

$$\left(Q + 2\omega\frac{\partial}{\partial\psi}\right) V_1(\psi, y) = -\alpha q \frac{1}{2\omega} p(y)\sin\psi. \tag{36}$$

It is not so difficult to ensure that by definition (8) $Qp(y) = -\lambda p(y)$. Therefore, we can look for a solution of (36) in the following form

$$V_1(\psi, y) = -\frac{\alpha p(y)}{2\omega} q[(C_1 \sin\psi + C_2 \cos\psi)]$$

with unknown coefficients C_1 and C_2. Substituting this function in (36) and equating the coefficients near $\sin\psi$ and $\cos\psi$ we can find a solution of the equation (36) as follows:

$$V_1(\psi, y) = -\frac{\alpha}{2\omega(\lambda^2 + 4\omega^2)} q p(y)(\lambda\sin\psi + 2\omega\cos\psi). \tag{37}$$

Now we should look for a solution of the equation

$$Q_0 V_2(\psi, y) = -1 - Q_2 q - Q_1 V_1(\psi, y).$$

Substitution using the formulae (31), (32) and (37) we have to look for a solution of the equation

$$Q_0 V_2(\psi, y) = G(\psi, y), \tag{38}$$

where

$$G(\psi, y) = -1 + \left\{2\alpha\delta[1-\cos\psi] + 2\delta\sin\psi\frac{\partial}{\partial\psi}\right\} q$$

$$\left\{\alpha\frac{1}{2\omega} p(y)\sin\psi + \frac{1}{\omega}[1+\cos\psi] p(y)\frac{\partial}{\partial\psi}\right\}$$

$$\left(\frac{\alpha}{2\omega(\lambda^2 + 4\omega^4)} q p(y)(\lambda\sin\psi + 2\omega\cos\psi)\right)$$

According to the Fredholm alternative this equation has a solution if and only if the right part in (38) satisfies equality:

$$\int_0^{2\pi}\int_0^1 G(\psi, y)\,dy\,d\psi = 0.$$

This equality permits to find an unknown constant q:

$$q = \alpha^{-1}\left[\delta - \frac{\lambda\sigma^2(\alpha+2)}{8\omega^2(\lambda^2 + 4\omega^2)}\right]^{-1}, \tag{39}$$

where $\sigma^2 = \int_0^1 p^2(y)dy$. Remember that we search for the Lyapunov function

$$F(r, \psi, y) := r^{\alpha} V^{\varepsilon}(\psi, y) = \varepsilon^{-2} r^{\alpha}(q + \varepsilon V_1(\psi, y) + \varepsilon^2 V_2(\psi, y)),$$

where functions $V_1(\psi, y)$ and $V_2(\psi, y)$ are bounded by definition and q is given by formula (39). Therefore, if parameter $\varepsilon > 0$ is sufficiently small, the solution of the Lyapunov equation satisfies inequality (33) if and only if $\delta > \dfrac{\lambda\sigma^2(\alpha+2)}{8\omega^2(\lambda^2 + 4\omega^2)}$.

As far as in the above formula α is an arbitrarily chosen positive number, we can ensure that there exists such a critical value for damping

$$\delta_{cr} = \frac{\lambda\sigma^2}{4\omega^2(\lambda^2 + 4\omega^2)}, \tag{40}$$

that $\mathrm{P}(\lim_{t\to\infty} r(t) = 0) = 1$, if $\delta > \delta_{cr}$ and $\mathrm{P}(\lim_{t\to\infty} r(t) = \infty) = 1$, if $\delta < \delta_{cr}$.

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After substitution of the series $u(t,x) = \sum_{n=1}^{\infty} T_n(t)\sin\left(\frac{\pi n x}{L}\right)$ in equation (1) we proceed to equations (7) for the amplitudes $\{T_n(t), n \in \mathbf{N}\}$ of the longitudinal oscillations

$$\ddot{T}_n(t) + 2\varepsilon^2\delta\dot{T}_n(t) + \omega_n^2 T_n(t) + \varepsilon h_n(y(t)) T_n(t) = 0, \tag{41}$$

where

$$\omega_n^2 = \frac{1}{m}\left(\frac{\pi n}{L}\right)^2\left(EJ\left(\frac{\pi n}{L}\right)^2 + P_0\right),\ h_n(y) = \frac{1}{\varepsilon m}\left(\frac{\pi n}{L}\right)^2 p(y),$$

$$\delta = \frac{D}{2\varepsilon^2 m} \tag{42}$$

Remember that $P_1(t) = p(y(t))$, where $y(t)$ is a piecewise constant stationary process with uniform $R(0,1)$ distribution and $\mathbf{E}\{p(y(t))\} = 0$, $\mathbf{E}\{p^2(y(t))\} = \sigma^2$. Now we can apply the necessary and sufficient condition, achieved in the previous section, for the almost sure asymptotic stability of the longitudinal oscillations in a form of inequality:

$$\delta > \frac{\lambda\gamma_n^2}{4\omega_n^2(\lambda^2+4\omega_n^2)} := \delta_n^{cr}, \tag{43}$$

where $\gamma_n^2 = \frac{2}{m}\left(\frac{\pi n}{L}\right)^4 \sigma^2$. Substituting (42) in this formula we can derive the necessary and sufficient condition for the longitudinal oscillations (41) exponential decay in a form of inequalities

$$D > D_n^{cr}(\lambda, L, P_0, \sigma^2, m) :=$$

$$:= \frac{\sigma^2}{2}\left(\frac{\pi n}{L}\right)^2 \frac{\lambda}{\left(EJ\left(\frac{\pi n}{L}\right)^2 + P_0\right)\left[\lambda^2 + \frac{4}{m}\left(\frac{\pi n}{L}\right)^2\left(EJ\left(\frac{\pi n}{L}\right)^2 + P_0\right)\right]}$$

(44)

for each $n \in \mathbf{N}$. It is not so difficult to ensure that $\max_n D_n^{cr}(\lambda, L, P_0, \sigma^2) = D_1^{cr}(\lambda, L, P_0, \sigma^2)$ for any $P_0 > 0$, $\lambda > 0$, $\sigma^2 > 0$, and $L > 0$. Therefore, the necessary and sufficient condition for beam stability may be written in a form of inequality for a dissipation parameter:

$$D > D_{cr}(\lambda, L, P_0, \sigma^2, m) :=$$

$$:= \frac{\sigma^2}{2}\left(\frac{\pi}{L}\right)^2 \frac{\lambda}{\left(EJ\left(\frac{\pi}{L}\right)^2 + P_0\right)\left[\lambda^2 + \frac{4}{m}\left(\frac{\pi}{L}\right)^2\left(EJ\left(\frac{\pi}{L}\right)^2 + P_0\right)\right]} \tag{45}$$

The critical dissipation $D_{cr}(\lambda, L, P_0, \sigma^2, m)$ is an increasing function of a mass parameter m and of a variance σ^2 of the longitudinal force, and is a decreasing function of the constant component P_0 of the longitudinal force. However, dependence of this function on switching intensity λ and length L has a form rather like a mountainous surface (mountain ridge):

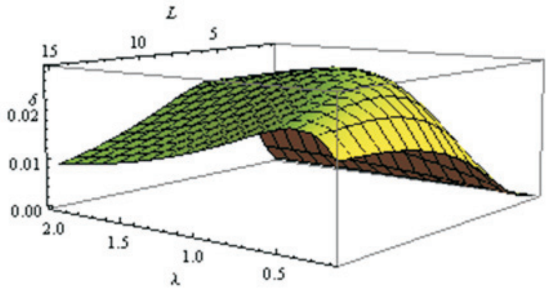

Fig. 1. Dependence of the critical dissipation border on length L and intensity λ ($P_0 = 1$, $\sigma^2 = 1$, $m = 1$).

To ensure beam stability under longitudinal impulse type perturbations of any intensity we need the critical value of dissipation

$$D_{cr}(\lambda, L, P_0, \sigma^2, m) := \max_{\lambda>0} D_{cr}(\lambda, L, P_0, \sigma^2, m) =$$

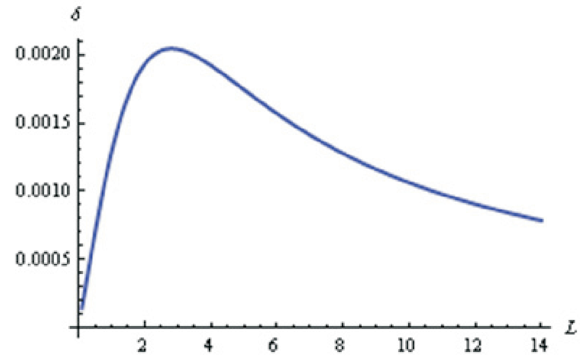

Fig. 2. Dependence of the maximal critical dissipation border on length L ($P_0 = 1$, $\sigma^2 = 1$, $m = 1$).

$$= \frac{\pi L^2 \sigma^2 \sqrt{m}}{8(EJ\pi^2 + P_0 L^2)^{3/2}} \tag{46}$$

It is not so difficult to be sure that for any values of the parameters P_0, σ^2, and m the critical value of dissipation (46) is a unimodal function on length L (see the example at Fig. 2) having the maximum for length $L = \pi\sqrt{2EJP_0^{-1}}$:

$$D > \hat{D}_{cr}(P_0, \sigma^2, m) := \max_{L>0} D_{cr}(\lambda, L, P_0, \sigma^2, m) =$$

$$= D_{cr}(\lambda, L, P_0, \sigma^2, m)\Big|_{L^2 = 2EJP_0^{-1}\pi^2} = \frac{\sigma^2\sqrt{m}}{12P_0\sqrt{3EJ}} \tag{47}$$

Therefore, if we have only the expected value and variance of the longitudinal force switched by random Markov process, given by statistical observations, we may be sure of the beam stability if and only if $D > \frac{\pi L^2 \sigma^2 \sqrt{m}}{8(EJ\pi^2 + P_0 L^2)^{3/2}}$. But if need to be ensure stability for a beam of any length we need more dissipation: $D > \frac{\sigma^2\sqrt{m}}{12P_0\sqrt{3EJ}}$.

REMARK

It should be mentioned that the linear equation (1) allows to analyse only small deformations of a beam. As it has been shown previously (Katafygiotis and Tsarkov, 1996), the solutions of the linear second order equations of type (41) for sufficiently small have an exponential behaviour. Therefore, if equilibrium of equation (1) is not stable the beam vibration amplitudes exponentially increase and we cannot assume the beam deformations to be small. In this case, we should apply non-linear Euler-Bernoulli beam theory including the effects of mid-plane stretching (Rao, 2007). This approach requires involving a non-linear term $-\frac{E}{2L}\frac{\partial^2 u}{\partial x^2}\int_0^L\left(\frac{\partial u}{\partial x}\right)^2 dx$ in equation (1) and we cannot analyse the resulting equation applying the substitution (6). We will return to the equilibrium instability problem later in another paper.

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GADĪJUMA PULSĒJOŠAI SLODZEI PAKĻAUTU AR ŠARNĪRIEM PIESTIPRINĀTU SIJU STABILITĀTE

Šajā rakstā pētīti stabilitātes nosacījumi ar šarnīriem piestiprinātās sijās, kurās garenvirziena spēks pakļauts gadījuma perturbācijām, modelējot to kā saliktu Puasona procesu ar mazām nejaušām amplitūdām. Pieņemot, ka amplitūdas ir savstarpēji neatkarīgas un nav atkarīgas arī lēcienu laiku momentos, mēs lietojām otrās Ļapunova metodes modifikāciju gandrīz droša līdzsvara asimptotiskās stabilitātes analīzei.