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The Upper and Lower Approximations in Rough Subgroupoid of a Groupoid

H. Tasbozan [1,a] and I. Icen [2,b]

Abstract. In this article, we introduce the concept of rough subgroupoid of a groupoid as a generalization of a rough subgroup and give some features about the lower and the upper approximations in a groupoid [1]. We give some of the characterization of them. **Key words and phrases.** rough set, rough groupoid, rough subgroupoid.

1. Introduction

The rough set is not only concerned with uncertainty. It is also effective in the calculation methods of soft clusters. These areas include artificial intelligence, data mining, pattern recognition, decision analysis and fault diagnosis. The concept of a rough set was introduced by Pawlak in [2]. The algebraic approach of rough set was studied by some authors [1, 3, 4, 5, 6, 7, 8, 9, 10]. Recently, the notion of rough group, rough subgroup and some properties were studied [1]. Groupoid was introduced by Brandt [11] on the composition of quadratic forms with four variables. Grothendick [12] used groupoid for the construction of module space. Also groupoids play an essential role in physics and mathematics as moduli space. In algebraic topology, the fundamental groupoid of a topological space has been exploited by R.Brown and other [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24].

Groupoid have been used in a wide different area of mathematic such as functional analysis, ergodic theory, algebraic topology, algebraic geometry, differential geometry and group theory.

In this article we define the concept of rough subgroupoid of a groupoid and discuss some features about it.

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[1] *Department of Mathematics, Hatay Mustafa Kemal University, Hatay, TURKEY.*
[2] *Department of Mathematics, Inn University, Malatya, TURKEY.*
[a] *e-mail: htasbozan@mku.edu.tr*
[b] *e-mail: ilhan.icen@inonu.edu.tr* .

2. Preliminaries

Definition 2.1. *Let U be a nonempty set and let R be an equivalence relation on U. Then $S = (U, R)$ is called the approximation space and U is called a universe* [2].

Definition 2.2. *Let $S = (U, R)$ be an approximation space. Suppose $\emptyset \neq X \subseteq U$. Then the sets*

$$\begin{aligned} \overline{R}(X) &= \{x \mid [x]_R \cap X \neq \emptyset\} \\ \underline{R}(X) &= \{x \mid [x]_R \subseteq X\} \end{aligned}$$

are called respectively lower and upper approximations of the set X in the approximation space S, where $[x]_R$ denotes the equivalence class of the relation R containing x. The set $R(X) = (\underline{R}(X), \overline{R}(X))$ is called the rough set of X in S. The set $\overline{R}(X) - \underline{R}(X)$ is called the boundary region of X. If the boundary region of X is not an empty set, then X is a rough set.

For a fixed approximation space $S = (U, R)$ and for a fixed nonempty subset X of U, the rough set of X, i.e. $R(X)$ is unique [2].

Definition 2.3. *A groupoid G consists of two sets G and G_0 called respectively the set of elements or morphisms and the set of objects of the groupoid, together with two maps $\alpha, \beta : G \to G_0$, called respectively the source and target maps, a map $1_{()} : G_0 \to G$, $x \mapsto 1_x$ called the object map and a partial multiplication*

$$G \times G \to G,\ (a, b) \mapsto a \circ b$$

defined on the fibre product set

$$G \times G = \{(a, b) \in G \times G :\ \alpha(b) = \beta(a)\}.$$

These maps are subject to the following conditions:

(1) $\alpha(b \circ a) = \alpha(a)$ and $\beta(b \circ a) = \beta(b)$ for all $a, b \in G$;
(2) $c \circ (b \circ a) = (c \circ b) \circ a$ for all $a, b, c \in G$ such that $\alpha(c) = \beta(b)$ and $\alpha(b) = \beta(a)$;
(3) $\alpha(1_x) = \beta(1_x) = x$ for all $x \in G_0$, where 1_x is the identity at x;
(4) $a \circ 1_{\alpha(a)} = a$ and $1_{\beta(a)} \circ a = a$ for all $a \in G$;
(5) each $a \in G$ has an inverse a^{-1}such that $\alpha(a^{-1}) = \beta(a)$, $\alpha(a) = \beta(a^{-1})$ and $a^{-1} \circ a = 1_{\alpha(a)}$,$a \circ a^{-1} = 1_{\beta(a)}$

If (G, G_0) is a groupoid then we say G is a groupoid on G_0 [13].

Definition 2.4. *Let G be a groupoid on G_0. A subgroupoid H of G is a pair of subsets $H \subseteq G$ and $H_0 \subseteq G_0$ such that $\alpha(H) \subseteq H_0, \beta(H) \subseteq H_0$, $1_x \in H$ for $x \in H_0$ and H is closed under the partial multiplication and inversion in G* [13].

Definition 2.5. *A normal subgroupoid N of G is subgroupoid N of G such that $N_0 = G_0$ and for each $x, y, z \in G_0, a \in G(z, y)$, $b \in G(x, z)$, $a \circ b \in G(x, z)$ and for $\alpha(a) = \beta(b)$ we have $a \circ N(z) = N(y) \circ a$ and $(a \circ N(z)) \circ (b \circ N(x)) = a \circ b \circ N(x)$. In other words, a subgroupoid N of G is called normal if $N_0 = G_0$ and for $x, y \in G_0$ and $a \circ N(z) \circ a^{-1} = N(y)$* [13, 25].

3. Lower and Upper Approximations in a Groupoid

Definition 3.1. *Let G be a groupoid, N be a normal subgroupoid of G and $N_x : x \to x$ be a morphism. Let A be a nonempty subset of G and A_0 is the set of end points of the morphisms in A. Then the sets*

$$\begin{aligned} \underline{N_x}(A) &= \{a \in G \mid a \circ N_x \subseteq A, \alpha(a), \beta(a) \in A_0, x \in G_0\}, \\ \underline{N}(A) &= \bigcup_{x \in G_0} \underline{N_x}(A) \end{aligned}$$

and

$$\begin{aligned} \overline{N_x}(A) &= \{a \in G \mid a \circ N_x \cap A \neq \emptyset, \alpha(a), \beta(a) \in A_0, x \in G_0\}, \\ \overline{N}(A) &= \bigcup_{x \in G_0} \overline{N_x}(A) \end{aligned}$$

are called respectively, lower and upper approximations of a set A with respect to the normal subgroupoid N of G. Then we say that the pair of $N(A) = \left(\underset{x \in G_0}{\cup} \underline{N_x}(A), \underset{x \in G_0}{\cup} \overline{N_x}(A) \right)$ is a rough set of A in G.

Definition 3.2. *Let $N(A) = (\underline{N}(A), \overline{N}(A))$ be a rough set of A in G. A nonempty subset A of a groupoid G is called an $\overline{N}$ **rough(normal) subgroupoid** of G if the upper approximation $\underset{x \in G_0}{\cup} \overline{N}(A)$ of A is a (normal)subgroupoid of G. Similarly a nonempty subset A of G is called an $\underline{N}$ **rough (normal) subgroupoid** of G if the lower approximation $\underset{x \in G_0}{\cup} \underline{N}(A)$ of A is a (normal)subgroupoid of G.*

Example 3.1. *$G = \{1_0, a, a^{-1}, 1_1\}$ is a groupoid with $a : 0 \to 1, a^{-1} : 1 \to 0$ and $N = \{1_0, 1_1\}$ is a normal subgroupoid of G. Let us take $N_0 = G_0$, N_0 is the set of the morphism with source objects of G and the set $A = \{a\} \subseteq G$. In this case, we have $a \circ N_0 = \{a\} \subseteq A$ and $a^{-1} \circ N_0 = \{a^{-1}\}$. Then, we have*

$$\underline{N_0}(A) = \{a \in G \mid a \circ N_0 \subseteq A\} = A,$$
$$\overline{N_0}(A) = \{a \in G \mid a \circ N_0 \cap A \neq \emptyset\} = A$$

and also

$$\underline{N}(A) = \underset{x \in G_0}{\cup} \underline{N_x}(A) = A,$$
$$\overline{N}(A) = \underset{x \in G_0}{\cup} \overline{N_x}(A) = A.$$

Because of $\overline{N}(A) - \underline{N}(A) = \emptyset$, A is not a $\overline{N}$ rough subgroupoid of G and a $\underline{N}$ rough subgroupoid of G.

Example 3.2. *Let $X = \{x, y, z\}$ be a set. Then there is a groupoid with object set X and set of arrows the product set $X \times X$ so that an arrow $x \to y$ is simply the ordered pair (y, x). The composition is then given by $(z, y) \circ (y, x) = (z, x)$. Then $R = X \times X$ is an equivalence relation on X. R is a groupoid with the composition* [14].

$$R = \{(x,x), (y,y), (z,z), (x,y), (y,x), (x,z), (z,x), (y,z), (z,y)\}$$

is a groupoid,

$$N = \{(x,x), (y,y), (z,z)\}$$

is a normal subgroupoid of N and let

$$A = \{(x,x), (y,y), (y,x), (z,x), (y,z)\} \subset R$$

be a set. We have

$$\begin{aligned} a \circ N_x &= \{(x,x), (x,y), (x,z)\}, \\ a \circ N_y &= \{(y,y), (y,x), (y,z)\}, \\ a \circ N_z &= \{(z,z), (z,x), (z,y)\}, \end{aligned}$$

and thus from the Definition 6, we obtain

$$\begin{aligned} \underline{N_x}(A) &= \emptyset, \\ \underline{N_x}(A) &= \{a \in G \mid a \circ N_y \subseteq A\} = \{(y,y), (y,x), (y,z)\}, \\ \underline{N_z}(A) &= \emptyset, \\ \underline{N}(A) &= \underline{N_x}(A) \cup \underline{N_y}(A) \cup \underline{N_z}(A) = \{(y,y), (y,x), (y,z)\}, \\ \overline{N}(A) &= \overline{N_x}(A) \cup \overline{N_y}(A) \cup \overline{N_z}(A) = R. \end{aligned}$$

A is a $\overline{N}$ rough subgroupoid of G and $\underline{N}$ rough subgroupoid of G.

Proposition 3.1. *Let N and H be normal subgroupoids of a groupoid G. Let A and B be any nonempty subsets of G. Then*

(1) $\underset{x \in G_0}{\cup} \underline{N_x}(A) \subseteq A \subseteq \underset{x \in G_0}{\cup} \overline{N_x}(A)$.

(2) $\underset{x \in G_0}{\cup} \overline{N_x}(A \cup B) = \left(\underset{x \in G_0}{\cup} \overline{N_x}(A) \right) \cup \left(\underset{x \in G_0}{\cup} \overline{N_x}(B) \right)$.

(3) $\underset{x\in G_0}{\cup}\underline{N_x}(A\cap B)=\left(\underset{x\in G_0}{\cup}\underline{N_x}(A)\right)\cap\left(\underset{x\in G_0}{\cup}\underline{N_x}(B)\right)$.
(4) $A\subseteq B$ implies $\underset{x\in G_0}{\cup}\underline{N_x}(A)\subseteq\underset{x\in G_0}{\cup}\underline{N_x}(B)$.
(5) $A\subseteq B$ implies $\underset{x\in G_0}{\cup}\overline{N_x}(A)\subseteq\underset{x\in G_0}{\cup}\overline{N_x}(B)$.
(6) $\underset{x\in G_0}{\cup}\underline{N_x}(A\cup B)\supseteq\left(\underset{x\in G_0}{\cup}\underline{N_x}(A)\right)\cup\left(\underset{x\in G_0}{\cup}\underline{N_x}(B)\right)$.
(7) $\underset{x\in G_0}{\cup}\overline{N_x}(A\cap B)\subseteq\left(\underset{x\in G_0}{\cup}\overline{N_x}(A)\right)\cap\left(\underset{x\in G_0}{\cup}\overline{N_x}(B)\right)$.
(8) $N\subseteq H$ implies $\underset{x\in G_0}{\cup}\underline{N_x}(A)\subseteq\underset{x\in G_0}{\cup}\underline{H_x}(A)$.

Proof. (1) If $\forall a\in\underset{x\in G_0}{\cup}\underline{N_x}(A)$, $\exists x\in G_0$, then $a=a\circ 1_x\in a\circ N_x\subseteq A$. Thus $\underline{N}(A)\subseteq A$. Next, if $\forall a\in A$, then $a=a\circ 1_x\in a\circ N_x$. Thus $a\in a\circ N_x\cap A$, that is, $\forall x\in G_0$, $a\circ N_x\cap A\neq\emptyset$. This implies $a\in\underset{x\in G_0}{\cup}\overline{N_x}(A)$, and so $A\subseteq\underset{x\in G_0}{\cup}\overline{N_x}(A)$.

(2) $a\in\underset{x\in G_0}{\cup}\overline{N_x}(A\cup B)\Leftrightarrow a\circ N_x\cap\ (A\cup B)\neq\emptyset,\forall x\in G_0$

$$\Leftrightarrow (a\circ N_x\cap A)\cup(a\circ N_x\cap B)\neq\emptyset,\ \forall x\in G_0$$
$$\Leftrightarrow (a\circ N_x\cap A)\neq\emptyset \text{ or } (a\circ N_x\cap B)\neq\emptyset$$
$$\Leftrightarrow a\in\underset{x\in G_0}{\cup}\overline{N_x}(A) \text{ or } a\in\underset{x\in G_0}{\cup}\overline{N_x}(B)$$
$$\Leftrightarrow a\in\left(\underset{x\in G_0}{\cup}\overline{N_x}(A)\right)\cup\left(\underset{x\in G_0}{\cup}\overline{N_x}(B)\right)$$

Thus $\underset{x\in G_0}{\cup}\overline{N_x}(A\cup B)=\left(\underset{x\in G_0}{\cup}\overline{N_x}(A)\right)\cup\left(\underset{x\in G_0}{\cup}\overline{N_x}(B)\right)$.

(3) $a\in\underset{x\in G_0}{\cup}\underline{N_x}(A\cap B)\Leftrightarrow a\circ N_x\subseteq A\cap B,\ \forall x\in G_0$

$$\Leftrightarrow a\circ N_x\subseteq A \text{ and } a\circ N_x\subseteq B$$
$$\Leftrightarrow a\in\underset{x\in G_0}{\cup}\underline{N_x}(A) \text{ and } a\in\underset{x\in G_0}{\cup}\underline{N_x}(B)$$
$$\Leftrightarrow a\in\left(\underset{x\in G_0}{\cup}\underline{N_x}(A)\right)\cap\left(\underset{x\in G_0}{\cup}\underline{N_x}(B)\right)$$

Thus $\underset{x\in G_0}{\cup}\underline{N_x}(A\cap B)=\left(\underset{x\in G_0}{\cup}\underline{N_x}(A)\right)\cap\left(\underset{x\in G_0}{\cup}\underline{N_x}(B)\right)$.

(4) Since $A\subseteq B$, $A\cap B=A$. With the help of (3),

$$\underset{x\in G_0}{\cup}\underline{N_x}(A)=\underset{x\in G_0}{\cup}\underline{N_x}(A\cap B)=\left(\underset{x\in G_0}{\cup}\underline{N_x}(A)\right)\cap\left(\underset{x\in G_0}{\cup}\underline{N_x}(B)\right)$$

gives us $\underset{x\in G_0}{\cup}\underline{N_x}(A)\subseteq\underset{x\in G_0}{\cup}\underline{N_x}(B)$.

(5) Since $A\subseteq B$, $AUB=B$. With the help of (2),

$$\underset{x\in G_0}{\cup}\overline{N_x}(A)=\underset{x\in G_0}{\cup}\overline{N_x}(A\cup B)=\left(\underset{x\in G_0}{\cup}\overline{N_x}(A)\right)\cup\left(\underset{x\in G_0}{\cup}\overline{N_x}(A)(B)\right)$$

gives us $\underset{x\in G_0}{\cup}\overline{N_x}(A)\subseteq\underset{x\in G_0}{\cup}\overline{N_x}(B)$.

(6) $A\subseteq A\cup B$, $B\subseteq A\cup B$ and with the help of (4), $\underset{x\in G_0}{\cup}\underline{N_x}(A)\subseteq\underset{x\in G_0}{\cup}\underline{N_x}(A\cup B)$ and $\underset{x\in G_0}{\cup}\underline{N_x}(B)\subseteq\underset{x\in G_0}{\cup}\underline{N_x}(A\cup B)$. Therefore

$$\left(\underset{x\in G_0}{\cup}\underline{N_x}(A)\right)\cup\left(\underset{x\in G_0}{\cup}\underline{N_x}(B)\right)\subseteq\underset{x\in G_0}{\cup}\underline{N_x}(A\cup B).$$

(7) $A \cap B \subseteq A$, $A \cap B \subseteq B$ and with the help of (5), $\underset{x\in G_0}{\cup}\overline{N_x}(A\cap B) \subseteq \underset{x\in G_0}{\cup}\overline{N_x}(A)$ and $\underset{x\in G_0}{\cup}\overline{N_x}(A\cap B) \subseteq \underset{x\in G_0}{\cup}\overline{N_x}(B)$. Therefore

$$\underset{x\in G_0}{\cup}\overline{N_x}(A\cap B) \subseteq \left(\underset{x\in G_0}{\cup}\overline{N_x}(A)\right) \cap \left(\underset{x\in G_0}{\cup}\overline{N_x}(B)\right).$$

(8) If $\forall c \in \underset{x\in G_0}{\cup}\overline{N_x}(A)$, then $\forall x \in G_0$, $\exists x \in c \circ N_x \cap A$, and so $x \in c \circ N_x \subseteq c \circ H_x$. Therefore, $x \in c \circ H_x \cap A$. This gives us $c \in \underset{x\in G_0}{\cup}\overline{H_x}(A)$, and $\underset{x\in G_0}{\cup}\overline{N_x}(A) \subseteq \underset{x\in G_0}{\cup}\overline{H_x}(A)$.

Proposition 3.2. *Let N be a normal subgroupoid of a groupoid G. Let $\emptyset \neq A \subseteq G$ and $\emptyset \neq B \subseteq G$ are subgroupoids of G. Then $\left(\underset{x\in G_0}{\cup}\overline{N_x}(A)\right) \circ \left(\underset{x\in G_0}{\cup}\overline{N_x}(B)\right) = \underset{x\in G_0}{\cup}\overline{N_x}(A\circ B)$.*

Proof. Let N be a normal subgroupoid of a groupoid G such that $N_0 = G_0$ and for each $x, y, z \in G_0, a \in G(z, y), b \in G(x, z)$, $a \circ b \in G(x, z)$ and for $\alpha(b) = \beta(a)$ we have $(a \circ N(z)) \circ (b \circ N(x)) = a \circ b \circ N(x)$. Let $c \in \underset{x\in G_0}{\cup}\overline{N_x}(A\circ B)$. Then $\forall x \in G_0$, $c \circ N_x \cap A \circ B \neq \emptyset$. Thus there exists an element m in G such that $m \in c \circ N_x \cap A \circ B$, and so $m \in c \circ N_x$ and $m \in c \circ A \circ B$. Then $m = a \circ b$ with $a \in A$ and $b \in B$. Since $c \in m \circ N_x = (a \circ b)\circ N_x = (a \circ N_z) \circ (b \circ N_x)$, we have $c = k \circ l$ with $k \in a \circ N_z$ and $l \in b \circ N_x$. Then $a \in k \circ N_z$, and so $a \in k \circ N_z \cap A$. Thus $k \in \underset{x\in G_0}{\cup}\overline{N_x}(A)$. We have same $l \in \underset{x\in G_0}{\cup}\overline{N_x}(B)$. Then $c = k \circ l \in \left(\underset{x\in G_0}{\cup}\overline{N_x}(A)\right) \circ \left(\underset{x\in G_0}{\cup}\overline{N_x}(B)\right)$, gives $\underset{x\in G_0}{\cup}\overline{N_x}(A\circ B) \subseteq \left(\underset{x\in G_0}{\cup}\overline{N_x}(A)\right) \circ \left(\underset{x\in G_0}{\cup}\overline{N_x}(B)\right)$. Contrarily, let $c \in \left(\underset{x\in G_0}{\cup}\overline{N_x}(A)\right) \circ \left(\underset{x\in G_0}{\cup}\overline{N_x}(B)\right)$, then $c = a \circ b$ with $a \in \underset{x\in G_0}{\cup}\overline{N_x}(A)$ and $b \in \underset{x\in G_0}{\cup}\overline{N_x}(A)$. Then there consists $r,s \in G$ with $r \in a \circ N_z \cap A$ and $s \in b \circ N_x \cap B$, and so $r \in a \circ N_z$, $r \in A$, $s \in b \circ N_x$, $s \in B$. Since N is normal, $r \circ s \in (a \circ N_z)(b \circ N_x) = a \circ b \circ N_x$, and $r \circ s \in A \circ B$. Thus $r \circ s \in a \circ b \circ N_x \cap A \circ B$, which yields that $c = a \circ b \in \underset{x\in G_0}{\cup}\overline{N_x}(A\circ B)$, and so $\left(\underset{x\in G_0}{\cup}\overline{N_x}(A)\right) \circ \left(\underset{x\in G_0}{\cup}\overline{N_x}(A)\right) \subseteq \underset{x\in G_0}{\cup}\overline{N_x}(A\circ B)$. For this reason we have $\left(\underset{x\in G_0}{\cup}\overline{N_x}(A)\right) \circ \left(\underset{x\in G_0}{\cup}\overline{N_x}(B)\right) = \underset{x\in G_0}{\cup}\overline{N_x}(A\circ B)$.

Proposition 3.3. *Let N be a normal subgroupoid of a groupoid G, $\emptyset \neq A \subseteq G$ and $\emptyset \neq B \subseteq G$ are subgroupoids of G. Then $\left(\underset{x\in G_0}{\cup}\underline{N_x}(A)\right) \circ \left(\underset{x\in G_0}{\cup}\underline{N_x}(B)\right) \subseteq \underset{x\in G_0}{\cup}\underline{N_x}(A\circ B)$.*

Proof. Let N be a normal subgroupoid of a groupoid G such that $N_0 = G_0$ and for each $x, y, z \in G_0, a \in G(z, y), b \in G(x, z)$, $a \circ b \in G(x, z)$ and for $\alpha(b) = \beta(a)$ we have $(a \circ N(z)) \circ (b \circ N(x)) = a \circ b \circ N(x)$. Let $c \in \left(\underset{x\in G_0}{\cup}\underline{N_x}(A)\right) \circ \left(\underset{x\in G_0}{\cup}\underline{N_x}(B)\right)$. Then $a \in \underset{x\in G_0}{\cup}\underline{N_x}(A)$ and $b \in \underset{x\in G_0}{\cup}\underline{N_x}(B)$ gives $c = a \circ b$. Then $a \circ N_z \subseteq A$ and $b \circ N_x \subseteq B$. Since N is normal, $c \circ N_x = a \circ b \circ N_x = (a \circ N_z) \circ (b \circ N_x) \subseteq A \circ B$, then $c \in \underset{x\in G_0}{\cup}\underline{N_x}(A\circ B)$. Therefore $\left(\underset{x\in G_0}{\cup}\underline{N_x}(A)\right) \circ \left(\underset{x\in G_0}{\cup}\underline{N_x}(B)\right) \subseteq \underset{x\in G_0}{\cup}\underline{N_x}(A\circ B)$.

Remark 3.1. *Let H and N be normal subgroupoids of a groupoid G. Clearly $H \cap N$ is also a normal subgroupoid of G.*

Proposition 3.4. *Let H and N be normal subgroupoids of a groupoid G and $\emptyset \neq A \subseteq G$. Then $\underset{x\in G_0}{\cup}\overline{(H\cap N)_x}(A) = \left(\underset{x\in G_0}{\cup}\overline{H_x}(A)\right) \cap \left(\underset{x\in G_0}{\cup}\overline{N_x}(A)\right)$.*

Proof. Let $c \in \bigcup_{x \in G_0} \overline{(H \cap N)_x}(A)$. Then $c \circ (H \cap N)_x \cap A \neq \emptyset$. This gives $a \in c \circ (H \cap N)_x \cap A$, $\forall x \in G_0$. Since $a \in c \circ (H \cap N)_x$ and $a \in A$, then $a \in c \circ H_x$, $a \in A$ and $a \in c \circ N_x$, $a \in A$, $\forall x \in G_0$.Thus $a \in c \circ H_x \cap A$ and $\exists a \in c \circ N_x \cap A$, $\forall x \in G_0$.Therefore $c \in \bigcup_{x \in G_0} \overline{H_x}(A)$ and $c \in \bigcup_{x \in G_0} \overline{N_x}(A)$.

Thus $\bigcup_{x \in G_0} \overline{(H \cap N)_x}(A) = \left(\bigcup_{x \in G_0} \overline{H_x}(A) \right) \cap \left(\bigcup_{x \in G_0} \overline{N_x}(A) \right)$.

Proposition 3.5. *Let H and N be normal subgroupoids of a groupoid G. Let A be a nonempty subset of G. Then $\bigcup_{x \in G_0} \underline{(H \cap N)_x}(A) = \left(\bigcup_{x \in G_0} \underline{H_x}(A) \right) \cap \left(\bigcup_{x \in G_0} \underline{N_x}(A) \right)$.*

Proof.
$$\begin{aligned} \forall c \in \bigcup_{x \in G_0} \underline{(H \cap N)_x}(A) &\Leftrightarrow c \circ (H \cap N)_x \subseteq A \\ &\Leftrightarrow c \circ H_x \subseteq A \text{ and } c \circ N_x \subseteq A, \forall x \in G_0 \\ &\Leftrightarrow c \in \bigcup_{x \in G_0} \underline{H_x}(A) \text{ and } c \in \bigcup_{x \in G_0} \underline{N_x}(A) \\ &\Leftrightarrow c \in \left(\bigcup_{x \in G_0} \underline{H_x}(A) \right) \cap \left(\bigcup_{x \in G_0} \underline{N_x}(A) \right) \end{aligned}$$

Thus $\bigcup_{x \in G_0} \underline{(H \cap N)_x}(A) = \left(\bigcup_{x \in G_0} \underline{H_x}(A) \right) \cap \left(\bigcup_{x \in G_0} \underline{N_x}(A) \right)$.

Proposition 3.6. *Let N be a normal subgroupoid of a groupoid G. If A is a subgroupoid of G, then it is an $\bigcup_{x \in G_0} \overline{N_x}$ rough subgroupoid of G.*

Proof. Let N be a normal subgroupoid of a groupoid G such that $N_0 = G_0$ and for each $x, y, z \in G_0, a \in G(z, y)$, $b \in G(x, z)$, $a \circ b \in G(x, z)$ and for $\alpha(b) = \beta(a)$ we have $(a \circ N(z)) \circ (b \circ N(x)) = a \circ b \circ N(x)$. Let 1_x be the identity of G. Since N and A are subgroups of G, $1_x \in$ A and $1_x = 1_x \circ\ 1_x \in 1_x \circ N_x$, and so $1_x \in 1_x \circ N_x \cap A$. Thus $1_x \circ N_x \cap A \neq \emptyset$. This implies that $1_x \in \bigcup_{x \in G_0} \overline{N_x}(A)$, $\forall x \in G_0$. Let a , $b \in \bigcup_{x \in G_0} \overline{N_x}(A)$. Then k , $l \in G$ so $k \in a \circ N_z \cap A$ and $l \in b \circ N_x \cap A$. Thus $k \in a \circ N_z$, $l \in b \circ N_x$ and $k \in A$, $l \in A$. Since A is a subgroupoid of G, $k \circ l \in A$. And since N is a normal subgroupoid of G, $k \circ l \in (a \circ N_z) \circ (b \circ N_x) = a \circ b \circ N_x$. Thus $k \circ l \in a \circ b \circ N_x \cap A$, and so $a \circ b \in \bigcup_{x \in G_0} \overline{N_x}(A)$. Let a be any element of $\bigcup_{x \in G_0} \overline{N_x}(A)$. Then $k \in a \circ N_z \cap A$ for some $k \in G$, that is, $k \in a \circ N_z$ and $k \in A$. Then since A is a subgroupoid of G, $k^{-1} \in A$. On the other hand, since $k = a \circ h$ for some $h \in N_z$, and since N is a normal subgroupoid of G, $h^{-1} \in N_z$ and it implies that $k^{-1} = (a \circ h)^{-1} = h^{-1} \circ a^{-1} \in N_z \circ a^{-1} = a^{-1} \circ N_y$. Thus $k^{-1} \in a^{-1} \circ N_y \cap A$, and so $a^{-1} \in \bigcup_{x \in G_0} \overline{N_x}(A)$. This show that $\bigcup_{x \in G_0} \overline{N_x}(A)$ is a subgroupoid of G.

Proposition 3.7. *Let N be a normal subgroupoid of a groupoid G. If A is a normal subgroupoid of G, then it is an $\bigcup_{x \in G_0} \overline{N_x}$ rough normal subgroupoid of G.*

Proof. We must indicate $\bigcup_{x \in G_0} \overline{N_x}(A)$ is normal. Let $a \in G(x, x) \in \bigcup_{x \in G_0} \overline{N_x}(A)$ and $k \in G(x, y) \in G$ for each $x, y \in G_0$. Then $l \in G$ such that $l \in a \circ N_x \cap A$, that is, $l \in a \circ N_x$ and $l \in A$. Since N is normal, $k \circ l \circ k^{-1}$ $\in k \circ (a \circ N_x) \circ k^{-1} = (k \circ a) \circ (N \circ k^{-1}) = (k \circ a) \circ (k^{-1} \circ N_y) = (k \circ a \circ k^{-1}) \circ N_y$. Since A is normal, $k \circ l \circ k^{-1} \in kAk^{-1} \subseteq A$. Thus $k \circ l \circ k^{-1} \in (k \circ a \circ k^{-1}) \circ N_y \cap A$, and so $k \circ a \circ k^{-1} \in \bigcup_{x \in G_0} \overline{N_x}(A)$. This implies that $\bigcup_{x \in G_0} \overline{N_x}(A)$ is normal.

Proposition 3.8. *Let N be a normal subgroupoid of a groupoid G. If A is a subgroupoid of G which $N \subseteq A$, then it is an $\bigcup_{x \in G_0} \underline{N_x}$ rough subgroupoid of G.*

Proof. Let N be a normal subgroupoid of a groupoid G such that $N_0 = G_0$ and for each $x, y, z \in G_0, a \in G(z, y)$,$b \in G(x, z)$, $a \circ b \in G(x, z)$ and for $\alpha(b) = \beta(a)$ we have $(a \circ N(z)) \circ (b \circ N(x)) = a \circ b \circ N(x)$. Since A and

N are subgroupoids of G, $1_x \in A$, $1_x \in N_x$. Since $1_z \circ N_z = N_z \subseteq A$, $1_z \in \underset{x\in G_0}{\cup} \underline{N_x}(A)$ and $1_x \circ N_x = N_x \subseteq A$, then $1_z \in \underline{N}(A)$ and $1_x \in \underset{x\in G_0}{\cup} \underline{N_x}(A)$. Let a, $b \in \underset{x\in G_0}{\cup} \underline{N_x}(A)$. Then $a \circ N_z \subseteq A$ and $b \circ N_x \subseteq A$. Since$N$ is a normal subgroupoid and A is a subgroupoid of G, $a \circ b \circ N_x = (a \circ N_z) \circ (b \circ N_x) \subseteq A \circ A \subseteq A$. So that $a \circ b \in \underset{x\in G_0}{\cup} \underline{N_x}(A)$. Let $a \in \underset{x\in G_0}{\cup} \underline{N_x}(A)$. Then $a = a \circ 1_z \in a \circ N_z \subseteq A$. Since A is a subgroupoid of G, $a^{-1} \in A$. Thus we have $a^{-1} \circ N_y \subseteq A \circ A \subseteq A$. This implies that $a^{-1} \in \underset{x\in G_0}{\cup} \underline{N_x}(A)$. Therefore, $\underset{x\in G_0}{\cup} \underline{N_x}(A)$ is a subgroupoid of G.

Proposition 3.9. *Let N be a normal subgroupoid of a groupoid G. If A is a normal subgroupoid of G such that $N \subseteq A$, then it is an $\underset{x\in G_0}{\cup} \underline{N_x}(A)$ rough normal subgroupoid of G.*

Proof. We must indicate $\underset{x\in G_0}{\cup} \underline{N_x}(A)$ is normal. Let $a \in G(x,x) \underset{x\in G_0}{\cup} \underline{N_x}(A)$ and $k \in G(x,y)$ for each $x, y \in G_0$. Then, $a \circ N_x \subseteq A$. Since N and A are normal,

$$(k \circ a \circ k)^{-l} \circ N_x = k \circ (a \circ N_x) \circ k^{-1} \subseteq k \circ A \circ k^{-1} \subseteq A$$

and so $k \circ A \circ k^{-1} \in \underset{x\in G_0}{\cup} \underline{N_x}(A)$. Thus $\underset{x\in G_0}{\cup} \underline{N_x}(A)$ is normal.

Proposition 3.10. *Let H and N be normal subgroupoids of a groupoid G. If A is a subgroupoid of G, then $\left(\underset{x\in G_0}{\cup} \overline{H_x}(A)\right) \circ \left(\underset{x\in G_0}{\cup} \overline{N_x}(A)\right) \subseteq \underset{x\in G_0}{\cup} \overline{(H \circ N)_x}(A)$.*

Proof. Let N be a normal subgroupoid of a groupoid G such that $N_0 = G_0$ and for each $x, y, z \in G_0$, $a \in G(z, y)$, $b \in G(x, z)$, $a \circ b \in G(x, z)$ and for $\alpha(b) = \beta(a)$ we have $(a \circ N(z)) \circ (b \circ N(x)) = a \circ b \circ N(x)$. Let $c \in \underset{x\in G_0}{\cup} \overline{H_x}(A) \circ \underset{x\in G_0}{\cup} \overline{N_x}(A)$. Then $c = a \circ b$ with $a \in \underset{x\in G_0}{\cup} \overline{H_x}(A)$ and $b \in \underset{x\in G_0}{\cup} \overline{N_x}(A)$. Then k , $l \in G$ which $k \in a \circ H_z \cap A$ and $l \in b \circ N_x \cap A$. Thus k $\in a \circ H_z$, $l \in b \circ N_x$, $k \in A$, $l \in A$. Then, since H is normal,

$$\begin{aligned} k \circ l &\in (a \circ H_z) \circ (b \circ N_x) = \{a \circ (H_z \circ b) \circ N_x\} = \{a \circ (b \circ H_x) \circ N_x\} = \{(a \circ b) \circ H_x\} \circ N_x \\ &= (a \circ b) \circ H_x \circ N_x = c \circ H_x \circ N_x. \end{aligned}$$

Since A is a subgroupoid of G, $k \circ l \in A$. Therefore, $k \circ l \in c \circ H_x \circ N_x \cap A$, and so we have $c \in \underset{x\in G_0}{\cup} \overline{(H \circ N)_x}(A)$. Thus $\left(\underset{x\in G_0}{\cup} \overline{H_x}(A)\right) \circ \left(\underset{x\in G_0}{\cup} \overline{N_x}(A)\right) \subseteq \underset{x\in G_0}{\cup} \overline{(H \circ N)_x}(A)$. This completes the proof.

Proposition 3.11. *Let H and N be normal subgroupoids of a groupoid G. If A is a subgroupoid of G, then $\left(\underset{x\in G_0}{\cup} \underline{H_x}(A)\right) \circ \left(\underset{x\in G_0}{\cup} \underline{N_x}(A)\right) \subseteq \underset{x\in G_0}{\cup} \underline{(H \circ N)_x}(A)$.*

Proof. Let N be a normal subgroupoid of a groupoid G such that $N_0 = G_0$ and for each $x, y, z \in G_0$, $a \in G(z, y)$, $b \in G(x, z)$, $a \circ b \in G(x, z)$ and for $\alpha(b) = \beta(a)$ we have $(a \circ N(z)) \circ (b \circ N(x)) = a \circ b \circ N(x)$. Let $c \in \underset{x\in G_0}{\cup} \underline{H_x}(A) \circ \underset{x\in G_0}{\cup} \underline{N_x}(A)$. Then $c = a \circ b$ with $a \in \underset{x\in G_0}{\cup} \underline{H_x}(A)$ and $b \in \underset{x\in G_0}{\cup} \underline{N_x}(A)$. Thus $a \circ H_z \subseteq A$ and $b \circ N_x \subseteq A$. Since H is a normal subgroupoid and $A$is a subgroupoid of G, we have $(a \circ b) \circ H_x \circ N_x = \{a \circ (b \circ H_x)\} \circ N_x = \{a \circ (H_z \circ b)\} \circ N_x = \{(a \circ H_z) \circ b\} \circ N_x = (a \circ H_z) \circ (b \circ N_x) \subseteq A \circ A \subseteq A$. Therefore, we have $c = a \circ b \in \underset{x\in G_0}{\cup} \underline{(H \circ N)_x}(A)$, and so $\left(\underset{x\in G_0}{\cup} \underline{H_x}(A)\right) \circ \left(\underset{x\in G_0}{\cup} \underline{N_x}(A)\right) \subseteq \underset{x\in G_0}{\cup} \underline{(H \circ N)_x}(A)$, which completes the proof.

4. Conclusion

In this article, we introduced some features of a rough subgroupoid of a groupoid. Let G be a groupoid, N be a normal subgroupoid of G. Let $\emptyset \neq A \subseteq G$. The sets $\underset{x\in G_0}{\cup}\underline{N_x}(A)$, $\underset{x\in G_0}{\cup}\overline{N_x}(A)$ are called lower and upper approximations of A respectively. $N(A)=\left(\underset{x\in G_0}{\cup}\underline{N_x}(A), \underset{x\in G_0}{\cup}\overline{N_x}(A)\right)$ is called a rough set of A in G.A nonempty subset A of a groupoid G is called an $\underset{x\in G_0}{\cup}\overline{N_x}$ **rough(normal) subgroupoid** of G if the upper approximation $\underset{x\in G_0}{\cup}\overline{N_x}(A)$ of A is a (normal)subgroupoid of G. Similarly a nonempty subset A of G is called an $\underset{x\in G_0}{\cup}\underline{N_x}$ **rough (normal) subgroupoid** of G if $\underset{x\in G_0}{\cup}\underline{N_x}(A)$ is a (normal)subgroupoid of G.

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