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Original research papers

AERONAUTICAL INSPIRATIONS IN BIOMECHANICS

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Abstract
Introduction. The goal of the paper is to show that some problems formulated in the dynamics of atmospheric flight are very similar to the problems formulated in the biomechanics of motion and medicine. Three problems were compared: minimum-heat transfer from the boundary layer to the ballistic missile skin, minimum-time ski descent, and the minimisation of the negative cumulated effect of the drug in cancer chemotherapy. ***Material and methods.*** All these problems are solved using the same method originally developed for aerospace systems – the method of Miele (the extremisation method of linear integrals via Green's theorem). ***Results.*** It is shown that the problems arising in different branches of knowledge are very similar in problem formulations, mathematical models, and solution methods used. ***Conclusions.*** There are no barriers between different disciplines.

Key words: minimum-time ski descent, optimal chemotherapy

Introduction

The paper is devoted to selected problems of optimal control in different branches of knowledge: flight dynamics, sports biomechanics, and medicine. It is shown that some problems are very similar in problem formulations and mathematical modelling. The point in common is the method used – the method of Miele. This method is a non-classical one. Since the paper is addressed to the general reader, the method in described in detail. Three problems were considered: the minimum-heat transfer problem during a re-entry manoeuvre, the minimum-time ski descent problem, and the minimisation of toxic effects during cancer chemotherapy.

Material and methods

Consider a minimised functional that is linear in the derivative y'

$$J=\int_A^B [\varphi(x,y)+y'\psi(x,y)]dx=\int_A^B [\varphi(x,y)dx+\psi(x,y)dy]. \qquad (1)$$

Symbols φ and ψ denote the known functions of two arguments (x,y). Symbols A and B stand for initial and final points. Let us assume that the process under consideration is represented by a curve y(x) joining points A and B in the (x,y)-plane and that all solutions are within or on the border of the admissible domain represented by a region limited by a closed curve ε(x,y) = 0. The points A and B also belong to this curve. Now we can determine the difference in values of the integral (1) for two arbitrarily taken curves AQB and APB

$$\Delta J=J_{AQB}-J_{APB}=\int_{AQB}(\varphi dx+\psi dy)-\int_{APB}(\varphi dx+\psi dy)=\oint_{AQBPA}(\varphi dx+\psi dy). \qquad (2)$$

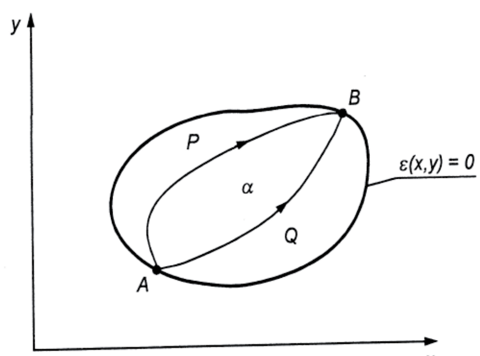


Figure 1. The graphical interpretation of different admissible strategies represented by arbitrarily taken curves AQB and APB

Now, employing Green's theorem, the cyclic integral (2) may be transformed into a surface integral

$$\Delta J=\iint_\alpha \omega(x,y)dx\,dy\,, \qquad (3)$$

where α is the area limited by these two curves. The symbol ω denotes the so-called fundamental function that is equal to

$$\omega(x,y)=\frac{\partial\psi}{\partial x}-\frac{\partial\varphi}{\partial y}. \tag{4}$$

Three cases are possible:

a) The fundamental function ω is identically equal to zero. That means $\Delta J = 0$, and the functional is independent of the curve in the (x,y)-plane – the process is irrespective of the strategy.

b) The fundamental function ω has the same sign, for example $\omega > 0$. In such a case, the inequality is valid: $\Delta J > 0$ or $J_{AQB} > J_{APB}$. It means that every curve to the left gives a smaller value of the functional. In the limit, the minimising curve belongs to the border of the admissible domain.

c) The fundamental function ω changes its sign. It means that there is a curve on which $\omega = 0$ that divides the admissible domain into two subregions, where $\omega > 0$ and $\omega < 0$, respectively. The optimal path contains subarcs on the border of the admissible domain and on the curve $\omega = 0$. This last subarc is a singular arc in the calculus of variations. Further details can be found in the work of Miele [1].

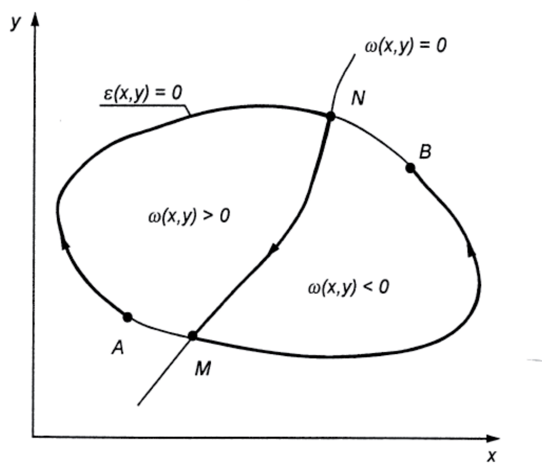


Figure 2. The fundamental function ω changes its sign. The optimal solution contains two subarcs on the border of the admissible domain AN, MB and the subarc within the admissible domain NM where $\omega = 0$

Results

Minimum-heat transfer strategy for the re-entry manoeuvre of a variable-geometry ballistic missile

The problem is formulated as follows (cf. [1] and [2]). The missile enters into the atmosphere without propulsion. The known re-entry velocity v_A must be reduced to another (lower) known velocity v_B over the given loss of altitude (h_A–h_B). The velocity is controlled by the dive brakes, so the position of the brakes may be regarded as a control variable that is an unknown function of the altitude.

The model assumptions are as follows:

a) The Earth is flat and non-rotating. The gravitational acceleration g is constant.

b) The trajectory is vertical, so the lifting force is zero. The results can be applied to arbitrary inclined paths by coordinate transformation. Other trajectories may be also considered [2].

c) The missile is regarded as a particle.

d) The missile velocity is greater than the speed of sound (the Mach number Ma > 1).

Now, the equation of motion of the missile derives from Newton's second law

$$\frac{dv}{dh}=\frac{0.5\rho S C_D v}{m}-\frac{g}{v}, \tag{5}$$

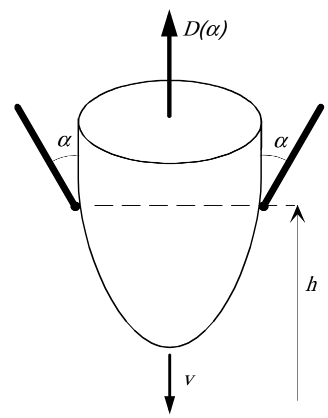
where: v – the velocity of the missile, h – the altitude, S – the reference area, ρ – the density of the air depending on the altitude, g – the gravitational acceleration, m – the mass of the missile, and $C_D(h)$ – the variable drag coefficient. This coefficient is regarded as a control variable that may be arbitrarily taken from the range

$$(C_D)_{min} \le C_D(h) \le (C_D)_{max}. \tag{6}$$

The lower limit corresponds to the brakes fully retracted and the upper limit to the brakes fully extended. Equation (5) should satisfy the boundary conditions

$$v(h_A) = v_A, \qquad v(h_B) = v_B, \tag{7}$$

where, v_A and v_B are the given velocities for the altitudes h_A and h_B, respectively.



D = the drag depending on the position of the brakes.

Figure 3. Model of a variable-geometry ballistic missile

For hypervelocity flight, the following heat transfer law is valid ([1] and [2])

$$\frac{dH}{dt}=K\rho\, v^3, \tag{8}$$

where: H – the heat transferred from the boundary layer to the missile skin, K – a characteristic constant, and t – the time. It means that the heat H is proportional to the work done by the resistive force (aerodynamic drag). The integral is equivalent to the above differential equation

$$H_B = -K\int_A^B \rho(h)\, v^2\, dh \Rightarrow \text{MIN} \tag{9}$$

and it is minimised. It should be remembered that dh is negative because the altitude decreases with time. The functional (9) depends on the unknown velocity v(h) that should be computed. The functional (9) may be expressed in the form of the linear integral (1), thus

$$H_B = \int_A^B 0\,dv + \left[-K\rho(h)v^2\right]dh\,. \tag{10}$$

For the problem under consideration, the fundamental function is as follows

$$\omega(v,h) = -2K\rho(h)v < 0\,. \tag{11}$$

The sign of the fundamental function is negative within the admissible domain, and it is irrespective of the parameters of the model. The admissible domain is given in Figure 4.

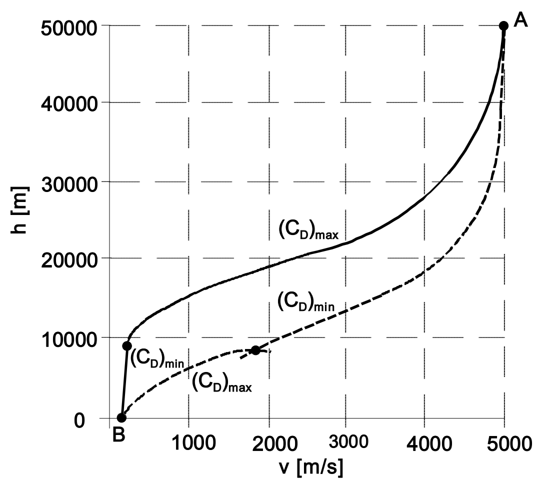
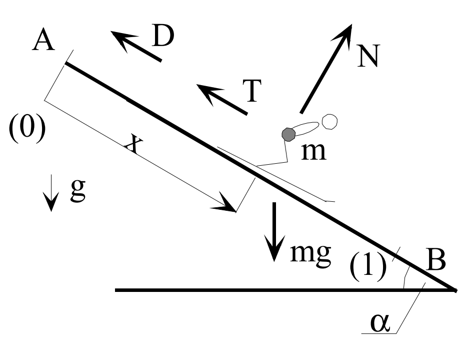


Figure 4. The admissible domain for the minimum-heat transfer problem for a variable-geometry ballistic missile. The upper curve minimises the functional

The optimal strategy requires that the brakes be fully extended at the beginning of the descent and then fully retracted. No middle position of the brakes is optimal.

Minimum-time ski descent problem

The problem is formulated in the following manner (cf. [3] and [4]). A skier should cover the distance from the given point A to the given point B in the minimum time. In both points, the velocities v_A and v_B are also given, and v_B is lower than the maximal one which could be achieved. This may stem from the fact that below point B, there is a difficult segment of the slope. The skier employs their aerodynamic drag for braking. They change their position from a dropped to an upright position.



D = the aerodynamic drag, T = the frictional force, N = the normal reaction of the ground, mg = the weighting force.

Figure 5. The particle model of the skier on the slope

The model assumptions are as follows:

a) The skier is reduced to their centre of gravity.
b) The slope is a straight line (a more general case is considered in Maroński's article, where the shape of the slope may be represented by any continuous function [4]).
c) Coulomb's law describes the friction T between the skis and the slope. This means that the frictional force is proportional to the normal reaction of the ground.

The equation of motion is as follows

$$\frac{dv}{dx} = -\frac{0.5\rho SC_D v}{m} + \frac{g(\sin\alpha - \mu\cos\alpha)}{v}, \tag{12}$$

where: x – the distance covered (independent variable), α – the inclination angle of the slope, and μ – the frictional coefficient. This equation is valid with the boundary conditions

$$v(x_A) = v_A, \qquad v(x_B) = v_B\,. \tag{13}$$

The control function ($SC_D(x)$) should belong to the range

$$(SC_D)_{min} \le SC_D(x) \le (SC_D)_{max} \tag{14}$$

which is individual for each skier and defines their abilities of minimum and maximum braking. The time of covering the distance (x_B-x_A) is minimised

$$t_B = \int_A^B \frac{1}{v}\,dx \Rightarrow \mathrm{MIN}, \tag{15}$$

and this is the difference compared with the previous problem. The linear integral to be minimised takes the form

$$t_B = \int_A^B 0\;dv + \frac{1}{v}\,dx\,. \tag{16}$$

The fundamental function

$$\omega(x,v) = \frac{1}{v^2} > 0 \tag{17}$$

is always greater than zero within the admissible domain (case (b)).

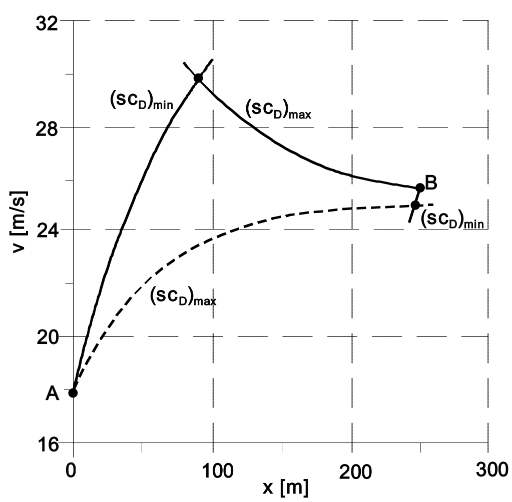


Figure 6. The admissible domain for the minimum-time ski descent problem. The upper curve minimises the time. The upper corner is the point of switching the control function

Minimisation of toxic effect in cancer chemotherapy

In this problem, two opposite goals should be achieved: the assumed destruction of the cancer cells at the end of the therapy and the minimised cumulative negative toxic effect of the drug on normal tissues. The assumptions are as follows (cf. [5] and [6]):

a) The model considers the dynamics of cancer cell population growth. It does not consider the dynamics of healthy cell population growth. The population is homogeneous.
b) Each mother-cell divides into two daughter-cells.
c) The stream of cells (the increase of cell number in the time unit) during division is proportional to the total number of cancer cells in the population. This is the fundamental assumption of the Malthusian model of growth.
d) The drug kills daughter-cells just after division. It does not kill mother-cells. The maximum dosage of the drug, which kills all daughter-cells, is known. When the drug is not administered, the process is not disturbed.
e) The number of cancer cells at the beginning of the therapy is known, and the number of cancer cells at the end (usually lower) is assumed. The model does not allow the destruction all cancer cells in the finite time of the therapy.
f) The functional proportional to the dose of the drug administered during the whole therapy is minimised.
g) The time of the therapy is given.

The dynamics of cancer cell proliferation as the drug is administered is described by the equation (cf. [5] and [6])

$$\frac{dN}{dt} = \alpha\ N(2u-1), \tag{18}$$

where: N – the number of cancer cells, t – the time, α – the intrinsic rate (an inverse of the average length of the cell cycle time – here assumed to be constant), and u(t) – the control function representing the probability of the youth cells surviving just after division belonging to the range

$$0 \le u(t) \le 1\,. \tag{19}$$

The value u(t) = 0 denotes that all daughter cells are killed – the maximal dosage of the drug is administered. The value u(t) = 1 means that all daughter cells are alive after division and the drug is not applied. In such a case, the number of cells N increases exponentially.

Let us assume that at the beginning of the therapy, the number of cancer cells N_A is given. For the given time of the therapy, this number should be reduced to the number N_B (or maintained at the same level), then

$$N(t_A)=N_A\ , \quad N(t_B)=N_B\,. \tag{20}$$

The cumulated negative toxic effect of the therapy is represented by the functional

$$J_B = \int_A^B [1-u(t)]\,dt \Rightarrow \text{MIN} \tag{21}$$

and it is minimised. The linear integral (1) takes the form

$$J_B = \int_A^B 0.5\ dt\ + \left[-\frac{1}{2\alpha N}\right] dN\,. \tag{22}$$

The fundamental function ω(t,N) is identically equal to zero within the admissible domain (case (a))

$$\omega(t,N) \equiv 0\,. \tag{23}$$

There is no unique solution of this problem in contrast to the previous examples. Each curve that joins the initial point A and the final point B is optimal.

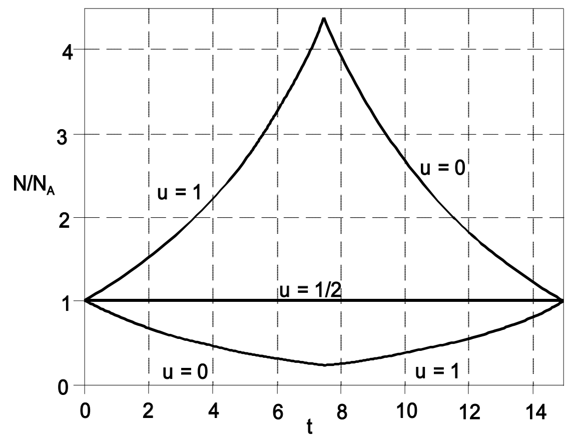


Figure 7. The admissible domain for the minimisation of the toxic effect in cancer chemotherapy. The optimal solution is non-unique

Conclusions

It is shown in the paper that some problems and methods developed in flight dynamics may be used in biomechanics. Three problems are tackled: the re-entry manoeuvre of a variable-geometry ballistic missile, the minimum-time ski descent, and the optimal cancer chemotherapy. The same method is applied – the method of Miele originally developed for flight dynamics. One can find further analogies: the minimum-fuel aircraft manoeuvre and the minimisation of the energy consumption in cycling [7]. The method of Miele is not the only one that may be used in biomechanics. Chebyshev's pseudospectral method, successfully applied for the minimisation of the fuel consumption of commercial aircraft and record vehicles ([8] and [9]), may be employed for the minimum-time running problem [10]. The conclusion is that there are no barriers between different disciplines. Development is more intensive in the areas related to military applications where expenditures and efforts are much higher comparing with other branches. However, the ideas and the methods may be successfully transferred.

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