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Obstacle parabolic equations in non-reflexive Musielak-Orlicz spaces

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Abstract. We prove existence of entropy solutions to general class of unilateral nonlinear parabolic equation in inhomogeneous Musielak-Orlicz spaces avoiding ceorcivity restrictions on the second lower order term. Namely, we consider

$$\begin{cases} u \geq \psi \quad \text{in} \quad Q_T, \\ \frac{\partial b(x,u)}{\partial t} - div(a(x,t,u,\nabla u)) = f + div(g(x,t,u)) \in L^1(Q_T). \end{cases} \tag{0.1}$$

The growths of the monotone vector field $a(x,t,u,\nabla u)$ and the non-coercive vector field $g(x,t,u)$ are controlled by a generalized nonhomogeneous N- function M (see (3.3)-(3.6)). The approach does not require any particular type of growth of M (neither Δ_2 nor ∇_2). The proof is based on penalization method.

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1. Introduction

Generalized Orlicz-spaces $L_{M(.)}$ have been studied since the 1940's. A major synthesis of functional analysis in these spaces is given in the 1983-monograph of Musielak [16], hence the alternative name Musielak-Orlicz spaces. These spaces are similar to Orlicz spaces, but defined by a more general function $M(x,t)$ which may vary with the location in space.

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Let Ω be a bounded open set of $\mathbb{R}^N$ $(N \geq 2)$, T is a positive real number and $Q_T = \Omega \times (0,T)$. Consider the following nonlinear Dirichlet equation:

$$\begin{cases} u \geq \psi & \text{in} \quad Q_T, \\ \frac{\partial b(x,u)}{\partial t} + A(u) - div(g(x,t,u)) = f & \text{in} \quad Q_T, \\ u(x,t) = 0 & on \quad \partial\Omega \times (0,T), \\ b(x,u)(t=0) = b(x,u_0) & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where $A(u) = -div(a(x,t,u,\nabla u))$ is a Leary-Lions operator defined on the inhomogeneous Musielak-Orlicz-Sobolev space $W_0^{1,x}L_M(Q_T)$, M is a Musielak-Orlicz function related to the growths of the Carathéodory functions $a(x,t,u,\nabla u)$ and $g(x,t,u)$ (see assumptions (3.3)-(3.6)), $b : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for every $x \in \Omega$, $b(x,.)$ is a strictly increasing $\mathcal{C}^1(\mathbb{R})$-function, the data f and $b(.,u_0)$ in $L^1(Q_T)$ and $L^1(\Omega)$ respectively and $u_0 \geq \psi$ with ψ is a measurable function with values in $\mathbb{R}$.
The parabolic problems have invaded several fields as well in mathematics, physics as in the economy. Among the first equations appears the transport equation where $b(x,u) = u$ and $g = 0$ and the solution is a fairly regular function. Since these problems have evolved over the last decades by adding other hypotheses and changing the space of functions solutions as needed. Several works dealing with this type of problem (1.1) in Classical Sobolev spaces, in orlicz spaces, lastly in Sobolev spaces with variable exponents and rarely in Musielak-Orlicz spaces.
Starting with the paper [8] where $g = 0$, the existence results have been proved in the framework of Classical Sobolev spaces in ([5], [7], [15]) where $g(x,t,u) = g(u)$ continuous function on u in the Orlicz spaces. For the lower order $g \neq 0$ depending on x,t and u and without coercivity condition, the problem (1.1) was treated firstly in [14] and recently in ([1]), [2], [9]) using the framework of renormalized solutions.
In Musielak spaces Gwiazda et al. in [11], have been proved the renormalized solution where the conjugate of Musielak-Orlicz function satisfies the Δ_2-condition and in [12] where $b(x,u) = u$ and $g = 0$.
The aim of this paper is to generalize [1, 11, 12] and reducing the hypotheses either for the lower nonlinear term g and the framework, i.e. the inhomogeneous space $W^{1,x}L_M(Q_T)$ without $\triangle_2$-condition on M and $\overline{M}$, which introduces some complexity understanding if the dual pairing. The difficulties that arise in problem (1.1) are due to the control of the term $\text{div}(g(x,t,u))$ which depend on x,t and u, lose of coercivity condition and the functional setting in these works involve the Musielak-Orlicz spaces which fail to be reflexive (no more approximation properties of spaces via Mazur's Lemma and Stokes formula) and any regularity on the obstacle.
An example of equations to which the present result can be applied

$$\begin{cases} u \geq \psi & \text{in} \quad Q_T, \\ \dfrac{\partial u}{\partial t} - \Delta_M(u) + uM(x,\nabla u) = f + c(x,t)\overline{M}_x^{-1}M(x, \dfrac{\alpha_0}{\delta}|u|) & \text{in} \quad Q_T, \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,T), \end{cases}$$

where $-\Delta_M(u) = -div(\dfrac{m(x,|\nabla u|)}{|\nabla u|}.\nabla u)$, m is the derivative of M with respect to t and ψ is an admissible obstacle function.
Our approach is to investigate the relationship between the obstacle problem (1.1) and some penalized sequence of approximate Equation (4.7). We study the possibility to find a solution of (1.1) (See Theorem 4.1) as limit of a subsequence u_n of solutions of (4.7).
This paper is organized as follows. In section 2, we recall some definitions, properties and technical Lemmas about Musielak-Orlicz-Sobolev spaces. The section 3 is devoted to specify the assumptions on b, g, f, u_0 and giving the definition of a entropy solution of (1.1) and statement of main results. In section 4, we give the proof of the theorem (4.1).

2. Inhomogeous Musielak-Orlicz space- Notation and properties

Let Ω be a bounded open subset in $\mathbb{R}^N$ $(N \geq 2)$ and let M be a real-valued function defined in $\Omega \times \mathbb{R}_+$ and satisfying the conditions:
(M_1): $M(x,.)$ is an N-function for all $x \in \Omega$,

$(M_2): M(.,t)$ is a measurable function for all $t \geq 0$.
A function M which satisfies the conditions (M_1) and (M_2) is called a Musielak-Orlicz function.
Let $M_x(t) = M(x,t)$, we associate its non-negative reciprocal function M_x^{-1}, with respect to t, that is $M_x^{-1}(M(x,t)) = M(x, M_x^{-1}(t)) = t$.
Let M and P be two Musielak-Orlicz functions, we say that P grows essentially less rapidly than M at 0 (resp. near infinity) and we write $P \prec\prec M$, if for every positive constant c, we have $\lim_{t\to 0}\big(\sup_{x\in\Omega}\dfrac{P(x,ct)}{M(x,t)}\big) = 0$ (resp. $\lim_{t\to\infty}\big(\sup_{x\in\Omega}\dfrac{P(x,ct)}{M(x,t)}\big) = 0\big)$.

Proposition 2.1. *(See[10]) Let $P \prec\prec M$ near infinity and for all $t > 0$, $\sup_{x\in\Omega} P(x,t) < \infty$, then for all $\epsilon > 0$, there exists $C_\epsilon > 0$ such that*

$$P(x,t) \leq M(x,\epsilon t) + C_\epsilon, \quad \forall t > 0, \quad \textit{for a.e. } x \in \Omega. \tag{2.1}$$

The Musielak-Orlicz function $M(x,t)$ is said to satisfy the Δ_2-condition if, there exists $k > 0$ and a nonnegative function $h(.) \in L^1(\Omega)$, such that

$$M(x,2t) \leq M(x,t) + h(x) \text{ a.e. in } \Omega.$$

for large values of t, or for all values of t.
The Musielak-Orlicz space $L_M(\Omega)$ is define as

$$L_M(\Omega) = \{u : \Omega \to \mathbb{R} \quad \text{mesurable} : \quad \varrho_{M,\Omega}(\frac{u}{\lambda}) < \infty, \quad \text{for some} \quad \lambda > 0\}.$$

where $\varrho_{M,\Omega}(u) = \displaystyle\int_\Omega M(x,|u(x)|),\, dx$, equipped with the Luxemburg norm

$$\|u\|_M = \inf\big\{\lambda > 0 : \int_\Omega M(x, \frac{|u(x)|}{\lambda})dx \leq 1\big\}.$$

Denote $\overline{M}(x,s) = \sup_{t\geq 0}(st - M(x,t))$ the conjugate Musielak-Orlicz function of M.
We define $E_M(\Omega)$ as the subset of $L_M(\Omega)$ of all measurable function $u : \Omega \mapsto \mathbb{R}$ such that $\displaystyle\int_\Omega M(x, \frac{|u(x)|}{\lambda})\,dx < \infty$ for all $\lambda > 0$. It is a separable space and $(E_M(\Omega))^* = L_{\overline{M}}(\Omega)$.
We define the Musielak-Orlicz-Sobolev space as

$$W^1L_M(\Omega) = \{u \in L_M(\Omega) : D^\alpha u \in L_M(\Omega), \quad \forall |\alpha| \leq 1\},$$

endowed with the norm

$$\|u\|^1_{M,\Omega} = \inf\{\lambda > 0 : \sum_{|\alpha|\leq 1} \varrho_{M,\Omega}(\frac{D^\alpha u}{\lambda}) \leq 1\}.$$

Lemma 2.1. [4]*(Approximation theorem) Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let M and $\overline{M}$ be two complementary Musielak-Orlicz functions which satisfy the following conditions:*

(1) There exists a constant $c > 0$ such that $\inf_{x\in\Omega} M(x,1) > c$,
(2) There exists a constant $A > 0$ such that for all $x, y \in \Omega$ with $|x-y| \leq \frac{1}{2}$, we have

$$\frac{M(x,t)}{M(y,t)} \leq |t|^{\left(\frac{A}{\log(\frac{1}{|x-y|})}\right)} \quad \textit{for all} \quad t \geq 1,$$

(3) $\displaystyle\int_K M(x,\lambda)dx < \infty, \quad \forall\lambda > 0$ and for every compact $K \subset \Omega$,
(4) There exists a constant $C > 0$ such that $\overline{M}(x,1) \leq C$ a.e. in Ω.

Under this assumptions $\mathcal{D}(\Omega)$ is dense in $L_M(\Omega)$ with respect to the modular topology, $\mathcal{D}(\Omega)$ is dense in $W^1_0L_M(\Omega)$ for the modular convergence and $\mathcal{D}(\overline{\Omega})$ is dense in $W^1L_M(\Omega)$ for the modular convergence.

Example 2.1. *We give some example for a Musielak-Orlicz functions of approximation theorem*

- $M(x,t)=|t|^{p(x)}$ *with* $p:\Omega\to[1,\infty)$ *a measurable function with Log-Hölder continuity*
$$\frac{M(x,t)}{M(x,t)}=|t|^{p(x)-p(y)}\le t^{\left(\frac{A}{\log(\frac{1}{|x-y|})}\right)}\quad \textit{for all}\quad t\ge 1.$$
- *The next Musielak-Orlicz function satisfying also the* $\triangle_2$*-condition* $M(x,t)=\alpha(x)(\exp(|t|)-1+|t|)$, $0<\alpha(x)\in L^\infty(\Omega)$.

Lemma 2.2. [3]*(Modular Poincaré inequality) Under the assumptions of Lemma 2.1, and by assuming that* $M(x,.)$ *decreases with respect to one of coordinate of* x*, there exists a constant* $\delta>0$ *which depends only on* Ω *such that*
$$\int_{Q_T}M(x,|u|)dxdt\le\int_{Q_T}M(x,\delta|\nabla u|)dxdt. \tag{2.2}$$

Inhomogeneous Musielak-Orlicz-Sobolev spaces :
Let M be an Musielak-Orlicz function. For each $\alpha\in\mathbb{N}^N$, denote by ∇_x^α the distributional derivative on Q_T of order α with respect to the variable $x\in\mathbb{R}^N$. The inhomogeneous Musielak-Orlicz-Sobolev spaces are defined as follows,
$$W^{1,x}L_M(Q_T)=\{u\in L_M(Q_T):\nabla_x^\alpha u\in L_M(Q_T),\forall\alpha\in\mathbb{N}^N,|\alpha|\le 1\},$$
$$W^{1,x}E_M(Q_T)=\{u\in E_M(Q_T):\nabla_x^\alpha u\in E_M(Q_T),\forall\alpha\in\mathbb{N}^N,|\alpha|\le 1\}.$$
The last space is a subspace of the first one and both are Banach spaces under the norm
$$\|u\|=\sum_{|\alpha|\le 1}\|\nabla_x^\alpha u\|_{M,Q_T}.$$
The space $W_0^{1,x}E_M(Q_T)$ is defined as the (norm) closure in $W^{1,x}E_M(Q_T)$ of $\mathcal{D}(Q_T)$. We can easily show as in [6], that when Ω has the segment property, then each element u of the closure of $\mathcal{D}(Q_T)$ with respect of the weak* topology $\sigma(\Pi L_M,\Pi E_{\overline{M}})$ is a limit, in $W_0^{1,x}E_M(Q_T)$, of some subsequence in $\mathcal{D}(Q_T)$ for the modular convergence. This implies that $\overline{\mathcal{D}(Q_T)}^{\sigma(\Pi L_M,\Pi E_{\overline{M}})}=\overline{\mathcal{D}(Q_T)}^{\sigma(\Pi L_M,\Pi L_{\overline{M}})}$. This space will be denoted by $W_0^{1,x}L_M(Q_T)$. Furthermore, $W_0^{1,x}E_M(Q_T)=W_0^{1,x}L_M(Q_T)\cap\Pi E_M$, and the dual space of $W_0^{1,x}E_M(Q_T)$ will be denoted by
$$W^{-1,x}L_{\overline{M}}(Q_T)=\Big\{f=\sum_{|\alpha|\le 1}\nabla_x^\alpha f_\alpha:\quad f_\alpha\in L_{\overline{M}}(Q_T)\Big\}.$$
This space will be equipped with the usual quotient norm $\|f\|=\inf\sum_{|\alpha|\le 1}\|f_\alpha\|_{\overline{M},Q_T}$.

Lemma 2.3. ([10]*) Let* $a<b\in\mathbb{R}$ *and* Ω *be a bounded open subset of* $\mathbb{R}^N$ *with the segment property, then* $\{u\in W_0^{1,x}L_M(\Omega\times(a,b))\cap L^1(\Omega\times(a,b)):\frac{\partial u}{\partial t}\in W^{-1,x}L_{\overline{M}}(\Omega\times(a,b))+L^1(\Omega\times(a,b))\}\subset\mathcal{C}([a,b],L^1(\Omega))$.

T_k, $k>0$, denotes the Truncation function at level k defined on $\mathbb{R}$ by $T_k(r)=\max(-k,\min(k,r))$.

3. Formulation of the problem and main results

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ $(N\ge 2)$ and let M and P be two Musielak-Orlicz functions such that M and its complementary $\overline{M}$ satisfies conditions of Lemma (2.2) and $P\prec\prec M$.
$$b:\Omega\times\mathbb{R}\to\mathbb{R}\text{ is a Carathéodory function such that for every }x\in\Omega, \tag{3.1}$$
$b(x,.)$ is a strictly increasing $\mathcal{C}^1(\mathbb{R})$-function and $b\in L^\infty(\Omega\times\mathbb{R})$ with $b(x,0)=0$.
There exists a constant $\lambda>0$ and functions $A\in L^\infty(\Omega)$ and $B\in L_M(\Omega)$ such that
$$\lambda\le\frac{\partial b(x,s)}{\partial s}\le A(x),\quad \Big|\nabla_x\Big(\frac{\partial b(x,s)}{\partial s}\Big)\Big|\le B(x)\quad\text{a.e. }x\in\Omega,\ \forall\, s\in\mathbb{R}. \tag{3.2}$$
$a:Q_T\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$ is Carathéodory function such that for a.e. $x\in\Omega$ and for all $s\in\mathbb{R}$, $\xi,\xi^*\in\mathbb{R}^N$, $\xi\ne\xi^*$ and there exists a constant $\nu>0$,
$$|a(x,t,s,\xi)|\le\nu(a_0(x,t)+\overline{M}_x^{-1}P(x,|s|))\quad\text{with}\quad a_0\in E_{\overline{M}}(Q_T), \tag{3.3}$$

$$(a(x,t,s,\xi)-a(x,t,s,\xi^*))(\xi-\xi^*)>0, \tag{3.4}$$
$$a(x,t,s,\xi).\xi \geq \alpha M(x,|\xi|). \tag{3.5}$$

$g : Q_T \times IR \times IR^N \to IR^N$ is a Carathéodory function such that

$$|g(x,t,s)| \leq q(x,t) + c(x,t)\overline{M}_x^{-1} M(x, \frac{\alpha_0}{\delta}|s|), \tag{3.6}$$

where $0 < \alpha_0 < 1$, $\|c(x,t)\|_{L^\infty(Q_T)} < \min(\frac{\alpha}{\alpha_0+1}; \frac{\lambda\alpha}{(2\alpha_0+1)\|A\|_{L^\infty}+\alpha_0})$,
$q : Q_T \to IR^+$ is positive function which belong to $E_{\overline{M}}(Q_T)$.

$$f \in L^1(Q_T), \tag{3.7}$$

$$u_0 \in L^1(\Omega) \text{ such that } b(x,u_0) \in L^1(\Omega). \tag{3.8}$$

Let ψ a measurable function with values in $\mathbb{R}$ such that

$$\psi \in W_0^1 E_M(Q_T) \cap L^\infty(Q_T), \frac{\partial \psi}{\partial t} \in L^1(Q_T) \quad \text{such that} \quad u_0 \geq \psi, \tag{3.9}$$

and let $K_\psi = \{u \in W_0^{1,x} L_M(Q_T) : u \geq \psi \text{ a.e. in } Q_T\}$.
Note that $<,>$ means for either the pairing between $W_0^{1,x}L_M(Q_T) \cap L^\infty(Q_T)$ and $W^{-1,x}L_{\overline{M}}(Q_T) + L^1(Q_T)$ or between $W_0^{1,x}L_M(Q_T)$ and $W^{-1,x}L_{\overline{M}}(Q_T)$.

Example 3.1. *Our framework admits the following examples*
$b(x,u) = b_0(x)u$, *where* $\lambda \leq b_0(x) \leq A(x)$, $|\nabla_x(b_0(x))| \leq B(x)$ *and* $b_0 \in \mathcal{C}^1(\Omega)$,
$A(x,t,u,\nabla u) = a_0(x,t)|\nabla u|^{p(x)-2}\nabla u + |u|^{p(x)}$, $g(x,t,u) = c(x,t)|\frac{\alpha_0}{\delta}u)|^{\frac{p(x)}{p'(x)}}$, *where* $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ *and the Musielak function is* $M(x,t) = |t|^{p(x)}$.

4. Definition of entropy solutions and statement of main results

Definition 4.1. *A measurable function u defined on Q_T is a entropy solution of problem (1.1), if it satisfies the following conditions:*

$$b(x,u) \in L^\infty(0,T;L^1(\Omega)), b(x,u)(t=0) = b(x,u_0) \quad in \quad \Omega,$$
$$T_k(u) \in W_0^{1,x}L_M(Q_T), \quad \forall k>0, \quad \forall t \in]0,T],$$
$$\left\{\begin{array}{l} \displaystyle\int_0^T \langle \frac{\partial v}{\partial s}; \int_0^u \frac{\partial b(x,z)}{\partial s} T_k'(z-v)dz\rangle ds + \int_\Omega \int_0^{u_0} \frac{\partial b(x,s)}{\partial s} T_k(s-v(0))dsdx \\ \displaystyle + \int_{Q_T} a(x,s,u,\nabla u)\nabla T_k(u-v)dxds + \int_{Q_T} g(x,s,u)\nabla T_k(u-v)dxds \\ \displaystyle \leq \int_{Q_T} fT_k(u-v)dxds, \quad \forall k>0, \forall v \in K_\psi \cap L^\infty(Q_T) \quad with \quad v(T)=0, \\ \quad such\ that \quad \frac{\partial v}{\partial t} \in L_{\overline{M}}(0,T;W^{-1}L_{\overline{M}}(\Omega)). \end{array}\right. \tag{4.1}$$

Theorem 4.1. *Assume that (3.1)-(3.9) hold true. Then there exists at least one entropy solution u of the problem (1.1) in the sense of the definition (4.1).*

Proof of theorem 4.1 Truncated problem .
For each $n > 0$, we define the following approximations

$$b_n(x,s) = b(x,T_n(s)), \quad \forall\, s \in \mathbb{R}, \tag{4.2}$$

$$a_n(x,t,s,\xi) = a(x,t,T_n(s),\xi) \quad \text{a.e. } (x,t) \in Q_T,\ \forall\, s \in \mathbb{R},\ \forall\, \xi \in \mathbb{R}^N, \tag{4.3}$$

$$g_n(x,t,s) = g(x,t,T_n(s)) \quad \text{a.e. } (x,t) \in Q_T,\ \forall\, s \in \mathbb{R}, \tag{4.4}$$

$\{f_n\}_n \in L^1(Q_T)$ be a sequence of smooth functions such that

$$f_n \to f \text{ strongly in } L^1(Q_T), \tag{4.5}$$

and

$$u_{0n} \in \mathcal{C}_0^\infty(\Omega) \text{ such that } b_n(x,u_{0n}) \to b(x,u_0) \text{ strongly in } L^1(\Omega). \tag{4.6}$$

Let us now consider the penalized approximate equations:

$$\begin{cases} \dfrac{\partial b_n(x,u_n)}{\partial t} - div(a_n(x,t,u_n,\nabla u_n)) \\ -div(g_n(x,t,u_n)) - nT_n(u_n-\psi)^- = f_n \ in \ Q_T, \\ u_n(x,t) = 0 \quad on \quad \partial\Omega \times (0,T), \\ b_n(x,u_n)(t=0) = b_n(x,u_{0n}) \quad in \quad \Omega. \end{cases} \tag{4.7}$$

For any fixed $n > 0$, let $u_n \in W_0^{1,x}L_M(Q_T)$, using (3.6) we get

$$|g_n(x,t,u_n)\nabla u_n| \le \overline{M}(x, \frac{q(x,t)}{\epsilon}) + \epsilon M(x, |\nabla u_n|)$$
$$+ \|c(.,.)\|_{L^\infty(Q_T)}(\overline{M}(x, \frac{1}{\epsilon}\overline{M}_x^{-1} M(x, \frac{\alpha_0}{\delta}|T_n(u_n)|) + \epsilon\overline{M}(x, |\nabla u_n|)),$$

then

$$|g_n(x,t,u_n)\nabla u_n| \le d_{n,\epsilon}(x,t) + \epsilon(1 + \|c(.,.)\|_{L^\infty(Q_T)})M(x, |\nabla u_n|),$$

by (3.5)

$$[a_n(x,t,u_n,\nabla u_n) + g_n(x,t,u_n)]\nabla u_n \ge [\alpha - \epsilon(1 + \|c(.,.)\|_{L^\infty(Q_T)})]M(x, |\nabla u_n|) - d_{n,\epsilon}(x,t).$$

Choosing ϵ such that $\alpha - \epsilon\Big(1 + \|c(.,.)\|_{L^\infty(Q_T)}\Big) > \frac{\alpha}{2}$, we obtain

$$[a_n(x,t,u_n,\nabla u_n) + g_n(x,t,u_n)]\nabla u_n \ge \frac{\alpha}{2} M(x, |\nabla u_n|) - d_{n,\epsilon}(x,t)$$

where $d_{n,\epsilon} \in L^1(Q_T)$. Then the operator $[a_n(x,t,u_n,\nabla u_n) + g_n(x,t,u_n)]$ satisfies the conditions of Theorem 1 of [13] and there exists at last one solution $u_n \in W_0^{1,x}L_M(Q_T)$ of (4.7).

Remark 4.1. *The explicit dependence in x and t of the functions a and g will be omitted so that $a(x,t,u,\nabla u) = a(u,\nabla u)$ and $g(x,t,u) = g(u)$ (resp. a_n and g_n).*

Proposition 4.1. *Let u_n be a solution of approximate equation (4.7) and there exists a measurable function u such that*

$$\begin{cases} T_k(u_n) \to T_k(u) \quad weakly \ in \quad W^{1,x}L_M(Q_T), u_n \to u \quad a.e. \ in \quad Q_T, \\ b_n(x,u_n) \to b(x,u) \quad a.e. \ in \quad Q_T \quad and \quad b(x,u) \in L^\infty(Q_T), \\ a(T_k(u_n), \nabla T_k(u_n))\nabla T_k(u_n) \rightharpoonup a(T_k(u), \nabla T_k(u))\nabla T_k(u) \quad weakly \ in \quad L^1(Q_T), \\ \nabla u_n \to \nabla u \quad a.e. \ in \quad Q_T. \end{cases} \tag{4.8}$$

Then u be a solution of problem (4.1).

Proof. Let $w \in K_\psi \cap L^\infty(Q_T)$ such that $\frac{\partial w}{\partial t} \in W_0^{-1,x}L_{\overline{M}}(Q_T)$, with $w(T) = 0$.
Pointwise multiplication of the approximation equation (4.7) by $T_k(u_n - w)$, we get

$$\begin{cases} \displaystyle\int_0^T < \frac{\partial b_n(x,u_n)}{\partial t}; T_k(u_n - w) > dt + \int_{Q_T} a_n(u_n, \nabla u_n))\nabla T_k(u_n - w)\, dx\, dt \\ \displaystyle + \int_{Q_T} g_n(u_n)\nabla T_k(u_n - w)\, dx\, dt - \int_{Q_T} nT_n(u_n - \psi)^- T_k(u_n - w)\, dx\, dt \\ \displaystyle = \int_{Q_T} f_n T_k(u_n - w)\, dx\, dt \end{cases} \tag{4.9}$$

Passing to the limit as in (4.9) $n \to +\infty$.
Limit of the first term of (4.9), The first term can be written

$$\int_0^T < \frac{\partial b_n(x,u_n)}{\partial t}; T_k(u_n - w) > dt = \int_0^T < \frac{\partial w}{\partial t}; \int_0^{u_n} \frac{\partial b_n(x,z)}{\partial t} T_k'(z - w)dz > dt$$
$$+ \int_\Omega \int_0^{u_n(T)} < \frac{\partial b_n(x,t)}{\partial t}; T_k(t - w(T)) > dtdx - \int_\Omega \int_0^{u_{0n}} < \frac{\partial b_n(x,t)}{\partial t}; T_k(t - w(0)) > dtdx$$

Since $w(T) = 0$ and $\dfrac{\partial b(x,t)}{\partial t} \geq 0$, then, $\displaystyle\int_\Omega \int_0^{u_n(T)} \frac{\partial b_n(x,t)}{\partial t} T_k(t)dtdx \geq 0$. Since u_{0n} converge to u_0 strongly in $L^1(\Omega)$, we obtain,

$$\lim_{n\to+\infty} \int_\Omega \int_0^{u_{0n}} \frac{\partial b_n(x,t)}{\partial t} T_k(t-\omega(0))dtdx = \int_\Omega \int_0^{u_0} \frac{\partial b(x,t)}{\partial t} T_k(t-\omega(0))dtdx.$$

On the other hand, we know that $T_m(u_n)$ converges to $T_m(u)$ weakly in $W_0^{1,x}L_M(Q_T)$, and

$$\begin{aligned}
&\lim_{n\to+\infty} \int_0^T < \frac{\partial w}{\partial t}; \int_0^{u_n} \frac{\partial b_n(x,z)}{\partial t} T_k'(z-w)dz > dt \\
&= \lim_{n\to+\infty} \int_0^T < \frac{\partial w}{\partial t}; \int_0^{T_m(u_n)} \frac{\partial b_n(x,z)}{\partial t} T_k'(z-w)dz > dt \\
&= \int_0^T < \frac{\partial w}{\partial t}; \int_0^{T_m(u)} \frac{\partial b(x,z)}{\partial t} T_k'(z-w)dz > dt
\end{aligned}$$

with $m = k + \|\omega\|_\infty$. We have

$$\begin{aligned}
&\int_0^T < \frac{\partial w}{\partial t}; \int_0^{T_m(u)} \frac{\partial b(x,z)}{\partial t} T_k'(z-w)dz > dt \\
&\leq \lim_{n\to+\infty} \int_0^T < \frac{\partial b_n(x,u_n)}{\partial t}; T_k'(u_n-w) > dt + \int_\Omega \int_0^{u_0} < \frac{\partial b(x,t)}{\partial t}; T_k'(t-w(0)) > dtdx.
\end{aligned}$$

Limit of $a_n(u_n,\nabla u_n)\nabla T_k(u_n-w)$:
We have $a_n(u_n,\nabla u_n)\nabla T_k(u_n-w) = a(T_m(u_n),\nabla T_m(u_n))\nabla T_k(T_m(u_n)-w)$ for $n > m$. By proposition 4.1 and the pointwise convergence of u_n to u as $n\to+\infty$, we get $a_n(u_n,\nabla u_n)\nabla T_k(u_n-w) \rightharpoonup a(u,\nabla u)\nabla T_k(u-w)$ weakly in $L^1(Q_T)$.
Limit of $g_n(u_n)\nabla T_k(u_n-w)$**:**
Since $g_n(u_n)\nabla T_k(u_n-w) = g(T_m(u_n))\nabla T_k(T_m(u_n)-w)$ a.e. in Q_T, with $m = k+\|\omega\|_\infty$, and the weakly convergence of $T_m(u_n)$ to $T_m(u)$ as $n\to+\infty$, allows us to have $g_n(u_n)\nabla T_k(u_n-w) \rightharpoonup g(u)\nabla T_k(u-w)$ weakly in $L^1(Q_T)$.
Limit of $f_nT_k(u_n-w)$**:**
Using (4.5) we can easily see that $f_nT_k(u_n-w)\to fT_k(u-w)$ in $L^1(Q_T)$.
Since $-nT_n(u_n-\psi)^-T_k(u_n-w)\geq 0$ since $w\in K_\psi$, then, as a consequence of the above convergence result, we are in a position to pass to the limit as $n\to+\infty$ in (4.9) to conclude that u satisfies (4.1).
• It remains to show that $b(x,u)$ satisfies the initial condition.
Firstly, remark that, in view of the definition of $B^m_{n,\xi}$ (see (4.21)), we have $B^m_{n,\xi}(x,u_n)$ is bounded in $L^\infty(Q_T)$. Secondly, by (4.9) we show that $\dfrac{\partial B^m_{n,\xi}(x,u_n)}{\partial t}$ is bounded in $L^1(Q_T)+W^{-1,x}L_{\overline{M}}(Q_T))$. As a consequence, $B^m_{n,\xi}(x,u_n)\in C^0([0,T];L^1(\Omega))$ (see Lemma 2.3).
It follows that, $B^m_{n,\xi}(x,u_n)(t=0)$ converges to $B^m_\xi(x,u)(t=0)$ strongly in $L^1(\Omega)$. On the other hand, the smoothness of ξ imply that $B^m_{n,\xi}(x,u_n)(t=0) = B^m_{n,\xi}(x,u_{0n})$ converges to $B^m_\xi(x,u)(t=0)$ strongly in $L^1(\Omega)$, we obtain $B^m_\xi(x,u)(t=0) = B^m_\xi(x,u_0)$ a.e. in Ω and for all $m>0$, now letting m to $+\infty$, we conclude that $b(x,u)(t=0) = b(x,u_0)$ a.e. in Ω.

Remark 4.2. *We focus our work to show the conditions of the proposition 4.8, then for this we go through 3 steps to arrive at our result.*

Step 1: In this step let us begin by showing

$$T_k(u_n)\to T_k(u) \quad \text{weakly in} \quad W^{1,x}L_M(Q_T), \tag{4.10}$$

and

$$u\geq\psi. \quad \text{a.e. in} \quad Q_T. \tag{4.11}$$

Fix $k > 0$,
Let a function $\omega_n = b_n(x, u_n) - b(x, \psi)$ in $L^1(Q_T)$, then

$$\nabla T_k(\omega_n) = \big[\frac{\partial b_n(x, u_n)}{\partial s}\nabla u_n + \frac{\partial b(x, \psi)}{\partial s}\nabla \psi\big]\chi_{\{|\omega_n| \le k\}} + \Big(\nabla_x b_n(x, u_n) - \nabla_x b(x, \psi)\Big)\chi_{\{|\omega_n| \le k\}}, \tag{4.12}$$

and

$$|u_n| \le \frac{1}{\lambda}|\omega_n| + \|\psi\|_{L^\infty(Q_T)}. \tag{4.13}$$

Let $\tau \in (0, T)$ and using $T_k(\omega_n)\chi_{(0,\tau)}$ as a test function in problem (4.7), we get

$$\begin{aligned}
&\int_\Omega B_k(\omega_n)(\tau)dx + \int_{Q_\tau} a_n(x, t, u_n, \nabla u_n)\nabla T_k(\omega_n)dxdt \\
&+ \int_{Q_\tau} g_n(x, t, u_n)\nabla T_k(\omega_n)dxdt - n\int_{Q_\tau} T_n(u_n - \psi)^- T_k(\omega_n)dxdt \\
&\le k\big(\|f_n\|_{L^1(Q_T)} + \|b(x, u_{0n}) - b(x, \psi(0))\|_{L^1(\Omega)}\big),
\end{aligned} \tag{4.14}$$

where $B_k(r) = \int_0^r T_k(s)ds$.

For the first right hand side of (4.14):
We know that $\frac{1}{2}|T_k(s)|^2 \le \frac{1}{2}sT_k(s) \le B_k(s) \le ks$, $\forall s \in \mathbb{R}$, then we obtain

$$\int_\Omega B_k(\omega_n)dx \ge \frac{1}{2}\int_\Omega |T_k(\omega_n)|^2 dx \ge 0, \quad \forall k > 0. \tag{4.15}$$

For the second right hand side of (4.14):

$$\begin{aligned}
\int_{Q_\tau} a_n(u_n, \nabla u_n)\nabla T_k(\omega_n)\,dx\,dt &= \int_{Q_\tau} a_n(u_n, \nabla u_n)\frac{\partial b_n(x, u_n)}{\partial s}\nabla u_n\chi_{\{|\omega_n| \le k\}}dxdt \\
&+ \int_{Q_\tau} a_n(u_n, \nabla u_n)\frac{\partial b_n(x, u_n)}{\partial s}\nabla \psi\chi_{\{|\omega_n| \le k\}}dxdt \\
&+ \int_{Q_\tau} a_n(u_n, \nabla u_n)\big(\nabla_x b_n(x, u_n) - \nabla_x b(x, \psi)\big)\chi_{\{|\omega_n| \le k\}}\,dx\,dt.
\end{aligned}$$

Using (3.5), we get

$$\int_{Q_\tau} a_n(u_n, \nabla u_n)\frac{\partial b_n(x, u_n)}{\partial s}\nabla u_n\chi_{\{|\omega_n| \le k\}}dxdt \ge \lambda\alpha\int_{Q_\tau} M(x, |\nabla u_n|)\chi_{\{|\omega_n| \le k\}}dxdt.$$

And using (3.3), (2.1), Young inequality and Lemma 2.2 for any $\epsilon > 0$,

$$\begin{aligned}
&\int_{Q_\tau} a_n(u_n, \nabla u_n)\frac{\partial b_n(x, u_n)}{\partial s}\nabla \psi\chi_{\{|\omega_n| \le k\}}dxdt \\
&\le v\|A\|_{L^\infty(\Omega)}\big[\int_{Q_\tau} \overline{M}(x, a_0(x, t))\,dx\,dt + \int_{Q_\tau} M(x, |\nabla \psi|)\,dx\,dt\big] \\
&+ \epsilon v\|A\|_{L^\infty(\Omega)}\int_{Q_\tau} M(x, |\nabla u_n|)\chi_{\{|\omega_n| \le k\}}dxdt + C_2.
\end{aligned}$$

Using again (3.3), (2.1) and Young inequality, we get

$$\begin{aligned}
&\int_{Q_\tau} a_n(u_n, \nabla u_n)\big(\nabla_x b_n(x, u_n) - \nabla b(x, \psi)\big)\chi_{\{|\omega_n| \le k\}}dxdt \\
\le\;& \nu\big[\int_{Q_T} \overline{M}(x, a_0(x, t))\,dx\,dt + \int_{Q_T} M(x, |\nabla_x b_n(x, u_n) - \nabla_x b(x, \psi)|)dxdt\big] \\
+\;& \nu\epsilon\int_{Q_T} M(x, |\nabla u_n|)\chi_{\{|\omega_n| \le k\}}\,dx\,dt + \nu\int_{Q_T} M(x, |\nabla_x b_n(x, u_n) - \nabla_x b(x, \psi)|)\,dx\,dt.
\end{aligned}$$

For the third right hand side of (4.14):
Thanks to (3.6), (2.1), Young inequality and Lemma 2.2, we get

$$\begin{aligned}
&\int_{Q_\tau} g_n(u_n)\nabla T_k(\omega_n)dxdt\\
&\leq \|A\|_{L^\infty}[\int_{Q_T}\overline{M}(x,\frac{q(x,t)}{\epsilon})\,dx\,dt+\epsilon\int_{Q_\tau}M(x,|\nabla u_n|)\chi_{\{|\omega_n|\leq k\}}\,dx\,dt]\\
&+\|A\|_{L^\infty}[\int_{Q_T}\overline{M}(x,q(x,t))\,dx\,dt+\int_{Q_T}M(x,|\nabla\psi|)\,dx\,dt]\\
&+\int_{Q_T}\overline{M}(x,q(x,t))\,dx\,dt+\int_{Q_T}M(x,|\nabla_x b_n(x,u_n)-\nabla_x b(x,\psi)|)\,dx\,dt\\
&+\|A\|_{L^\infty}\|c(x,t)\|_{L^\infty(Q_T)}(\alpha_0+1)\int_{Q_\tau}M(x,|\nabla u_n|)\chi_{\{|\omega_n|\leq k\}}\,dx\,dt\\
&+\|A\|_{L^\infty}\|c(x,t)\|_{L^\infty(Q_T)}[\alpha_0\int_{Q_\tau}M(x,|\nabla u_n|)\chi_{\{|\omega_n|\leq k\}}\,dx\,dt+\int_{Q_T}M(x,|\nabla\psi|)\,dx\,dt]\\
&+\|c(x,t)\|_{L^\infty(Q_T)}[\alpha_0\int_{Q_\tau}M(x,|\nabla u_n|)\chi_{\{|\omega_n|\leq k\}}dxdt\\
&+\int_{Q_T}M(x,|\nabla_x b_n(x,u_n)-\nabla_x b(x,\psi)|)\,dx\,dt].
\end{aligned}$$

Finally we obtain

$$\begin{aligned}
&\lambda\alpha\int_{Q_\tau}M(x,|\nabla u_n|)\chi_{\{|\omega_n|\leq k\}}\,dx\,dt-n\int_{Q_\tau}T_n(u_n-\psi)^-T_k(\omega_n)dxdt\\
&\leq v\|A\|_{L^\infty(\Omega)}[\int_{Q_T}\overline{M}(x,a_0(x,t))\,dx\,dt+\int_{Q_T}M(x,|\nabla\psi|)\,dx\,dt]\\
&+\epsilon v\|A\|_{L^\infty(\Omega)}\int_{Q_\tau}M(x,|\nabla u_n|)\chi_{\{|\omega_n|\leq k\}}dxdt+C_2\\
&+\nu[\int_{Q_T}\overline{M}(x,a_0(x,t))\,dx\,dt+\int_{Q_T}M(x,|\nabla_x b_n(x,u_n))-\nabla_x b(x,\psi)|)dxdt]\\
&+\nu\epsilon\int_{Q_\tau}M(x,|\nabla u_n|)\chi_{\{|\omega_n|\leq k\}}\,dx\,dt+\nu\int_{Q_T}M(x,|\nabla_x b_n(x,u_n))-\nabla_x b(x,\psi)|)dxdt\\
&+\|A\|_{L^\infty}[\int_{Q_T}\overline{M}(x,\frac{q(x,t)}{\epsilon})\,dx\,dt+\epsilon\int_{Q_\tau}M(x,-|\nabla u_n|)\chi_{\{|\omega_n|\leq k\}}\,dx\,dt]\\
&+\|A\|_{L^\infty}[\int_{Q_T}\overline{M}(x,q(x,t))\,dx\,dt+\int_{Q_T}M(x,|\nabla\psi|)\,dx\,dt]\\
&+\int_{Q_T}\overline{M}(x,q(x,t))\,dx\,dt+\int_{Q_T}M(x,|\nabla_x b_n(x,u_n)-\nabla_x b(x,\psi)|)dxdt\\
&+\|A\|_{L^\infty}\|c(x,t)\|_{L^\infty(Q_T)}(\alpha_0+1)\int_{Q_\tau}M(x,|\nabla u_n|)\chi_{\{|\omega_n|\leq k\}}\,dx\,dt\\
&+\|A\|_{L^\infty}\|c(x,t)\|_{L^\infty(Q_T)}[\alpha_0\int_{Q_\tau}M(x,|\nabla u_n|)\chi_{\{|\omega_n|\leq k\}}\,dx\,dt+\int_{Q_T}M(x,|\nabla\psi|)\,dx\,dt]\\
&+\|c(x,t)\|_{L^\infty(Q_T)}[\alpha_0\int_{Q_\tau}M(x,|\nabla u_n|)\chi_{\{|\omega_n|\leq k\}}dxdt\\
&+\int_{Q_\tau}M(x,|\nabla_x b_n(x,u_n)-\nabla_x b(x,\psi)|)\chi_{\{|\omega_n|\leq k\}}\,dx\,dt]\\
&+k\big(\|f_n\|_{L^1(Q_T)}+\|b(x,u_{0n})-b(x,\psi(0))\|_{L^1(\Omega)}\big),
\end{aligned}$$

Using (3.5), we get,

$$\frac{1}{C'}\int_{Q_\tau} M(x,\nabla T_k(u_n))\chi_{\{|\omega_n|\le K\}}dxdt - n\int_{Q_\tau} T_n(u_n-\psi)^- T_k(\omega_n)dxdt \le kC_1 + C_\epsilon.$$

where $\frac{1}{C'} = \{\lambda\alpha - \epsilon[\nu\|A\|_{L^\infty} + \nu + \|A\|_{L^\infty}] - \|c(x,t)\|_{L^\infty(Q_T)}[(2\alpha_0+1)\|A\|_{L^\infty} + \alpha_0]\}$.
Choosing ϵ such that,

$$\epsilon < \frac{\lambda\alpha - \|c(x,t)\|_{L^\infty(Q_T)}[(2\alpha_0+1)\|A\|_{L^\infty} + \alpha_0]}{\nu\|A\|_{L^\infty} + \nu + \|A\|_{L^\infty}},$$

we deduce

$$\int_{Q_\tau} M(x,|\nabla u_n|)\chi_{\{|\omega_n|\le k\}}dxdt \le kC'C_1 + C'C_2, \tag{4.16}$$

and

$$0 < -n\int_{Q_\tau} T_n(u_n-\psi)^- T_k(\omega_n)dxdt \le kC_1 + C_2, \tag{4.17}$$

passing to limit as $k \to 0$, we get,

$$0 \le \int_{Q_\tau} T_n(u_n-\psi)^- dxdt \le \frac{C_2}{n}.$$

By letting $n \to +\infty$, we obtain $\int_{Q_t}(u-\psi)^-\,dx\,dt = 0$, then we conclude (4.11).
On the other hand since $\{|u_n| \le \beta\} \subset \{|b_n(x,u_n) - b(x,\psi)| \le k\}$ for all $\beta > 0$
and $k = \|A\|_{L^\infty}(\beta + \|\psi\|_{L^\infty})$ and using (4.16), we have

$$\int_{Q_\tau} M(x,|\nabla T_\beta(u_n)|)dxdt \le \int_{Q_\tau} M(x,|\nabla u_n|)\chi_{\{|\omega_n|\le k\}}dxdt \le \beta C_3 + C_4,$$

where $C_3 = kC'C_1$ and $C_4 = C'C_2$.
Using (2.2), we have

$$\begin{aligned}\inf_{x\in\Omega} M(x,\frac{\beta}{\delta})meas\{|u_n|>\beta\} &\le \int_{\{|u_n|>\beta\}} M(x,\frac{|T_\beta(u_n)|}{\delta})dxdt \\ &\le \int_{Q_\tau} M(x,|\nabla T_\beta(u_n)|)\,dx\,dt \le \beta C_3 + C_4,\end{aligned}$$

then

$$meas\{|u_n|>\beta\} \le \frac{\beta C_3 + C_4}{\inf_{x\in\Omega} M(x,\frac{\beta}{\delta})}, \quad \text{for all} \quad n \quad \text{and for all} \quad \beta.$$

Assuming that there exists a positive function m such that $\lim\limits_{t\to\infty}\frac{m(t)}{t} = +\infty$
and $m(t) \le ess\inf\limits_{x\in\Omega} M(x,t)$, $\forall t \ge 0$. Thus, we get $\lim\limits_{\beta\to\infty} meas\{|u_n|>\beta\} = 0$.
Now we turn to prove the almost every convergence of u_n , $b_n(x,u_n)$
and convergence of $a_n(x,t,T_k(u_n),\nabla T_k(u_n))$.

Proposition 4.2. *Let u_n be a solution of the approximate problem, then there exsits a measurable function u such that*

$$u_n \to u \quad a.e\ in \quad Q_T. \tag{4.18}$$

$$b_n(x,u_n) \to b(x,u) \quad a.e\ in \quad Q_T \quad and \quad b(x,u) \in L^\infty(0,T,L^1(\Omega)). \tag{4.19}$$

$$a_n(T_k(u_n),\nabla T_k(u_n)) \rightharpoonup \sigma_k \quad in \quad (L_{\overline{M}}(Q_T))^N, \quad for \quad \sigma(\Pi L_{\overline{M}},\Pi E_M), \tag{4.20}$$

for some $\sigma_k \in (L_{\overline{M}}(Q_T))^N$.

Proof.
Proof of (4.18) and (4.19): Consider a function non decreasing $\xi_k \in C^2(\mathbb{R})$ such that $\xi_k(s) = s$ for $|s| \leq \frac{k}{2}$ and $\xi_k(s) = k$ for $|s| \geq k$ and multiplying the approximate equation (4.7) by $\xi_k'(u_n)$, we get

$$\frac{\partial B^k_{n,\xi}(x,u_n)}{\partial t} = div(\xi_k'(u_n)a_n(x,t,u_n,\nabla u_n)) - \xi_k''(u_n)a_n(x,t,u_n,\nabla u_n)\nabla u_n$$
$$+div(\xi_k'(u_n)g_n(x,t,u_n)) - \xi_k''(u_n)g_n(x,t,u_n)\nabla u_n + f_n\xi_k'(u_n) \quad \text{in} \quad \mathcal{D}(Q_T), \tag{4.21}$$

where $B^k_{n,\xi}(x,r) = \int_0^r \frac{\partial b_n(x,s)}{\partial s}\xi_k'(s)ds$.
As a consequence of (4.16), we deduce that $\xi_k'(u_n)$ is bounded in $W_0^{1,x}L_M(Q_T)$ and $\frac{\partial B^k_{n,\xi}(x,u_n)}{\partial t}$ is bounded in $W^{-1,x}L_{\overline{M}}(Q_T) + L^1(Q_T)$. Using (3.2) we get $\frac{\partial \xi_k(u_n)}{\partial t}$ is bounded in $W^{-1,x}L_{\overline{M}}(Q_T) + L^1(Q_T)$, then $\xi_k(u_n)$ is compact in $L^1(Q_T)$. We conclude that for each k, the sequence $T_k(u_n)$ converges almost everywhere in Q, which implies that the sequence u_n converges almost everywhere to some measurable function u in Q_T.
To prove that $b(x,u) \in L^\infty(0,T,L^1(\Omega))$, proceeding as in [1], it is easy to show that if we use (4.21) and (4.16), we deduce $\int_\Omega B^k_{n,\xi}(x,u_n)dx \leq kC_4 + C_5$, for almost any t in $(0,T)$. Using the pointwise convergence of u_n and $B^k_{n,\xi}(x,u_n)$ and passing to limit as $k \to +\infty$ allows to show that $|b(x,u)| = \int_0^u sg(s)\frac{\partial b(x,s)}{\partial s}ds \leq C_4$. Thus $b(x,u) \in L^\infty(0,T,L^1(\Omega))$.
Proof of 4.20 : Using (3.3) and (4.16), we allows us to prove that $\{a_n(T_k(u_n),\nabla T_k(u_n))\}_n$ is bounded in $\left(L_{\overline{M}}(\Omega)\right)^N$ for all $k > 0$ and we conclude (4.20).

Step 2:
This technical Lemma will help us in the step 3 of the demonstration,

Lemma 4.1. *If the subsequence u_n satisfies (4.7), then*

$$\lim_{m\to+\infty}\limsup_{n\to+\infty}\int_{\{m\leq|u_n|\leq m+1\}} a(u_n,\nabla u_n)\nabla u_n dxdt = 0. \tag{4.22}$$

Proof.
Multiplying the approximating equation (4.7) by the test function
$Z_m(u_n) = T_{m+1}(u_n) - T_m(u_n)$, we get

$$\int_\Omega B_m(x,u_n(T))dx + \int_{Q_T} a_n(u_n,\nabla u_n)\nabla Z_m(u_n)dx\,dt + \int_{Q_T} g_n(u_n)\nabla Z_m(u_n)dx\,dt$$
$$-n\int_{Q_T} T_n(u_n-\psi)^- Z_m(u_n)dxdt = \int_{Q_T} f_n Z_m(u_n)\,dx\,dt + \int_\Omega B_m(x,u_{0n})dx, \tag{4.23}$$

where $B_m(x,r) = \int_0^r Z_m(s)\frac{\partial b_n(x,s)}{\partial s}ds$.
By (3.6), we have

$$\begin{aligned}\int_{Q_T} g_n(u_n)\nabla Z_m(u_n)dx\,dt &\leq \int_{\{m\leq|u_n|\leq m+1\}} M(x,\frac{q(x,t)}{\epsilon})dx\,dt \\ &+ [\|c\|_{L^\infty}(\alpha_0+1)+\epsilon]\int_{\{m\leq|u_n|\leq m+1\}} M(x,|\nabla u_n|)dx\,dt.\end{aligned}$$

Since $-n\int_{Q_T} T_n(u_n-\psi)^- Z_m(u_n)dxdt \geq 0$, $\int_\Omega B_m(x,u_n(T))dx \geq 0$ and following the same techniques in step 2, we obtain

$$\int_{Q_T} M(x,|\nabla Z_m(u_n)|)dxdt \leq C(\int_{\{m\leq|u_n|\leq m+1\}} M(x,\frac{q(x,t)}{\epsilon})dx\,dt + \int_\Omega B_m(x,u_{0n})dx$$
$$+\int_{Q_T} f_n Z_m(u_n)dxdt),$$

where $C = \frac{1}{\alpha-\epsilon-(\alpha_0+1)\|c\|_{L^\infty}}$ and taking ϵ such that $\epsilon < \alpha - (\alpha_0+1)\|c\|_{L^\infty}$.
Passing to limit as $n \to +\infty$, since the pointwise convergence of u_n and the strongly convergence in $L^1(Q_T)$ of f_n and $B_m(x, u_{0n})$, we get

$$\lim_{n\to+\infty}\int_{Q_T} M(x, \nabla Z_m(u_n))dxdt \le C\Big(\int_{\{m\le|u|\le m+1\}} M(x, \frac{q(x,t)}{\epsilon})dx\,dt + \int_\Omega B_m(x,u_0)dx + \int_{Q_T} fZ_m(u)dxdt\Big)$$

By Lebesgue's theorem and passing to limit as $m \to +\infty$, in the all term of the right-hand side, we get

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{Q_T} M(x, |\nabla Z_m(u_n)|)dxdt = 0, \tag{4.24}$$

On the other hand, we have

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\Big|\int_{Q_T} g_n(u_n)\nabla Z_m(u_n)dxdt\Big| \le \lim_{m\to+\infty}\lim_{n\to+\infty}\int_{Q_T} M(x, |\nabla Z_m(u_n)|)dxdt, + \lim_{m\to+\infty}\lim_{n\to+\infty}\int_{\{m\le|u_n|\le m+1\}} \overline{M}(x, |g_n(u_n)|)dxdt.$$

Using the pointwise convergence of u_n and by Lebesgue's theorem, in the second term of the right side, we get

$$\lim_{n\to+\infty}\int_{\{m\le|u_n|\le m+1\}} \overline{M}(x, |g_n(u_n)|)dxdt = \int_{\{m\le|u|\le m+1\}} \overline{M}(x, |g(u)|)dxdt,$$

and by Lebesgue's theorem

$$\lim_{m\to+\infty}\int_{\{m\le|u|\le m+1\}} \overline{M}(x, |g(u)|)\,dx\,dt = 0, \tag{4.25}$$

By (4.24) and (4.25), we have $\lim\limits_{m\to+\infty}\lim\limits_{n\to+\infty}\int_{Q_T} g_n(u_n)\nabla Z_m(u_n)dxdt = 0$, then passing to the limit in (4.23), we get the (4.22).

Step 3: We will concentrate on the following last two conditions of proposition 4.1.

$$a(T_k(u_n), \nabla T_k(u_n))\nabla T_k(u_n) \rightharpoonup a(T_k(u), \nabla T_k(u))\nabla T_k(u) \quad \text{weakly in} \quad L^1(Q_T), \tag{4.26}$$

and

$$\nabla u_n \to \nabla u \quad \text{a.e. in} \quad Q_T. \tag{4.27}$$

This step is devoted to introduce for $k \ge 0$ fixed a time regularization of the function $T_k(b(x,u) - b(x,\psi))$ in order to perform the monotonicity method.
Let v_0^μ be a sequence of function in $L^\infty(\Omega) \cap W_0^1 L_M(\Omega))$ such that $\|v_0^\mu\|_{L^\infty(\Omega)} \le k$ for all $\mu > 0$, and v_0^μ converge to $T_k(b(x,u_0) - b(x,\psi))$ a.e. in Ω and $\frac{1}{\mu}\|v_0^\mu\|_{L_M(\Omega)} \to 0$ as $\mu \to 0$.
Let us consider the unique solution $(T_k(b(x,u) - b(x,\psi)))_\mu \in L^\infty(Q_T) \cap W_0^{1,x} L_M(Q_T)$ of the monotone problem:

$$\begin{cases} \frac{\partial (T_k(b(x,u)-b(x,\psi)))_\mu}{\partial t} + \mu((T_k(b(x,u) - b(x,\psi)))_\mu - T_k(b(x,u) - b(x,\psi))) = 0 \text{ in } \mathcal{D}'(Q_T) \\ (T_k(b(x,u) - b(x,\psi)))_\mu(t=0) = w_0^\mu \quad \text{in} \quad \Omega. \end{cases} \tag{4.28}$$

Remark that

$$(T_k(b(x,u) - b(x,\psi)))_\mu \quad \text{converge to} \quad T_k(b(x,u) - b(x,\psi)) \tag{4.29}$$

a.e. in Q_T, weakly-* in $L^\infty(Q_T)$ and in $W_0^{1,x}L_M(Q_T)$ for the modular convergence as $\mu \to +\infty$, and we have

$$\|(T_k(b(x,u) - b(x,\psi)))_\mu\|_{L^\infty(Q_T)} \le \max(\|T_k(b(x,u) - b(x,\psi))\|_{L^\infty(Q_T)}, \|v_0^\mu\|_{L^\infty(\Omega)}) \le k.$$

Let S_m be a sequence of increasing C^∞-function such that $S_m(r) = r$, $|r| \le m$; $supp(S'_m) \subset [-2m, 2m]$ and $\|S''_m\|_{L^\infty(\mathbb{R})} \le \frac{3}{m}$ for any $m \ge 1$.
Denote $\epsilon(m,\mu,n)$ the value, such that: $\lim\limits_{m\to+\infty}\lim\limits_{\mu\to+\infty}\lim\limits_{n\to+\infty}\epsilon(n,\mu,m) = 0$.

Lemma 4.2. *Let S be a C^∞-function such that $S(r) = r$ for $|r| \le k$, and $supp(S')$ is compact. The subsequence u_n satisfies for $k \ge 0$*

$$\liminf_{\mu\to+\infty}\lim_{n\to+\infty}\int_0^T < \frac{\partial b(x,u_n)}{\partial t}, S'(b(x,u_n)-b(x,\psi))T_k(b(x,u_n)-b(x,\psi)) -(T_k(b(x,u)-b(x,\psi)))_\mu > dt \ge 0,$$

Proof. see Appendix

where $< .,. >$ denotes the duality pairing between $L^1(Q_T) + W^{-1}L_M(Q_T)$ and $L^\infty(Q_T) \cap W^1L_M(Q_T)$. We prove the following Lemma which is the critical point in the development of the monotonicity method.

Lemma 4.3. *The subsequence u_n satisfies for $k \ge 0$*

$$\limsup_{n\to+\infty}\int_{Q_T} a(u_n,\nabla T_k(u_n))\nabla T_k(u_n)dxdt \le \int_{Q_T}\sigma_k\nabla T_k(u_n)dxdt.$$

Proof.
We use the sequence $(T_k(u))_\mu$ of approximation of $T_k(u)$ and plug the test function $S'_m(\omega_n)(T_k(\omega_n) - (T_k(\omega_n))_\mu)$ for $m > 0$ and $\mu > 0$, where $\omega_n = b(x,u_n) - b(x,\psi)$.
For fixed $m \ge 0$, let $W_n^\mu = T_k(\omega_n) - (T_k(\omega))_\mu$, we obtain:

$$\left\{\begin{array}{l} \displaystyle\int_0^T < \frac{\partial b_n(x,u_n)}{\partial t}, S'_m(\omega_n)W_n^\mu > dt + \int_{Q_T} a_n(u_n,\nabla u_n))S'_m(\omega_n)\nabla W_n^\mu dxdt \\ \displaystyle + \int_{Q_T} a_n(u_n,\nabla u_n)S''_m(\omega_n)\nabla(\omega_n)W_n^\mu dxdt + \int_{Q_T} g_n(u_n)S'_m(\omega_n)\nabla W_n^\mu dxdt \\ \displaystyle + \int_{Q_T} g_n(u_n)S''_m(\omega_n)\nabla(\omega_n)W_n^\mu dxdt - \int_{Q_T} nT_n(u_n-\psi)^- S'_m(\omega_n)W_n^\mu dxdt \\ \displaystyle = \int_{Q_T} f_nS'_m(\omega_n)W_n^\mu dxdt \end{array}\right. \tag{4.30}$$

Now we pass to the limit in as $n \to +\infty$,$\mu \to +\infty$ and then $m \to +\infty$. In order to perform this task we prove below the following results for any fixed $k \ge 0$.

$$\liminf_{\mu\to+\infty}\lim_{n\to+\infty}\int_0^T < \frac{\partial b_n(x,u_n)}{\partial t}; S'_m(\omega_n)W_n^\mu > dt \ge 0 \quad \text{for any} \quad m \ge k, \tag{4.31}$$

$$\lim_{\mu\to+\infty}\lim_{n\to+\infty}\int_{Q_T} g_n(u_n))S'_m(\omega_n)\nabla W_n^\mu dxdt = 0 \quad \text{for any} \quad m \ge 1, \tag{4.32}$$

$$\lim_{\mu\to+\infty}\lim_{n\to+\infty}\int_0^T\int_{Q_T} g_n(u_n)S''_m(\omega_n)\nabla(\omega_n)W_n^\mu dxdt = 0, \tag{4.33}$$

$$\lim_{m\to+\infty}\limsup_{\mu\to+\infty}\limsup_{n\to+\infty}\int_{Q_T} a_n(u_n,\nabla u_n))S''_m(\omega_n)\nabla(\omega_n)W_n^\mu dxdt = 0, \tag{4.34}$$

$$\lim_{\mu\to+\infty}\lim_{n\to+\infty}\int_{Q_T} nT_n(u_n-\psi)^- S'_m(\omega_n)W_n^\mu dxdt = 0, \tag{4.35}$$

$$\lim_{\mu\to+\infty}\lim_{n\to+\infty}\int_{Q_T} f_nS'_m(\omega_n)W_n^\mu dxdt = 0. \tag{4.36}$$

Proof of (4.31):
The function $S_m \in L^\infty(\mathbb{R})$ and is increasing. We have $m \ge k$, $S_m(r) = r$ for $|r| \le k$, and $supp(S'_m)$ is compact. In view of the definition of W_n^μ, Lemma 4.2 applies with $S = S_m$ for fixed $m \ge k$. As a consequence (4.31) holds true.
Proof of (4.32):
Let us recall the main properties of W_n^μ.
For fixed $\mu > 0$, W_n^μ converge to $W^\mu = T_k(b(x,u) - b(x,\psi)) - (T_k(b(x,u) - b(x,\psi)))_\mu$ weakly in $W_0^{1,x}L_M(Q_T)$ as $n \to +\infty$.
Remark that

$$\|W_n^\mu\|_{L^\infty(Q_T)} \le 2k \quad \text{for any} \quad n,\mu > 0, \tag{4.37}$$

then we deduce that

$$W_n^\mu \to W^\mu \quad \text{a.e. in } Q_T \quad \text{and} \quad L^\infty(Q_T), \tag{4.38}$$

weakly-* when $n \to +\infty$. One had $suppS'_m \subset [-2m,-m] \cap [m,2m]$ for any fixed $m \geq 1$ and $n > m+1$, then $g_n(u_n)S'_m(\omega_n)\nabla W_n^\mu = g_n(u_n)S'_m(T_{2m}(\omega_n))\nabla W_n^\mu$ a.e. in Q_T since $suppS'_m \subset [-2m,2m]$.
On the other hand $g_n(u_n)S'_m(T_{2m}(\omega_n)) \to g(u)S'_m(T_{2m}(b(x,u)-b(x,\psi)))$ a.e. in Q_T and
$|g_n(u_n))S'_m(T_{2m}(\omega_n))| \leq 2m[q(x,t)+c(x,t)\overline{M}_x^{-1}M(x,\frac{\alpha_0}{\lambda}m)]$ for $m \geq 1$.
As W_n^μ converge to W^μ weakly in $W_0^{1,x}L_M(Q_T)$ as $\mu \to +\infty$, we obtain (4.32).
Proof of (4.33):
For any fixed $m \geq 1$ and $n > 2m$

$$g_n(u_n)S''_m(\omega_n)\nabla(\omega_n)W_n^\mu = g_n(u_n)S''_m(T_{2m}(\omega_n))\nabla T_{2m}(\omega_n)W_n^\mu, \quad \text{a.e. in} \quad Q_T.$$

As in the previous step it is possible to pass to the limit for $n \to +\infty$ since by (4.37) and (4.38).

$$g_n(u_n)S''_m(T_{2m}(\omega_n))W_n^\mu \to g(u))S''_m(T_{2m}(b(x,u)-b(x,\psi)))W^\mu, \quad \text{a.e. in} \quad Q_T.$$

Since

$$|g(u)S''_m(T_{2m}(b(x,u)-b(x,\psi)))W^\mu| \leq \frac{3}{m}[c(x,t)\overline{M}_x^{-1}M(x,\frac{\alpha_0}{\lambda}m)] \quad \text{a.e. in} \quad Q_T.$$

And W^μ converge to 0 in $W_0^{1,x}L_M(Q_T)$ for the modular convergence, we obtain (4.33).
Proof of (4.34):
In view of the definition of S_m, we have
$suppS'_m \subset [-2m,-m] \cap [m,2m]$ for any fixed $m \geq 1$, as a consequence

$$\begin{aligned}
&|\int_{Q_T} a_n(u_n,\nabla u_n))S''_m(\omega_n)\nabla(\omega_n)W_n^\mu dxdt| \\
&\leq \|S''_m\|_{L^\infty(\mathbb{R})}\|W_n^\mu\|_{L^\infty(Q_T)}\int_{m\leq|(\omega_n|\leq 2m} a_n(u_n,\nabla u_n))\nabla(\omega_n)dxdt \\
&\leq 3\|W_n^\mu\|_{L^\infty(Q_T)}\frac{1}{m}\int_{|(\omega_n|\leq 2m} a_n(u_n,\nabla u_n))\nabla(\omega_n)\,dx\,dt \\
&\leq 3\|W_n^\mu\|_{L^\infty(Q_T)}\frac{1}{m}\int_{|u_n|\leq s} a_n(u_n,\nabla u_n)\nabla u_n dxdt,
\end{aligned}$$

with $s = \frac{2m}{\lambda} + \|\psi\|_{L^\infty(Q_T)}$ for any $m \geq 1$, any $n > 2m$ and any $\mu > 0$.
By Lemma (4.22) it is possible to establish (4.34).
proof of (4.35): By (4.5), the pointwise convergence of u_n and W_n^μ and its boundlessness it is possible to pass to the limit for $n \to +\infty$ for any $\mu > 0$ and any $m \geq 1$

$$\lim_{n\to+\infty}\int_{Q_T} f_n S'_m(\omega_n)W_n^\mu dxdt = \int_{Q_T} fS'_m(b(x,u)-b(x,\psi))W^\mu dxdt.$$

Now for fixed $m \geq 1$, using that
$\|(T_k(u))_\mu\|_{L^\infty(Q_T)} \leq \max(\|T_k(u)\|_{L^\infty(Q_T)}, \|w_0^\mu\|_{L^\infty(\Omega)}) \leq k$, $\forall\mu > 0, \forall k > 0$, it is easy to deduce (4.35).
Proof of (4.36): Similar to (4.35).
Finally we adopt the same technics used in [17] to obtain the Lemma 4.4.

Lemma 4.4. *The subsequence of u_n satisfies for any $k \geq 0$*

$$\lim_{n\to+\infty}\int_{Q_T}\big(a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u))\big)\big(\nabla T_k(u_n) - \nabla T_k(u)\big)dxdt = 0, \tag{4.39}$$

$$\sigma_k = a(T_k(u),\nabla T_k(u)) \quad \text{a.e. in} \quad Q_T, \quad \text{as} \quad n \to +\infty \tag{4.40}$$

$$a(T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n) \rightharpoonup a(T_k(u),\nabla T_k(u))\nabla T_k(u), \quad \text{weakly in} \quad L^1(Q_T). \tag{4.41}$$

Remark 4.3. *Note the (4.39) allows us to deduce that $\nabla u_n \to \nabla u$ a.e. in Q_T. This complete the proof of proposition 4.1 and Theorem 4.1.*

Appendix

Proof of Lemma 4.2

Let $w_n = \beta(x,u_n) - \beta(x,\psi)$ and $w = \beta(x,u) - \beta(x,\psi)$, then

$$\int_0^T \langle \frac{\partial \beta(x,u_n)}{\partial t}, S'(w_n)(T_k(w_n) - (T_k(w))_\mu)\rangle dt$$
$$= \int_0^T \langle \frac{\partial w_n}{\partial t}, S'(w_n)(T_k(w_n) - (T_k(w))_\mu)\rangle dt$$
$$+ \int_0^T \langle \frac{\partial \beta(x,\psi)}{\partial t}, S'(w_n)(T_k(w_n) - (T_k(w))_\mu)\rangle dt. \tag{.42}$$

On the first time we use (4.18), (4.19), $supp(S')$ is compact and tends n to $+\infty$ and we have

$$\lim_{n\to+\infty} \int_0^T \langle \frac{\partial \beta(x,\psi)}{\partial t}, S'(w_n)(T_k(w_n) - (T_k(w))_\mu)\rangle dt$$
$$= \int_0^T \langle \frac{\partial \beta(x,\psi)}{\partial t}, S'(w)(T_k(w) - (T_k(w))_\mu)\rangle dt.$$

On the second time we use the fact that $(T_k(w))_\mu \to T_k(w)$ for the modular convergence as $\mu \to +\infty$ in $W^{1,x}L_M(Q_T)$, then

$$\liminf_{\mu\to+\infty} \lim_{n\to+\infty} \int_0^T \langle \frac{\partial \beta(x,\psi)}{\partial t}, S'(w_n)(T_k(w_n) - (T_k(w))_\mu)\rangle dt = 0 \tag{.43}$$

On the other hand we write the first term on the right-hand side of (.42) as

$$\int_0^T \langle \frac{\partial w_n}{\partial t}, S'(w_n)(T_k(w_n) - (T_k(w))_\mu)\rangle dt = \int_0^T \langle \frac{\partial w_n}{\partial t}, S'(w_n)T_k(w_n)\rangle dt$$
$$- \int_0^T \langle \frac{\partial w_n}{\partial t}, S'(w_n)(T_k(w))_\mu\rangle dt$$
$$= I_1^n + I_2^{n,\mu}.$$

We denote $B_{S,k}(r) = \int_0^r S'(\sigma)T_k(\sigma)d\sigma$ and $B_S(r) = \int_0^r S'(\sigma)d\sigma$.

Then, for I_1^n we can pass to limit as $n \to +\infty$ and we deduce

$$\lim_{n\to+\infty} I_1^n = \lim_{n\to+\infty} \int_\Omega [B_{S,k}(w_n(T)) - B_{S,k}(w_n(0))]dx$$
$$= \int_\Omega [B_{S,k}(w(T)) - B_{S,k}(w(0))]dx. \tag{.44}$$

By the definition of $(T_k(w))_\mu$, the second integral $I_2^{n,\mu}$ can be written as

$$I_2^{n,\mu} = -\int_0^T \langle \frac{\partial w_n}{\partial t}, S'(w_n)(T_k(w))_\mu\rangle dt$$
$$= -\int_\Omega [B_S(w_n(T))(T_k(w))_\mu(T)dx - B_S(w_n(0))(T_k(w))_\mu(0)]dx$$
$$+ \int_\Omega \int_0^T B_S(w_n)\frac{\partial (T_k(w))_\mu}{\partial t} dtdx$$
$$= -\int_\Omega [B_S(w_n(T))(T_k(w))_\mu(T)dx - B_S(w_n(0))(T_k(w))_\mu(0)]dx$$
$$+ \mu \int_\Omega \int_0^T B_S(w_n)(T_k(w) - (T_k(w))_\mu)dtdx.$$

On the same way, we pass to limit as $n \to +\infty$, we obtain

$$\lim_{n\to+\infty} I_2^{n,\mu} = -\int_\Omega [B_S(w(T))(T_k(w))_\mu(T) - B_S(w(0))(T_k(w))_\mu(0)]dx$$

$$+\mu \int_\Omega \int_0^T B_S(w)(T_k(w)-(T_k(w))_\mu)dtdx. \tag{.45}$$

Now by the limit (4.29) letting $\mu \to +\infty$, we get

$$\begin{aligned}\lim_{\mu\to+\infty} &\int_\Omega [B_S(w(T))(T_k(w))_\mu(T)-\beta(w(0))(T_k(w))_\mu(0)]dx \\ &= \int_\Omega [B_S(w(T))T_k(w)(T)-\beta(w(0))T_k(w)(0)]dx. \end{aligned}\tag{.46}$$

The right-hand of (.45) can be written as

$$\begin{aligned}\mu \int_\Omega \int_0^T & B_S(w)(T_k(w))-(T_k(w))_\mu))dxdt \\ &= \mu \int_\Omega \int_0^T (B_S(w)-B_S(T_k(w))(T_k(w)-(T_k(w))_\mu)dxdt \\ &+ \mu \int_\Omega \int_0^T (B_S(T_k(w))-B_S((T_k(w))_\mu)(T_k(w)-(T_k(w))_\mu)dxdt \\ &+ \mu \int_\Omega \int_0^T B_S((T_k(w))_\mu)(T_k(w)-(T_k(w))_\mu)dxdt \\ &= J_1^\mu + J_2^\mu + J_3^\mu,\end{aligned}$$

where

$$\begin{aligned} J_1^\mu &= \mu \int_\Omega \int_0^T (B_S(w)-B_S(T_k(w)))(T_k(w))-(T_k(w))_\mu)dxdt \\ &= \mu \int_{\{w>k\}} \int_0^T (B_S(w)-B_S(k))(k-(T_k(w))_\mu)dxdt \\ &\quad +\mu \int_{\{w<-k\}} \int_0^T (B_S(w)-B_S(-k))(-k-(T_k(w))_\mu)dxdt,\end{aligned}$$

as $B_S(z)$ is non-decreasing for z and $-k \le T_k(z) \le k$. It follows that $J_1^\mu \ge 0$ and also $J_2^\mu \ge 0$.
For the integral J_3^μ, one has

$$\begin{aligned} J_3^\mu &= \mu \int_\Omega \int_0^T B_S((T_k(w))_\mu)(T_k(w)-(T_k(w))_\mu)dtdx \\ &= \int_\Omega \int_0^T B_S((T_k(w))_\mu)\frac{\partial T_k(w))_\mu}{\partial t}dtdx \\ &= \int_\Omega [\overline{B}_S((T_k(w))_\mu)(T)-\overline{B}_S((T_k(w))_\mu)(0)]dx,\end{aligned}$$

where $\overline{B}_S(z) = \int_0^z B_S(s)ds$.
Since $(T_k(w))_\mu \to T_k(w)$ a.e. in Q_T and $|(T_k(w))_\mu| \le k$, the Lebesgue's convergence theorem shows that

$$\lim_{\mu\to+\infty} J_3^\mu = \int_\Omega [\overline{B}_S(x,T_k(w))(T)-\overline{B}_S(x,T_k(w))(0)]dx. \tag{.47}$$

As a consequence of (.46) and (.47), we obtain

$$\begin{aligned}\lim_{\mu\to+\infty}\lim_{n\to+\infty} I_2^{n,\mu} &\ge \int_\Omega [B_S(x,T_k(w))(T)-B_S(x,T_k(w))(0)]dx \\ &+ \int_\Omega [\overline{B}_S(x,T_k(w))(T)-\overline{B}_S(x,T_k(w))(0)]dx.\end{aligned}\tag{.48}$$

Finally, by (.43), (.44), (.47) and (.48) we deduce

$$\lim_{\mu\to+\infty}\lim_{n\to+\infty}\int_0^T \langle \frac{\partial\beta(x,u_n)}{\partial t}, S'(w_n)(T_k(w_n)-(T_k(w))_\mu)\rangle dt$$

$$\geq \int_\Omega [B_{S,k}(w(T)) - B_{S,k}(w(0)]dx$$
$$- \int_\Omega [B_S(w(T))T_k(w)(T) - B_S(w(0))T_k(w))(0)]dx$$
$$+ \int_\Omega [\overline{B}_S(T_k(w))(T) - \overline{B}_S(T_k(w)(0)]dx, \quad (.49)$$

and we know that

$$\overline{B}_S(T_k(z)) = B_S(T_k(z))T_k(z) - B_{S,k}(T_k(z))$$

Indeed

$$\begin{aligned}\overline{B}_S(T_k(z)) &= \int_0^{T_k(z)} B_S(s)ds = \int_0^{T_k(z)} \int_0^s S'(\sigma)d\sigma ds\\ &= [s\int_0^s S'(\sigma)d\sigma]_0^{T_k(z)} - \int_0^{T_k(z)} rS'(r)dr\\ &= B_S(T_k(z))T_k(z) - B_{S,k}(T_k(z)),\end{aligned}$$

then using the definition of truncation $T_k(z)$, one has

$$\int_\Omega [B_{S,k}(x, w(T)) - B_{S,k}(x, w(0)]dx - \int_\Omega [B_S(w(T))T_k(w)(T) - B_S(w(0))T_k(w))(0)]dx$$
$$+ \int_\Omega [\overline{B}_S(T_k(w))(T) - \overline{B}_S(T_k(w))(0)]dx = 0.$$

This completes the proof.

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