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PRACTICAL ASPECTS CONCERNING NONLINEAR DYNAMIC ANALYSIS OF CABLE STRUCTURES

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Abstract: The paper is about some aspects concerning the nonlinear dynamic analysis of prestressed cable structures. A method for the assessment of the tangent stiffness matrix and of the nonlinear parameters is proposed. The methodology is similar to the one described by P. Krishna. The Newmark method is used to integrate the motion equation. In the final section of the paper a comparison between the output supplied by the software of the presented method is made, with constant stiffness matrix(linear) and with the non-linear matrix updated step by step (geometric non-linear). The elements used for comparison are the displacement and velocity response of a given pretensioned cable structure.

Keywords: prestressed cable, dynamic analysis, stiffness matrix, Newmark method

1. Introduction

Dynamic structural analysis can be classified in two categories: linear and nonlinear structural analysis.

The differential equation of movement for a nonlinear behavioral model is [1]:

$$M \cdot \ddot{U} + g(\dot{U}) + f(U) = P(t)\,, \tag{1}$$

or:

$$M \cdot \ddot{U} + g_1(U)\dot{U} + f(U) = P(t) \tag{1'}$$

where: $g(\dot{U}) = [g_1(\dot{u}_1,\cdots,\dot{u}_n)\ldots g_n(\dot{u}_1,\cdots,\dot{u}_n)]^T$ is the damping function;

$f(U) = [f_1(u_1,\cdots,u_n)\ldots f_n(u_1,\cdots,u_n)]^T$ is the stiffness function and:

$P(t) = [p_1(t)\ldots p_n(t)]^T$ is the stimulus function.

The solution to equation (1) is called dynamic response in terms of displacements $U(t)$, velocities $\dot{U}(t)$ and accelerations $\ddot{U}(t)$.

Nonlinearity can be classified into geometric nonlinearity and/ or physical non-linearity (from the composing material). In the case of cable structures, geometric nonlinearity is always present.

In the following sections different cases of nonlinearity will be discussed, especially geometric nonlinearity.

2. Computation of the Tangent Stiffness Matrix

For equation (1), namely:

$$M \cdot \ddot{U} + g(\dot{U}) + f(U) = P(t)$$

f(U) is taken under the following form:

$$f(U) = K \cdot U + R(U) \tag{2}$$

K is the tangent stiffness matrix; U is the displacements vector in the node, R is the residual terms vector which contains the non-linear terms from U.
The elements of the stiffness matrix are computed using a formula similar to the one presented in [2].

$$\begin{aligned}
&K_{i,j}^{xx} = \frac{1}{l_{i,j}}\,[F_i + (EA_i - F_i)\cdot(\theta_x^i)^2]; \quad K_{i,j}^{yy} = \frac{1}{l_{i,j}}[F_i + (EA_i - F_i)\cdot(\theta_y^i)^2] \\
&K_{i,j}^{xy} = \frac{(EA_i - F_i)}{l_{i,j}}\cdot(\theta_x^i)\cdot(\theta_y^j); \quad K_{i,j}^{zz} = \frac{1}{l_{i,j}}[F_i + (EA_i - F_i)\cdot(\theta_z^i)^2] \\
&K_{i,j}^{xz} = \frac{(EA_i - F_i)}{l_{i,j}}\cdot(\theta_x^i)\cdot(\theta_z^j); \quad K_{i,j}^{yz} = \frac{(EA_i - F_i)}{l_{i,j}}\cdot(\theta_y^i)\cdot(\theta_z^j) \\
&l_{i,j} = l_{q,p}; l_{q,p} = \sqrt{(x_q - x_p)^2 + (y_q - y_p)^2 + (z_q - z_p)^2}
\end{aligned} \tag{3}$$

The directing cosines θ_d^i $(d = x,y,z)$ are computed through the nodes coordinates.

F – prestress force;

E –modulus of elasticity;

A – cross-sectional area of cable;

θ – directing cosines;

l – length of the elements .

$$\theta_{q,p} = \left(\frac{x_q - x_p}{l_{q,p}};\frac{y_q - y_p}{l_{q,p}};\frac{z_q - z_p}{l_{q,p}}\right)$$

The displacements vector is:

$$U = \begin{bmatrix} U_1 \\ U_2 \\ \cdots \\ U_n \end{bmatrix} = [u_{1x} \quad u_{1y} \quad \vdots \quad u_{2x} \quad u_{2y} \quad \vdots \quad \cdots \quad u_{nx} \quad u_{ny} \quad u_{nz}]^T$$

$$= [u_1 \quad u_2 \quad \vdots \quad u_4 \quad u_5 \quad \vdots \quad \cdots \quad u_{n-2} \quad u_{n-1} \quad u_n]^T$$

The residual terms vector from (3) has the following form:

$$R = [R_1 \quad R_2 \quad \cdots \quad R_n]^T = [r_{1x} \quad r_{1y} \quad \cdots \quad r_{nx}]^T = [r_1 \quad r_2 \quad \cdots \quad r_n]^T$$

$$r_{i,j}^{xx} = (EA_i - F_i)\cdot\left[\frac{u_{ix} - u_{jx}}{l_{i,j}}\cdot c_{i,j}^{xx} + \frac{x_i - x_j}{2\cdot l_{i,j}}\cdot d_{i,j}^{xx}\right]$$

$$r_{i,j}^{yy} = (EA_i - F_i)\cdot\left[\frac{u_{iy} - u_{jy}}{l_{i,j}}\cdot c_{i,j}^{yy} + \frac{y_i - y_j}{2\cdot l_{i,j}}\cdot d_{i,j}^{yy}\right]$$

$$a_{i,j}^{xx} = \frac{1}{l_{i,j}^2}\cdot[(x_i - x_j)\cdot(u_{ix} - u_{jx})]; \quad a_{i,j}^{yy} = \frac{1}{l_{i,j}^2}\cdot[(y_i - y_j)\cdot(u_{iy} - u_{jy})]$$

$$b_{i,j}^{xx} = \frac{1}{l_{i,j}^2}\cdot[(u_{ix} - u_{jx})^2]; \quad b_{i,j}^{yy} = \frac{1}{l_{i,j}^2}\cdot[(u_{iy} - u_{jy})^2]$$

$$c_{i,j}^{xx} = a_{i,j}^{xx} + \frac{1}{2}\cdot b_{i,j}^{xx} - \frac{3}{2}\cdot(a_{i,j}^{xx})^2; c_{i,j}^{yy} = a_{i,j}^{yy} + \frac{1}{2}\cdot b_{i,j}^{yy} - \frac{3}{2}\cdot(a_{i,j}^{yy})^2$$

$$d_{i,j}^{xx} = b_{i,j}^{xx} - 3 \cdot (a_{i,j}^{xx})^2 - 3 \cdot a_{i,j}^{xx} \cdot b_{i,j}^{xx} + 5 \cdot (a_{i,j}^{xx})^3$$

$$d_{i,j}^{yy} = b_{i,j}^{yy} - 3 \cdot (a_{i,j}^{yy})^2 - 3 \cdot a_{i,j}^{yy} \cdot b_{i,j}^{yy} + 5 \cdot (a_{i,j}^{yy})^3$$

The Jacobian of f in initial configuration is:

$$A(U) = K + A_R \tag{4}$$

For evaluating the Jacobian A_R, the numerical derivation through devised differences procedure from [3] was used. At step (k) the expression of the differential coefficient is:

$$\left.\frac{\partial r_i}{\partial u_j}\right|_{u^{(k)}} = \frac{r_i(u_1, \dots, u_j + h^{(k)}, \dots, u_n) - r_i(u_1, \dots, u_j, \dots, u_n)}{h^{(k)}}$$

where h is small (the order 10-3-10-5).The movement equation is solved using the Newmark [4] integration operator described in [5].

For the Newmark formulae [6] in the case of geometrical nonlinearity the following algorithm is used:

1. The X_0 coordinates are read (at time t_0)
2. Initialization of $i = 0$
3. Computation of l_i, θ_i (the length, the directing cosines at t_i)
4. Computation if the stiffness matrix according to (3) $K(U_i)$ (for each t_i)
5. By integration U_{i+1}, $\dot{U}_{i+1}$ are obtained
6. $X_{i+1} = X_i + U_{i+1}$ (in the Newmark formula $U_{i+1} = \overline{U}_{i+1} + \beta h^2 W$)
7. $i = i+1$
8. If $i < TT/h$ (where TT is the Total Time) go to 3, else go to 9
9. Write the results

3. Newmark Integration Operator and Resolution Method

This is one of the most used operators due to its precision and stability characteristics.

The formulae for a single equation proposed by Newmark are [4]:

$$u_{i+1} = u_i + h\dot{u}_i + (\tfrac{1}{2} - \beta)h^2\ddot{u}_i + \beta h^2 \ddot{u}_{i+1} \tag{5}$$

$$\dot{u}_{i+1} = \dot{u}_i + (1-\gamma)h\ddot{u}_i + \gamma h\ddot{u}_{i+1} \tag{6}$$

In the case: $\gamma \neq \frac{1}{2}$, the method introduces an artificial damping of the displacement response, which is proportional to $\gamma - \frac{1}{2}$. Considering $\gamma = \frac{1}{2}$, equation (6) becomes:

$$\dot{u}_{i+1} = \dot{u}_i + \tfrac{1}{2}h\ddot{u}_i + \tfrac{1}{2}h\ddot{u}_{i+1} \tag{6'}$$

Formulae (5, 6’) are controlled by parameter β. For this reason, the method is called β-Newmark method. The operator is implicit because it contains $\ddot{u}_{i+1}$ in the second member. Formulae (5, 6’) can be rewritten as:

$$u_{i+1} = u_i + h\dot{u}_i + \tfrac{1}{2}h^2\ddot{u}_i + \beta h^2 \Delta\ddot{u}_{i+1} \tag{7a}$$

$$\dot{u}_{i+1} = \dot{u}_i + h\ddot{u}_i + \gamma h \Delta \ddot{u}_{i+1} \tag{7b}$$

$$\ddot{u}_{i+1} = \ddot{u}_i + \Delta \ddot{u}_{i+1} \tag{7c}$$

where $\Delta \ddot{u}_{i+1} = \ddot{u}_{i+1} - \ddot{u}_i$ represent the increase in acceleration at the end of the interval. Thus, the formulae estimate the remainder in Taylor series of the functions u and $\dot{u}$ [5].

For a system, formulae (7 a-c) are:

$$U_{i+1} = \overline{U}_{i+1} + \beta \Delta \ddot{U}_{i+1} \tag{8a}$$

$$\dot{U}_{i+1} = \overline{\dot{U}}_{i+1} + \gamma \Delta \ddot{U}_{i+1} \tag{8b}$$

$$\ddot{U}_{i+1} = \ddot{U}_i + \Delta \ddot{U}_{i+1} \tag{8c}$$

where the functions with a bar superscript represent the truncated Taylor series, namely:

$$\overline{U}_{i+1} = U_i + h\dot{U}_i + \frac{1}{2} h^2 \ddot{U}_{i+1} \tag{9a}$$

$$\overline{\dot{U}}_{i+1} = \dot{U}_i + h\ddot{U}_i \tag{9b}$$

3.1. Integration of Equation (2)

In order or simplify the notation, we denote with 1 the index of the current step and with 0 the index of the previous ($t_i = t_0, t_{i+1} = t_1$). By replacing (8 a, b, c) in (2), we obtain:

$$M \cdot (\ddot{U}_0 + \Delta \ddot{U}_1) + g(\overline{\dot{U}}_1 + \gamma h \Delta \ddot{U}_1) + f(\overline{U}_1 + \beta h^2 \Delta \ddot{U}_1) = P(t_1) \tag{10}$$

and denoting:

$$W = \Delta \ddot{U}_1,$$

equation (10) becomes:

$$F(W) = 0 \tag{11a}$$

Using the notations in § 2, $f(U) = K \cdot U + R(U)$. Thus we have:

$$F(W) = M \cdot W + g(\overline{\dot{U}}_1 + \gamma h W) + K \cdot (\overline{U}_1 + \beta h^2 W) + R(\overline{U}_1 + \beta h^2 W) + \\ + M \cdot \ddot{U}_0 - P(t_1) \tag{11b}$$

Equation (11a) is solved using the Newton method or using the fixed point iteration. Each integration step in the solution is performed in order to compute: $W = \Delta \ddot{U}_{i+1}$.

3.1.1 Newton Method

We denote with $J(W)$ the Jacobian of function F, namely:

$$J(W) = M + \gamma h B(\overline{\dot{U}}_1 + \gamma h W) + \beta h^2 K + \beta h^2 A_R(\overline{U}_1 + \beta h^2 W)$$

where B and A_R are the Jacobians of functions g and R. K is the rigidity matrix that appears in the formula due to the fact that during the evaluation of Jacobian, terms K and R were considered. The iteration scheme is:

$$J(W_n)\delta_{n+1} = -F(W_n) \tag{12a}$$

$$W_{n+1} = W_n + \delta_{n+1}; \quad W_0 = 0 \tag{12b}$$

Iteration (12) is considered until one of the following conditions is met: $||\delta_{n+1}|| \le \varepsilon$,

number of iterations $\le$ LNIT, where EPS and LNIT are previously chosen. Generally, a reduced number of iterations is sufficient.

4. CASE-STUDY STRUCTURES

The analyzed structure is a saddle surface (hyperbolic paraboloid) [7]. The properties of the structure are presented in Figure 1 and the coordinates in the table below.

Table 1

Coordinates of the structure

Node no	X [m]	Y [m]	Z [m]
1	-15.24	-19.05	-1.2954
2	0.0	-19.05	-1.905
3	15.24	-19.05	-1.2954
4	-15.24	0.0	0.6096
5	0.0	0.0	0.0
6	15.24	0.0	0.6096
7	-15.24	19.05	-1.2954
8	0.0	19.05	-1.905
9	15.24	19.05	-1.2954
10	-15.24	-38.1	-7.0104
11	0.0	-38.1	-7.62
12	15.24	-38.1	-7.0104
13	-30.48	-19.05	0.5334
14	30.48	-19.05	0.5334
15	-30.48	0.0	2.4384
16	30.48	0.0	2.4384
17	-30.48	19.05	0.5334
18	30.48	19.05	0.5334
19	-15.24	38.1	-7.0104
20	0.0	38.1	-7.62
21	15.24	38.1	-7.0104

The following types of cables were used for obtaining the structure:

In the direction of X: Φ 66.60 mm $A_x = 34.84\ cm^2$

In the direction of Y: Φ 88.80 mm $A_y = 61.935\ cm^2$

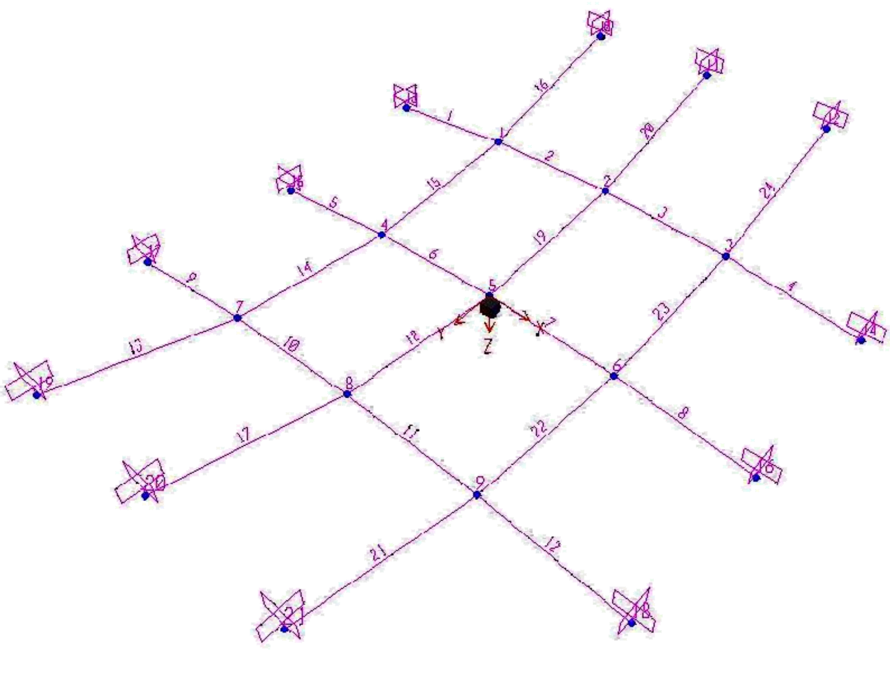

Fig. 1 - The analized structure

The characteristic curve was considered linear (even if the authors use a material with nonlinear behavior), where the modulus of elasticity is 15858 kN/cm^2.

The masses are considered lumped in the nodes. The equivalent node masses were computed according to the proper weight 960 N/m^2 given in Ma & Leonard example [7], thus resulting a forced of 288kN applied to the node, which represents a mass concentration mc = 28.41 kNs^2/m.

$$q = 960\ N/m^2 \ \Rightarrow\ P_{nod} = 15.24 \cdot 19.05 \cdot 0.960 = 278.71\ kN/node$$

$$cm = 278.71/9.81 = 28.41\ kN \cdot s^2/m$$

A very important phase during the design and execution of cable structures is the prestress which assures the proper behavior [8] to different types of dynamic stresses (wind, earthquake) over the structure exploitation period.
The prestressing forces are approximately equal in all the composing elements such that the ratio between the sag/ span is reduced (the structure can be considered flat shaped). The cables in the X direction were prestressed with a force of 28 kN /cm^2, thus resulting a prestressing force

$T_x^0 = 975.52\ kN$,

T_y^0 = 1781.87 kN– for the intermediate elements;

T_y^0 = 1858.05 kN – for the edge elements.

The prestressing forces are synthesized in Table 2.

Table 2

	Prestressing forces (kN/cm^2)	
Cables on direction	Marginal elements	Intermediate elements
X	28.0	28.0
Y	30.0	28.77

It is considered that the system is in equilibrium under the action of its own weight and the prestressing forces. This structure is subject to a uniformly distributed load of 27.54 daN/m^2 which is transformed to a equivalent node load of 501 kN, applied in each node.

5. Results

In the following, the authors will present graphically the displacements and velocities for node 5 on Z axis, with $t_0 = 0$, $TT = 6$ seconds and $h = 10\text{-}2$.

The differences computed through direct numerical integration using both the linear and non-linear Newmark operators can be observed.

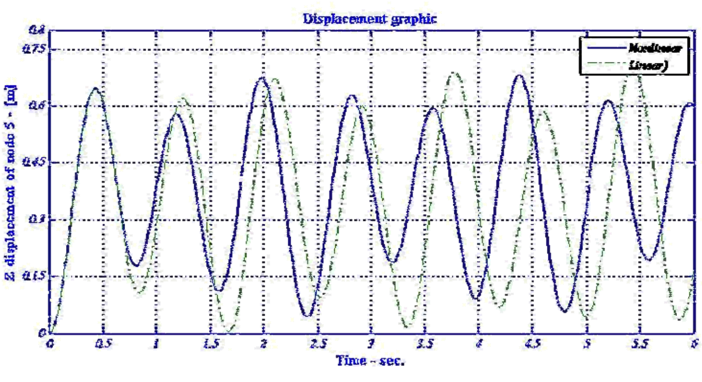

Fig 2 - Displacement graphic

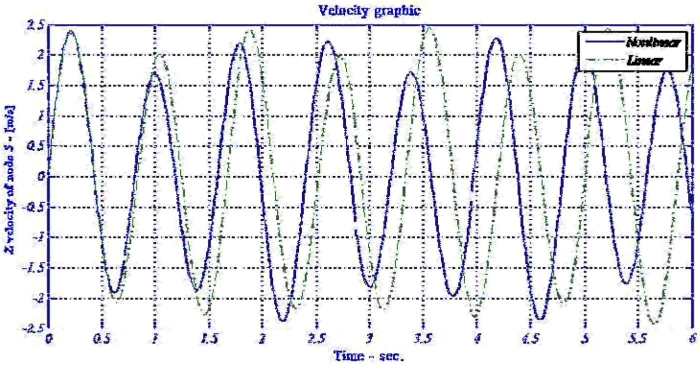

Fig. 3 - Velocity graphic

6. Conclusions

In the graphics above an interesting difference can be observed between the linear and non-linear computations. In both cases the Newmark method was used, with the difference that for the linear case the integration using the constant stiffness matrix was used and for the non-linear the matrix was updated step by step. The differences are not large but significant. It is clear that the processing time is higher in the case on non-linear analysis. Naturally in the case of the displacements the differences are larger than in the case of the velocities. The final conclusion is that structures built of trailing (suspended) cables must be non-linearly analyzed for obtaining accurate results or at least geometrically non-linear analyzed.

References

[1] Clough, R.W., Penzien, J., (1975), *Dynamics of structures*, McGraw-Hill, New York.
[2] Krishna, P., (1978), *Cable – Suspended Roofs*, McGraw-Hill Inc.
[3] Chisăliță, A., (2002), *Numerical Analysis*, Universitatea Tehnică din Cluj Napoca.
[4] Newmark, N. M., (1959), *A Method of Computation for Structural Dynamics*, ASCE Journal of the Engineering Mechanics Division, Vol. 85 No. EM3.
[5] Chisăliță, A., Chisăliță, F., (1993), *Analyse de l'operateur de Newmark,* Studia Univ. Babeş-Bolyai, Mathemathica, XXXVIII(4)
[6] Bathe, K. J., (1996), *Finite Element Procedures*, Prentice-Hall Inc.
[7] Ma, D., and Leonard, J., (1979), Slack-Elasto-Plastic Dynamics of Cable Systems, *ASCE Journal of the Engineering Mechanics Division*, Vol. 105 No. EM2.
[8] Chisăliță, A., (2007), *Program DINSAS – Manual de utilizare*, Universitatea Tehnică din Cluj Napoca, (Romanian)