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NUMERICAL RESULTS FOR LOTKA-VOLTERRA MODEL USING APPROXIMATE INERTIAL MANIFOLDS

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Abstract: A numerical study on the classical Lotka-Volterra model is performed, using approximate inertial manifolds. An analytical study consisting in the inertial form of the system is made. The construction of the approximate inertial manifolds is based on the identification of the absorbing domains using the graphical representations of the phase portraits. The hypotheses of the Jolly-Rosa-Temam algorithm are verified for certain values of parameters and the approximate inertial manifolds are constructed. Errors of approximation are computed.

Keywords: inertial form, inertial manifold, absorbing domain, dynamical system, prey-predator.

1. Introduction

During the last thirty years a new study on the infinite dynamical systems has evolved. It concernes the dynamics in the presence of the global attractor. Usually, the attractors have complicated geometrical structure and they may attract the orbits very slowly. The phase dynamics may be reduced close to the global attractor, on the inertial manifold. This concept was introduced by Foiaş and Constantin and much used by Nicolaenko, Sell and Temam[1-3].

An inertial manifold M is a finite dimensional Lipschitz manifold, positively invariant (i.e. S(t) M$\subseteq$ M, t $\geq$ 0) exponentially attracting all phase trajectories of the semidynamical system associated withthe Cauchy problem $u(0)=u_0$ for an evolution equation $\dot{u}=F(u)$ [3].

Lotka-Volterra model (prey-predator)[4] is the Cauchy problem $x(0)=x_0$, $y(0)=y_0$ for the s.o.d.e.

$$\begin{cases} \dot{x}=ax-bxy, \\ \dot{y}=-cy+dxy. \end{cases} \tag{1}$$

It was conceived by Lotka (1925) and Volterra (1931) observing the variation of fish population. The parameters signify: a is the intrinsec rate of growing for prey population, b predation rate coefficient, c predator population mortality, d reproduction rate of predatorsperunit of prey eaten. The variables, x, y represent, density of prey and predators respectively. Values of x, y and a, b, c, d also must be positive.

System (1) represents as well a model of economic dynamics, namely Goodwin's model ofincome distribution, based on class struggle and inspired by the Lotka-Volterra model. In the economic case, x is the rate of employment, i.e. workers/work force, y workers' shareof national income, $0 \leq x, y \leq 1$.

The Lotka Volterra model is very important in population modeling. It describes the population dynamics of two interacting species (prey-predator or parasite-host pairs). The analysis of the system may be used, in particular, to predict the probability of extinction. It describes the dynamics in models from ecology, molecular biology, ecosystems, and chemical systems (for example a model for oxygen depletion in a system of sewage could be developed). It can also be used in the detection of failures in civil structures.

Lotka-Volterra hasapplicability in hydraulics, for example the food-chain in rivers can be modelled using this system. It also describes the evolution of the dispersion of pollutants. It is known that nature can absorb a pollutant up to certain limits (threshold value). Experiments show that the dependence between the emitted quantity of a pollutant and the remaining quantity can be described by a certain function. If some quantity of the pollutant is emitted regularly, we obtain an iterative process for a sequence of functions describing the dependence between the emitted and remaining quantities of the pollutant. Using this discrete functional model, a system of two differential equations of Lotka-Volterra types may be constructed.

We present various results of numerical simulation for our model. In this paper we make an approximate study of the system using the approximate inertial manifolds, which is in fact a nonlinear Galerkin method. Inertial manifolds are very important since they describe the large time behavior of the system and they are finite-dimensional. Study of the dynamics on an inertial manifold produces a significant simplification in the study of the dynamics of the initial system.

In this paper we study numerical and analitical the Lotka-Volterra system using approximate inertial manifolds.

2. Inertial form

System (1) can be written as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} + A\begin{pmatrix} x \\ y \end{pmatrix} = f(x,y), \tag{2}$$

where

$$A = \begin{pmatrix} -a & 0 \\ 0 & c \end{pmatrix} \text{ and } f(x,y) = \begin{pmatrix} -bxy \\ dxy \end{pmatrix}. \tag{3}$$

Let us consider the projectors $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. We denote $\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ the projection of $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ on $P\mathbf{R}^2$, i.e. $\mathbf{p} = P\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p_1 \\ 0 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = Q\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ q_2 \end{pmatrix}$.

Thus, $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{p} + \mathbf{q} = \begin{pmatrix} p_1 \\ q_2 \end{pmatrix}$.

System (2) projected on $P\mathbf{R}^2$ and on $\mathbf{R}^2 \setminus P\mathbf{R}^2$ respectively is written as follows

$$\begin{aligned} \dot{p}_1 &= ap_1 - bp_1q_2, \\ \dot{q}_2 &= -cq_2 + dp_1q_2. \end{aligned} \tag{4}$$

The first equation from (4) represents the inertial formof the system.

Solving the second equation from (4) in q_2, we obtain

$$q_2(p,t) = q_2(0)e^{\int_0^t (-c+dp_1(\tau))d\tau}. \tag{5}$$

Replacing it in the first equation, we get

$$\dot{p}_1 = ap_1 - bp_1q_2(0)e^{\int_0^t -c+dp_1(\tau))d\tau},$$

equation which describesthe dinamics on the inertial manifold.

3. Jolly-Rosa-Temam algorithm for Lotka-Volterra model

In [5-7] an algorithm is developed for the construction of a sequence of approximate inertial manifolds, with the same dimension, which converges to the exact inertial manifold.

In [8,9] we verify the assumptions of Jolly-Rosa-Temam algorithm [5,6] for the Lotka-Volterra system (2) truncated to a disk of radius ρ in two cases according to the values of the parameters. Thus, we obtain the prepared equation

$$\frac{d\mathbf{u}}{dt} + A\mathbf{u} = f_\rho(\mathbf{u}), \tag{6}$$

where $\mathbf{u} = (x, y)$, $\quad f_\rho(\mathbf{u}) = \chi_\rho(r) f(\mathbf{u}), \chi_\rho(r) = \chi(\frac{r^2}{\rho^2}), \chi \in C^1(\mathbf{R}_+)$,

$\chi /_{[0,1]} = 1, \chi /_{[2,\infty)} = 0, 0 \le \chi(s) \le 1, \forall s \in [1,2]$.

As in [8], it results that the first condition is satisfied

$$\left\| f_\rho(u_1) - f_\rho(u_2) \right\| \le \max\{b,d\}(6\rho \left\| u_1 - u_2 \right\| + 2\sqrt{2} \left\| u_1 - u_2 \right\|) = M_\rho \left\| u_1 - u_2 \right\|,$$

where $M_\rho = \max\{b,d\}(6\rho + 2\sqrt{2})$ is the Lipschitz constant for the prepared equation.

For the verification of the third condition, we notice that $e^{-tA} = \begin{pmatrix} e^{at} & 0 \\ 0 & e^{-ct} \end{pmatrix}$.

We distinguish several cases, according to the values of parameters a and c.

Case I. $a > 0$ and $c > 0$ It is the only case with biological reality. The other cases have only theoretical interest, but they may be used in the study of bifurcation.

Since the condition $0 < \lambda_n \le \Lambda_n$ must be satisfied,this caseis divided in two situations

a) If $a \le c$, we can choose the projectors $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and thus $\left\| e^{-tA} P \right\| = e^{at}$ and $\left\| e^{-tA} Q \right\| = e^{-ct}$. Let us chose $\lambda_n = a, \Lambda_n = c, K_1 = 1, K_2 = 1, \alpha = 0$. The third condition is satisfied since $\left\| e^{-tA} P \right\| = e^{at} \le e^{-\lambda_n t}$ for all $t \le 0$ and $\left\| e^{-tA} Q \right\| = e^{-ct} \le e^{-\Lambda_n t}$ for all $t \ge 0$.

For the fifth condition of the algorithm, we take $K_3 = 1$, and the inequality becomes $|-a| \le \lambda_n$, which is satisfied.

Since $\gamma_\alpha = 0$, a sufficient condition of gap in the spectrumis satisfied if

$$c - a > 6\max\{b,d\}(6\rho + 2\sqrt{2}). \tag{7}$$

If the parameters verify (7), then all the hypothesis are satisfied, thusJolly, Rosa and Temam's algorithm can be applied to the prepared equation Lotka-Volterra (6). A sequence of approximate inertial manifolds, for this case, will be constructed in the next section.

b) If $a > c$, using the same projectors P and Q as in case Ia)and $K_1 = 1, K_2 = 1, \alpha = 0$, to have $\| e^{-tA} P \| = e^{at} \le e^{-\lambda_n t}$ for all $t \le 0$, it is enough to have $\lambda_n \ge -a$, condition which is satisfied for all $\lambda_n > 0$. In order to have $\| e^{-tA} Q \| = e^{-ct} \le e^{-\Lambda_n t}$ for all $t \ge 0$, we choose $\Lambda_n = c$. Since $0 < \lambda_n \le \Lambda_n$, we choose λ_n small enough, such as $\lambda_n \le c$ and thus the gap condition is satisfied. For example, if $\lambda_n = \frac{1}{10}$, is required to have

$$c - \frac{1}{10} > 6\max\{b,d\}(6\rho + 2\sqrt{2}). \tag{8}$$

The fifth condition is written as $a \le K_3 \dfrac{1}{10}$, which means we can choose K_3, independent of n.

Case II. $a < 0$, $c > 0$. There are also two situations

a) If $-a \le c$ we can choose the projectors P and Q as in case Ia), and $\lambda_n = -a, \Lambda_n = c, K_1 = 1, K_2 = 1, \alpha = 0$. Thus $\|e^{-tA}P\| = e^{at} \le e^{-\lambda_n t}$ for all $t \le 0$, and $\|e^{-tA}Q\| = e^{-ct} \le e^{-\Lambda_n t}$ for all $t \ge 0$.

For $K_3 = 1$, the fifth condition is written as $|-a| \le \lambda_n \Leftrightarrow -a \le -a$, which is satisfied.

For $\gamma_\alpha = 0$, the gap condition is

$$c + a > 6\max\{b, d\}(6\rho + 2\sqrt{2}). \tag{9}$$

b) For $-a > c$, we take $P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. For $\lambda_n = c, \Lambda_n = -a, K_1 = 1, K_2 = 1, \alpha = 0$, we have $\|e^{-tA}P\| = e^{-ct} \le e^{-\lambda_n t}$ for all $t \le 0$ and $\|e^{-tA}Q\| = e^{at} \le e^{-\Lambda_n t}$ for all $t \ge 0$.

We choose $K_3 = 1$ and, as $\|AP\| = c$, the fifth condition is verified.

Since $\gamma_\alpha = 0$, the gap condition is

$$-a - c > 6\max\{b, d\}(6\rho + 2\sqrt{2}). \tag{10}$$

Case III. $a > 0$, $c < 0$. In this case, we prove that for the projectors we used before, the Jolly-Rosa-Temam algorithm can not be applied.

a) For $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, the third condition becomes $\left\| e^{-tA}P \right\| = e^{at} \le K_1 e^{-\lambda_n t}$ for all $t \le 0$, which, for $K_1 = 1$ is satisfied for all $\lambda_n \ge -a$. But $-a < 0$, so we can choose λ_n arbitrarily, but small enough to have the gap condition also satisfied, e.g. $\lambda_n = \dfrac{1}{10}$. Thus, the first part of the third condition is verified.

The second part of this condition is written as $\|e^{-tA}Q\| = e^{-tc} \le K_2 e^{-\Lambda_n t}$ for all $t \ge 0$, i.e. $\Lambda_n t \le lnK_2 + tc$. For all $t > 0$, we must have $\Lambda_n \le \dfrac{1}{t} lnK_2 + c$. But for $t \to \infty$, this inequality becomes $\Lambda_n \le c$, which is imposible since $c < 0$. Therefore, the second part of this condition is not satisfied.

b) For $P=\begin{pmatrix}0&0\\0&1\end{pmatrix}$, $Q=\begin{pmatrix}1&0\\0&0\end{pmatrix}$, the third condition is $\|e^{-tA}P\| = e^{-ct} \le K_1 e^{-\lambda_n t}$ for all $t \le 0$, which, for $K_1 = 1$ is true for all $\lambda_n \ge c$. Nevertheless, $c < 0$, so we can choose λ_n arbitrarily, but small enough to have also the seventh condition satisfied, e.g. $\lambda_n = \frac{1}{10}$.

The second part of the third condition becomes $\|e^{-tA}Q\| = e^{at} \le K_2 e^{-\Lambda_n t}$ for all $t \ge 0$, i.e. $\Lambda_n t \le lnK_2 - ta$. For all $t > 0$, we must have $\Lambda_n \le \frac{1}{t} lnK_2 - a$. For t small, i.e. $t \to 0$, this reads $\Lambda_n \le -a < 0$, which is impossible. Therefore, the third condition is not satisfied neither for these projectors P and Q.

Case IV. $a < 0$, $c < 0$. For $P=\begin{pmatrix}0&0\\0&1\end{pmatrix}$, $Q=\begin{pmatrix}1&0\\0&0\end{pmatrix}$, the third condition is accomplished.It becomes $\|e^{-tA}P\| = e^{-ct} \le K_1 e^{-\lambda_n t}$ for all $t \le 0$, which, for $K_1 = 1$ is true for all $\lambda_n \ge c$, but as $c < 0$, we can choose λ_n arbitrarily, small enough to have the seventh condition verified, e.g. $\lambda_n = \frac{1}{10}$.

The second part of the third condition reads $\|e^{-tA}Q\| = e^{at} \le K_2 e^{-\Lambda_n t}$ for all $t \ge 0$, which is verified for $K_2 = 1$ and $\Lambda_n = -a > 0$.

The fifth condition is $\|AP\| = |c| \le K_3 \frac{1}{10}$, thus we can find K_3.

The gap condition reads

$$-a - \frac{1}{10} > 6\max\{b,d\}(6\rho + 2\sqrt{2}). \tag{11}$$

Case V. For $a = 0$ or $c = 0$, matrix A is not invertible, so the sixth condition is not verified.

In conclusion, Jolly-Rosa-Temam method can be applied in case I, II and IV.

4. Approximate inertial manifolds for the prepared equation Lotka-Volterra. Estimation of errors

We plot $q_2 = |Qu|$ as function of time, on the approximate inertial manifold, for Lotka-Volterra model. We approximate the distances between the orbits on the exact inertial manifold and those on the approximate inertial manifold.

We implementJolly-Rosa-Temam algorithm [5,6] in Scilab [10]. We plot the approximate inertial manifolds and compute errors of approximation. The numerical computations are made in Scilab and verified in Maple 10.

For $\rho = \sqrt{2}, b = 0.001, d = 0.002$, we find c such as the spectral gap condition to be accomplished (9). Thus, $c - a > 0.13632$.

We consider $a = 0.01$ and $c = 0.85$, to have the condition verified there must be $\sigma \in (0.41896;0.44104)$. For $\sigma_0 = 0.42$ and $\sigma = 0.44$, $k_{n;0.44} = k_{n;0.42} = 0.6495 < \frac{2}{3}$. For $\tau_j = c_1 2^{-j}$, $N_j = \frac{c_2}{c_1} j2^j$, the algorithm can be applied.

For an initial prey population of 1000 and 5 predators, in fig. 1 we represent $|Qu|$ as function of time for 3, 4, 5 and 6 iterations.

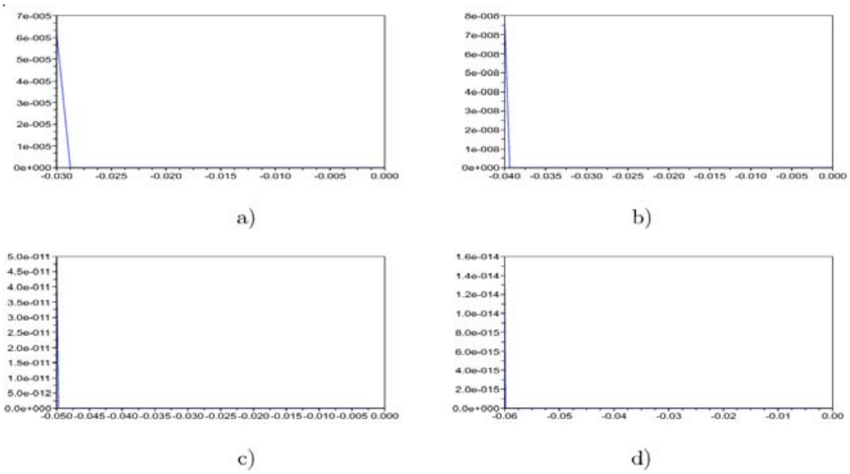

Fig. 1–Graphycal representation of $q_2 = |Qu|$ on the a.i.m., for a) 3 iterations, b) 4 iterations, c) 5 iterations, d) 6 iterations; $a = 0.01$, b =0.001 , $c = 0.85$, d =0.002 ; $x_0 = 1000; y_0 = 5$.

As $M_0 = 0$, results $\beta_{n,\sigma} = \beta_{n,\sigma_0} = 0.8349$. The error is $R_1(\tau) = 0.8349\tau e^{-0.42t}(1+|p_0|)$. The second evaluation of errors is

$$R_2(\sigma_0, N\tau) = R_2(0.42; N\tau) = 0.6495\|\varphi(p_0) - \psi\| + (41.745e^{-0.02N\tau} + 0.8349\tau)(1+|p_0|)$$

The error, i.e. the distance between the aproximative orbitsandthe exact orbitsobtained for a maximum of tweenty iterations is plotted in fig. 2. We notice that the errors are large even for more iterations.

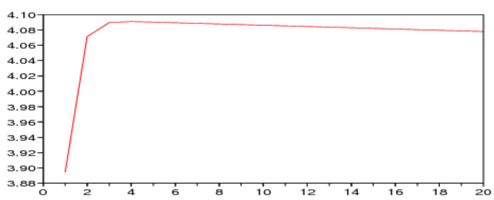

Fig. 2– Errors for Lotka-Volterra modelfor $a = 0.01$, b =0.001 , $c = 0.85$, d =0.002 ; $x_0 = 1000; y_0 = 5$.

For the initial point $x_0 = 426$, $y_0 = 0$, which is close to the center, we make the graphycal representations for $|Qu|$ as function of time, for 6 iterations in fig. 3.

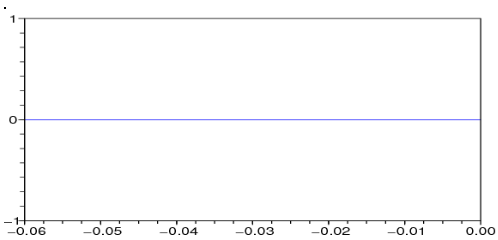

Fig. 3–Graphycal representation as function of t for $q_2 = |Qu|$
$a = 0.01$, $b = 0.001$, $c = 0.85$, $d = 0.002$; $x_0 = 426; y_0 = 0$.

For the computation of errors, as $M_\rho = 0.02262741699$, σ must be in $(0.0778823, 0.7821177)$. For $\sigma_0 = 0.1$ and $\sigma = 0.7$, we have $k_{n;0.7} = 0.1836428 < \frac{2}{3}$ and $k_{n;0.1} = 0.2815856 < \frac{2}{3}$. The errors we obtain are in Table 1. In fig. 4 the errors are plotted, as function of number of iterations. We observe that the error is reducingvery slowly, after four iterations, thus the six iterations from fig. 3. are enough. The errors are much smaller in this case when we choose the initial point from Ox axis, compared to the first case, when we take it arbitrarily in the plane.

Table 1

Errors of approximation for $a = 0.01$, $b = 0.001$, $c = 0.85$, $d = 0.002$; $x_0 = 426; y_0 = 0$

No. of iterations	1	2	3	4	5	6
Error	0.6007883	0.2601108	0.1965531	0.1839417	0.1792331	0.1792331

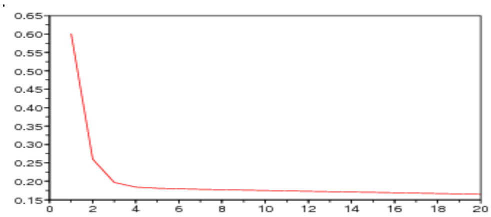

Fig. 4 – Errors for Lotka-Volterra model for
$a = 0.01$, $b = 0.001$, $c = 0.85$, $d = 0.002$; $x_0 = 426; y_0 = 0$

For the parameters $a = -20$, $b = 1$, $c = 1$, $d = 1$, we are in case II b). We take $\rho = \frac{1}{20}$ and the gap condition is satisfied ($19 > 18.77056274$). From the above considerations, we conclude that the other hypotheses are verified, thus the Jolly-Rosa-Temam can be applied. For the initial point $x_0 = 0.005, y_0 = 0$, which is in the absorbing domain of the attractive node, we plot $|Qu|$ as function oftime, for 6 iterationsin fig. 5. The initial point must be in this case on Oy.

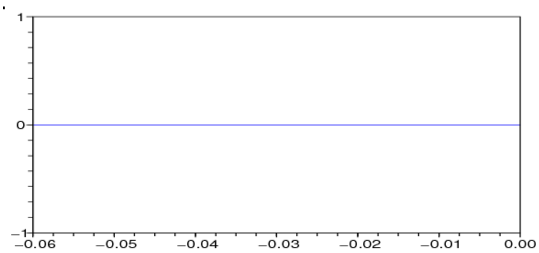

Fig. 5 – Graphycal representations of $q_2 = |Qu|$ on the a.i.m., for Lotka-Volterra model, for 6 iterations;
$a = -20$, $b = 1$, $c = 1$, $d = 1$; $x_0 = 0; y_0 = 0.04$

To compute errors, since $M_\rho = 3.128427124$, σ must be in $(10.385281, 10.614719)$. For $\sigma_0 = 10.386$ and $\sigma = 10.613$, we have $k_{n;10.386} = 0.6587111 < \frac{2}{3}$ and $k_{n;10.613} = 0.6587094 < \frac{2}{3}$ and the results for errors are in Table 2. In fig. 6,we plot errors as function of the number of iterations, for 200 iterations. Errors are large and they decrease very slowly.

Table 2

Errors of approximation for a =-20, b =1 , c = 1, d =1 ; $x_0 = 0; y_0 = 0.04$

No.of iterations	1	2	3	4	5	6
Error	110.31718	178.82244	223.6767	252.96749	272.01457	284.31836

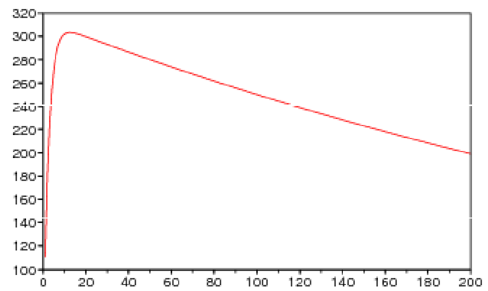

Fig. 6 – Errors for Lotka-Volterra model in case a =-20, b =1, c = 1, d =1; x_0 =0, y_0 =0.04.

5. Conclusions

The study on inertial manifolds is very important since it reduces the study of differential equations to a space with a lower dimension, simplifing the asymptotic study (for large time). The construction of an exact inertial manifold is possible very rarely, thus the approximation methods are very important.

The conditions that must be fulfilled to have approximate inertial manifolds according to Jolly-Rosa-Temam algorithm are quite restrictive and the values of the parameters are difficult to be found.

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