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Opial type inequalities for double Riemann-Stieltjes integrals

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Abstract. In this paper, we establish some Opial type inequalities for Riemann-Stieltjes integrals of functions with two variables. The obtained inequalities generalize those previously demonstrated (see [2]) **2010 Mathematics Subject Classification.** Primary: 26D15, 26D10, 26B15.
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1. Introduction

Opial's inequality and its generalizations, extensions and discretizations, play a fundamental role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations as well as difference equations.

In 1960, Z. Opial established the following interesting integral inequality which is known as Opial inequality in the literature[12]:

Theorem 1.1. *Let $x(t) \in C^{(1)}[0,h]$ be such that $x(0) = x(h) = 0$, and $x(t) > 0$ in $(0,h)$. Then, the following inequality holds*

$$\int_0^h |x(t)x'(t)|\, dt \leq \frac{h}{4} \int_0^h (x'(t))^2\, dt \tag{1.1}$$

The constant $h/4$ is the best possible

Over the years, a large number of papers have been appeared in the literature which deals various generalizations, as well as discrete form of Opial inequality and its generalizations, for some of them see [4], [5], [8], [9], [13]-[16], [26]-[28].

On the other hand, G. S. Yang gave the following generalization (1.1) for the function of two variables [29]

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Theorem 1.2. *If $f(t,s)$, $f_1(t,s)$ and $f_{12}(t,s)$ are continuous functions on $[a,b]\times[c,d]$ and if $f(a,s)=f(b,s)=f_1(t,c)=f_1(t,d)=0$ for $a\le t\le b$, $c\le s\le d$, then*

$$\int_a^b\int_c^d |f(t,s)|\,|f_{12}(t,s)|\,dsdt \le \frac{(b-a)\,(d-c)}{8}\int_a^b\int_c^d |f_{12}(t,s)|^2\,dsdt \tag{1.2}$$

where

$$f_1(t,s)=\frac{\partial}{\partial t}f(t,s) \text{ and } f_{12}(s,t)=\frac{\partial^2}{\partial t\partial s}f(t,s).$$

Over the years, many articles are dedicated to the generalizations of inequality (1.2). B. G. Pachpatte published some papers which focus on the generalizations of the inequality (1.2). For some of these generalizations, see [17]-[22]. Moreover using two functions and their partial derivatives, W. S. Cheung established some generalizations of the inequality (1.2) in [6]. For the other Opial type inequalities in higher dimension, see [1]-[3], [7], [23]-[25].

The following Lemma proved by Moricz in [11] is usefull to obtain our main result:

Lemma 1.1. *(Integrating by parts) If $f(t,s)$ is continuous on renctangle $Q=[a,b]\times[c,d]$ and $\alpha(t,s)\in BV_H(Q)$, then $\alpha(t,s)$ is integrable with respect to $f(t,s)$ over Q in the Riemann-Stieltjes sense, and*

$$\begin{aligned}\int_a^b\int_c^d f(t,s)d_td_s\alpha(t,s) &= \int_a^b\int_c^d \alpha(t,s)d_td_sf(t,s)\\ &\quad-\int_a^b \alpha(t,d)d_tf(t,d)+\int_a^b \alpha(t,c)d_tf(t,c)\\ &\quad-\int_c^d \alpha(b,s)d_sf(b,s)+\int_c^d \alpha(a,s)d_sf(a,s)\\ &= f(b,d)\alpha(b,d)-f(b,c)\alpha(b,c)-f(a,d)\alpha(a,d)+f(a,c)\alpha(a,c).\end{aligned}$$

2. Generalized Opial type inequalities

In this section, we establish some Opial type inequalities for double Riemann-Stieltjes integrals.

Theorem 2.1. *Let $f(s,t)$, $f_1(s,t)$, $f_{12}(s,t)$, $g(s,t)$, $g_1(s,t)$, $g_{12}(s,t)$ and $\alpha(x,y)$ be continuous on $\Delta:=[a,b]\times[c,d]$ and let $f(x,y)$ and $g(x,y)$ be integrable with respect to $\alpha(x,y)$ over Δ in the Riemann-Stieltjes sense. If $g(a,s)=g(b,s)=g_1(t,c)=g_1(t,d)=0$ for $(t,s)\in\Delta$, then for all $(x,y)\in\Delta$ we have*

$$\begin{aligned}&\int_a^b\int_c^d |f_{12}(t,s)g(t,s)|\,d_td_s\alpha(t,s) \qquad (2.1)\\ \le\;&\frac{1}{4}\left[(b-a)\int_a^b\int_c^d Q(s,y)\,|f_{12}(t,s)|^2\,d_td_s\alpha(t,s)+(d-c)\int_a^b\int_c^d P(t,x)\,|f_{12}(t,s)|^2\,d_td_s\alpha(t,s)\right]^{\frac{1}{2}}\\ &\times\left[\int_a^b\int_c^d |\alpha(b,y)-\alpha(a,y)-\alpha(b,s)+\alpha(a,s)|\,|g_{12}(t,s)|^2\,dsdt\right.\\ &\left.+\int_a^b\int_c^d |\alpha(t,d)-\alpha(x,d)-\alpha(t,c)+\alpha(x,c)|\,|g_{12}(t,s)|^2\,dsdt\right]^{\frac{1}{2}}\end{aligned}$$

$$\leq \frac{1}{8}\left[(b-a)\int_a^b\int_c^d Q(s,y)\left|f_{12}(t,s)\right|^2 d_t d_s\alpha(t,s)+(d-c)\int_a^b\int_c^d P(t,x)\left|f_{12}(t,s)\right|^2 d_t d_s\alpha(t,s)\right.$$

$$+\int_a^b\int_c^d \left|\alpha(b,y)-\alpha(a,y)-\alpha(b,s)+\alpha(a,s)\right|\left|g_{12}(t,s)\right|^2 dsdt$$

$$\left.+\int_a^b\int_c^d \left|\alpha(t,d)-\alpha(x,d)-\alpha(t,c)+\alpha(x,c)\right|\left|g_{12}(t,s)\right|^2 dsdt\right]$$

where

$$P(t,x)=\begin{cases} t-a, & a\leq t\leq x \\ b-t, & x\leq t\leq b\end{cases} \quad and \quad Q(s,y)=\begin{cases} s-c, & c\leq s\leq y \\ d-s, & y\leq s\leq d.\end{cases}$$

Proof. In order to prove Theorem 2.1, we consider the following four cases:

Case I: Let $g(a,s)=g_1(t,c)=0$ for $(t,s)\in[a,b]\times[c,d]$.

Since $g(a,s)=g_1(t,c)=0$, we can write

$$g(t,s)=\int_a^t\int_c^s g_{12}(u,v)dudv,$$

then we have

$$\int_a^b\int_c^d \left|f_{12}(t,s)g(t,s)\right| d_t d_s\alpha(t,s) \tag{2.2}$$

$$=\int_a^b\int_c^d (t-a)^{\frac{1}{2}}(s-c)^{\frac{1}{2}}\left|f_{12}(t,s)\right|(t-a)^{-\frac{1}{2}}(s-c)^{-\frac{1}{2}}\left|g(t,s)\right| d_t d_s\alpha(t,s)$$

$$=\int_a^b\int_c^d (t-a)^{\frac{1}{2}}(s-c)^{\frac{1}{2}}\left|f_{12}(t,s)\right|(t-a)^{-\frac{1}{2}}(s-c)^{-\frac{1}{2}}\left|\int_a^t\int_c^s g_{12}(u,v)dudv\right| d_t d_s\alpha(t,s).$$

By using Cauchy-Schwarz inequality for Riemann integrals, we get

$$(t-a)^{-\frac{1}{2}}(s-c)^{-\frac{1}{2}}\left|\int_a^t\int_c^s g_{12}(u,v)dudv\right| \tag{2.3}$$

$$\leq (t-a)^{-\frac{1}{2}}(s-c)^{-\frac{1}{2}}\left(\int_a^t\int_c^s dudv\right)^{\frac{1}{2}}\left(\int_a^t\int_c^s \left|g_{12}(u,v)\right|^2 dudv\right)^{\frac{1}{2}}$$

$$=\left(\int_a^t\int_c^s \left|g_{12}(u,v)\right|^2 dudv\right)^{\frac{1}{2}}.$$

Substituting the inequality (2.3) in (2.2), we obtain

$$\int_a^b\int_c^d \left|f_{12}(t,s)g(t,s)\right| d_t d_s\alpha(t,s) \tag{2.4}$$

$$\leq \int_a^b \int_c^d (t-a)^{\frac{1}{2}} (s-c)^{\frac{1}{2}} |f_{12}(t,s)| \left(\int_a^t \int_c^s |g_{12}(u,v)|^2 \, dudv \right)^{\frac{1}{2}} d_t d_s \alpha(t,s).$$

By using Cauchy-Schwarz inequality for Riemann-Stieltjes integrals, we obtain

$$\begin{aligned} & \int_a^b \int_c^d |f_{12}(t,s)g(t,s)| \, d_t d_s \alpha(t,s) \qquad (2.5) \\ \leq & \left(\int_a^b \int_c^d (t-a)(s-c) |f_{12}(t,s)|^2 d_t d_s \alpha(t,s) \right)^{\frac{1}{2}} \left(\int_a^b \int_c^d \left(\int_a^t \int_c^s |g_{12}(u,v)|^2 \, dudv \right) d_t d_s \alpha(t,s) \right)^{\frac{1}{2}}. \end{aligned}$$

By the integration by parts for Riemann-Stieltjes integrals (Lemma 1.1), we establish

$$\begin{aligned} & \int_a^b \int_c^d \left(\int_a^t \int_c^s |g_{12}(u,v)|^2 \, dudv \right) d_t d_s \alpha(t,s) \qquad (2.6) \\ = & -\int_a^b \int_c^d \alpha(t,d) |g_{12}(t,v)|^2 \, dvdt - \int_a^b \int_c^d \alpha(b,s) |g_{12}(u,s)|^2 \, dsdu \\ & + \alpha(b,d) \int_a^b \int_c^d |g_{12}(u,v)|^2 \, dvdu + \int_a^b \int_c^d \alpha(t,s) |g_{12}(t,s)|^2 \, dsdt \\ = & \int_a^b \int_c^d [-\alpha(t,d) - \alpha(b,s) + \alpha(b,d) + \alpha(t,s)] |g_{12}(t,v)|^2 \, dsdt. \end{aligned}$$

By the using the equality (2.6) in (2.4), we obtain the following inequality

$$\begin{aligned} & \int_a^b \int_c^d |f_{12}(t,s)g(t,s)| \, d_t d_s \alpha(t,s) \qquad (2.7) \\ \leq & \left(\int_a^b \int_c^d (t-a)(s-c) |f_{12}(t,s)|^2 d_t d_s \alpha(t,s) \right)^{\frac{1}{2}} \\ & \times \left(\int_a^b \int_c^d [-\alpha(t,d) - \alpha(b,s) + \alpha(b,d) + \alpha(t,s)] |g_{12}(t,v)|^2 \, dsdt \right)^{\frac{1}{2}}. \end{aligned}$$

Case II: Let $g(a,s) = g_1(t,d) = 0$ for $(t,s) \in [a,b] \times [c,d]$.

We get

$$g(t,s) = -\int_a^t \int_s^d g_{12}(u,v) dudv$$

for $(t,s) \in [a,b] \times [c,d]$. Then by Cauchy-Schwarz inequality for Riemann integrals, it follows that

$$\int_a^b \int_c^d |f_{12}(t,s)g(t,s)| \, d_t d_s \alpha(t,s)$$

$$= \int_a^b \int_c^d (t-a)^{\frac{1}{2}} (d-s)^{\frac{1}{2}} |f_{12}(t,s)| (t-a)^{-\frac{1}{2}} (d-s)^{-\frac{1}{2}} \left| \int_a^t \int_s^d g_{12}(u,v) dudv \right| d_t d_s \alpha(t,s)$$

$$\leq \int_a^b \int_c^d (t-a)^{\frac{1}{2}} (d-s)^{\frac{1}{2}} |f_{12}(t,s)| \left(\int_a^b \int_c^d |g_{12}(u,v)|^2 dudv \right)^{\frac{1}{2}} d_t d_s \alpha(t,s).$$

By applying Cauchy-Schwarz inequality for Riemann-Stieltjes integrals, we get

$$\int_a^b \int_c^d |f_{12}(t,s) g(t,s)| d_t d_s \alpha(t,s)$$

$$\leq \left(\int_a^b \int_c^d (t-a)(d-s) |f_{12}(t,s)|^2 d_t d_s \alpha(t,s) \right)^{\frac{1}{2}} \left(\int_a^b \int_c^d \left(\int_a^b \int_c^d |g_{12}(u,v)|^2 dudv \right) d_t d_s \alpha(t,s) \right)^{\frac{1}{2}}.$$

By integration by parts for Riemann-Stieltjes integrals, one can show that

$$\int_a^b \int_c^d \left(\int_a^b \int_c^d |g_{12}(u,v)|^2 dudv \right) d_t d_s \alpha(t,s)$$

$$= \int_a^b \int_c^d [\alpha(t,c) + \alpha(b,s) - \alpha(b,c) - \alpha(t,s)] |g_{12}(u,v)|^2 dsdt.$$

Thus, we have

$$\int_a^b \int_c^d |f_{12}(t,s) g(t,s)| d_t d_s \alpha(t,s) \tag{2.8}$$

$$\leq \left(\int_a^b \int_c^d (t-a)(d-s) |f_{12}(t,s)|^2 d_t d_s \alpha(t,s) \right)^{\frac{1}{2}}$$

$$\times \left(\int_a^b \int_c^d [\alpha(t,c) + \alpha(b,s) - \alpha(b,c) - \alpha(t,s)] |g_{12}(u,v)|^2 dsdt \right)^{\frac{1}{2}}.$$

Case III: Let $g(b,s) = g_1(t,c) = 0$ for $(t,s) \in [a,b] \times [c,d]$. Then we have

$$g(t,s) = -\int_t^b \int_c^s g_{12}(u,v) dudv.$$

Case IV: Let $g(b,s) = g_1(t,d) = 0$ for $(t,s) \in [a,b] \times [c,d]$. we can write

$$g(t,s) = \int_t^b \int_s^d g_{12}(u,v) dudv.$$

By following similar to those in proof of (2.7) and (2.8), but with suitable modifications, we establish the following inequalities in Case III and Case IV:

$$\int_a^b\int_c^d |f_{12}(t,s)g(t,s)|\, d_t d_s\alpha(t,s) \tag{2.9}$$
$$\leq \left(\int_a^b\int_c^d (b-t)\,(s-c)\,|f_{12}(t,s)|^2\, d_t d_s\alpha(t,s)\right)^{\frac{1}{2}}$$
$$\times\left(\int_a^b\int_c^d [\alpha(a,s)+\alpha(t,d)-\alpha(a,d)-\alpha(t,s)]\,|g_{12}(t,s)|^2\, dsdt\right)^{\frac{1}{2}},$$

$$\int_a^b\int_c^d |f_{12}(t,s)g(t,s)|\, d_t d_s\alpha(t,s) \tag{2.10}$$
$$\leq \left(\int_a^b\int_c^d (b-t)\,(d-s)\,|f_{12}(t,s)|^2\, d_t d_s\alpha(t,s)\right)^{\frac{1}{2}}$$
$$\times\left(\int_a^b\int_c^d [-\alpha(t,c)-\alpha(a,s)+\alpha(a,c)+\alpha(t,s)]\,|g_{12}(t,s)|^2\, dsdt\right)^{\frac{1}{2}},$$

respectively.

Since $g(a,s)=g_1(t,c)=0$ for $(t,s)\in[a,b]\times[c,d]$, if we write the inequality (2.7) for the rectangles $[a,b]\times[c,y]$ and $[a,x]\times[c,d]$ for $(x,y)\in[a,b]\times[c,d]$, then we have

$$\int_a^b\int_c^y |f_{12}(t,s)g(t,s)|\, d_t d_s\alpha(t,s) \tag{2.11}$$
$$\leq \left(\int_a^b\int_c^y (t-a)\,(s-c)\,|f_{12}(t,s)|^2\, d_t d_s\alpha(t,s)\right)^{\frac{1}{2}}$$
$$\times\left(\int_a^b\int_c^y [-\alpha(t,y)-\alpha(b,s)+\alpha(b,y)+\alpha(t,s)]\,|g_{12}(t,v)|^2\, dsdt\right)^{\frac{1}{2}}.$$

and

$$\int_a^x\int_c^d |f_{12}(t,s)g(t,s)|\, d_t d_s\alpha(t,s) \tag{2.12}$$
$$\leq \left(\int_a^x\int_c^d (t-a)\,(s-c)\,|f_{12}(t,s)|^2\, d_t d_s\alpha(t,s)\right)^{\frac{1}{2}}$$

$$\times\left(\int_a^x\int_c^d [-\alpha(t,d)-\alpha(x,s)+\alpha(x,d)+\alpha(t,s)]\,|g_{12}(t,v)|^2\,dsdt\right)^{\frac{1}{2}}.$$

respectively.

As $g(a,s)=g_1(t,d)=0$ for $(t,s)\in[a,b]\times[c,d]$, if we apply the inequality (2.8) for the rectangles $[a,b]\times[y,d]$ and $[a,x]\times[c,d]$ for $(x,y)\in[a,b]\times[c,d]$, then we get

$$\int_a^b\int_y^d |f_{12}(t,s)g(t,s)|\,d_td_s\alpha(t,s) \tag{2.13}$$

$$\leq \left(\int_a^b\int_y^d (t-a)\,(d-s)\,|f_{12}(t,s)|^2\,d_td_s\alpha(t,s)\right)^{\frac{1}{2}}$$

$$\times\left(\int_a^b\int_y^d [\alpha(t,y)+\alpha(b,s)-\alpha(b,y)-\alpha(t,s)]\,|g_{12}(u,v)|^2\,dsdt\right)^{\frac{1}{2}}.$$

and

$$\int_a^x\int_c^d |f_{12}(t,s)g(t,s)|\,d_td_s\alpha(t,s) \tag{2.14}$$

$$\leq \left(\int_a^x\int_c^d (t-a)\,(d-s)\,|f_{12}(t,s)|^2\,d_td_s\alpha(t,s)\right)^{\frac{1}{2}}$$

$$\times\left(\int_a^x\int_c^d [\alpha(t,c)+\alpha(b,s)-\alpha(b,c)-\alpha(t,s)]\,|g_{12}(u,v)|^2\,dsdt\right)^{\frac{1}{2}}.$$

Similarly, since $g(b,s)=g_1(t,c)=0$ for $(t,s)\in[a,b]\times[c,d]$, if we write the inequality (2.9) for the rectangles $[a,b]\times[c,y]$ and $[x,b]\times[c,d]$ for $(x,y)\in[a,b]\times[c,d]$, then we have

$$\int_a^b\int_c^y |f_{12}(t,s)g(t,s)|\,d_td_s\alpha(t,s) \tag{2.15}$$

$$\leq \left(\int_a^b\int_c^y (b-t)\,(s-c)\,|f_{12}(t,s)|^2\,d_td_s\alpha(t,s)\right)^{\frac{1}{2}}$$

$$\times\left(\int_a^b\int_c^y [\alpha(a,s)+\alpha(t,y)-\alpha(a,y)-\alpha(t,s)]\,|g_{12}(t,s)|^2\,dsdt\right)^{\frac{1}{2}},$$

and

$$\int_x^b\int_c^d |f_{12}(t,s)g(t,s)|\,d_td_s\alpha(t,s) \tag{2.16}$$

$$\leq \left(\int_x^b\int_c^d (b-t)\,(s-c)\,|f_{12}(t,s)|^2\, d_t d_s\alpha(t,s)\right)^{\frac{1}{2}}$$

$$\times\left(\int_x^b\int_c^d [\alpha(x,s)+\alpha(t,d)-\alpha(x,d)-\alpha(t,s)]\,|g_{12}(t,s)|^2\, dsdt\right)^{\frac{1}{2}},$$

Finally, as $g(b,s) = g_1(t,d) = 0$ for $(t,s) \in [a,b]\times[c,d]$, if we apply the inequality (2.10) for the rectangles $[a,b]\times[y,d]$ and $[x,b]\times[c,d]$ for $(x,y)\in[a,b]\times[c,d]$, then we have

$$\int_a^b\int_y^d |f_{12}(t,s)g(t,s)|\, d_t d_s\alpha(t,s) \tag{2.17}$$

$$\leq \left(\int_a^b\int_y^d (b-t)\,(d-s)\,|f_{12}(t,s)|^2\, d_t d_s\alpha(t,s)\right)^{\frac{1}{2}}$$

$$\times\left(\int_a^b\int_y^d [-\alpha(t,y)-\alpha(a,s)+\alpha(a,y)+\alpha(t,s)]\,|g_{12}(t,s)|^2\, dsdt\right)^{\frac{1}{2}},$$

and

$$\int_x^b\int_c^d |f_{12}(t,s)g(t,s)|\, d_t d_s\alpha(t,s) \tag{2.18}$$

$$\leq \left(\int_x^b\int_c^d (b-t)\,(d-s)\,|f_{12}(t,s)|^2\, d_t d_s\alpha(t,s)\right)^{\frac{1}{2}}$$

$$\times\left(\int_x^b\int_c^d [-\alpha(t,c)-\alpha(x,s)+\alpha(x,c)+\alpha(t,s)]\,|g_{12}(t,s)|^2\, dsdt\right)^{\frac{1}{2}},$$

Let $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ be reel numbers. Then we have the following Cauchy-Scwarz inequality

$$a_1b_1 + a_2b_2 + ... + a_nb_n \leq \left(a_1^2+a_2^2+....+a_n^2\right)^{\frac{1}{2}}\left(b_1^2+b_2^2+....+b_n^2\right)^{\frac{1}{2}}. \tag{2.19}$$

If we add the inequalities (2.11)-(2.18), then by using the Cauchy-Schwarz inequality (2.19), we obtain

$$4\int_a^b\int_c^d |f_{12}(t,s)g(t,s)|\, d_t d_s\alpha(t,s)$$

$$\leq \left[\int_a^b\int_c^y (b-a)\,(s-c)\,|f_{12}(t,s)|^2\, d_t d_s\alpha(t,s) + \int_a^x\int_c^d (t-a)\,(d-c)\,|f_{12}(t,s)|^2\, d_t d_s\alpha(t,s)\right.$$

$$\left. + \int_a^b\int_y^d (b-a)\,(d-s)\,|f_{12}(t,s)|^2\, d_t d_s\alpha(t,s) + \int_x^b\int_c^d (b-t)\,(d-c)\,|f_{12}(t,s)|^2\, d_t d_s\alpha(t,s)\right]^{\frac{1}{2}}$$

$$
\begin{aligned}
&\times \left[\int_a^b \int_c^y [\alpha(b,y) - \alpha(a,y) - \alpha(b,s) + \alpha(a,s)] \, |g_{12}(t,s)|^2 \, dsdt \right. \\
&+ \int_a^x \int_c^d [\alpha(t,c) - \alpha(x,c) - \alpha(t,d) + \alpha(x,d)] \, |g_{12}(t,s)|^2 \, dsdt \\
&+ \int_a^b \int_y^d [\alpha(b,s) - \alpha(b,y) - \alpha(a,s) + \alpha(a,y)] \, |g_{12}(t,s)|^2 \, dsdt \\
&\left. + \int_x^b \int_c^d [\alpha(t,d) - \alpha(x,d) - \alpha(t,c) + \alpha(x,c)] \, |g_{12}(t,s)|^2 \, dsdt \right]^{\frac{1}{2}} \\
= \quad & \left[(b-a) \int_a^b \int_c^d Q(s,y) \, |f_{12}(t,s)|^2 \, d_t d_s \alpha(t,s) + (d-c) \int_a^b \int_c^d P(t,x) \, |f_{12}(t,s)|^2 \, d_t d_s \alpha(t,s) \right]^{\frac{1}{2}} \\
&\times \left[\int_a^b \int_c^d |\alpha(b,y) - \alpha(a,y) - \alpha(b,s) + \alpha(a,s)| \, |g_{12}(t,s)|^2 \, dsdt + \right. \\
&\left. + \int_a^b \int_c^d |\alpha(t,d) - \alpha(x,d) - \alpha(t,c) + \alpha(x,c)| \, |g_{12}(t,s)|^2 \, dsdt \right]^{\frac{1}{2}} .
\end{aligned}
$$

This proves the first inequality in (2.1).

The proof of the second inequality in (2.1) is obvious from the fact that $\sqrt{pq} \leq \frac{1}{2}(p+q)$, for $p, q > 0$.

Corollary 2.1. *Assume that assumptions of Theorem 2.1 hold. If $w : [a,b] \times [c,d] \to [0,\infty)$ is continuous and we chose $\alpha(t,s) = \int_a^t \int_c^t w(u,v) dvdu$, then we obtain the following weighted inequalities*

$$
\begin{aligned}
&\int_a^b \int_c^d |f_{12}(t,s) g(t,s)| \, w(t,s) dsdt \\
\leq \quad & \frac{1}{4} \left[(b-a) \int_a^b \int_c^d Q(s,y) \, |f_{12}(t,s)|^2 \, w(t,s) dsdt + (d-c) \int_a^b \int_c^d P(t,x) \, |f_{12}(t,s)|^2 \, w(t,s) dsdt \right]^{\frac{1}{2}} \\
&\times \left[\int_a^b \int_c^d \left(\left| \int_a^b \int_s^y w(u,v) dvdu \right| + \left| \int_t^x \int_c^d w(u,v) dvdu \right| \right) |g_{12}(t,s)|^2 \, dsdt \right]^{\frac{1}{2}} \\
\leq \quad & \frac{1}{8} \left[(b-a) \int_a^b \int_c^d Q(s,y) \, |f_{12}(t,s)|^2 \, w(t,s) dsdt + (d-c) \int_a^b \int_c^d P(t,x) \, |f_{12}(t,s)|^2 \, w(t,s) dsdt \right. \\
&\left. + \int_a^b \int_c^d \left(\left| \int_a^b \int_s^y w(u,v) dvdu \right| + \left| \int_t^x \int_c^d w(u,v) dvdu \right| \right) |g_{12}(t,s)|^2 \, dsdt \right]
\end{aligned}
$$

where $Q(s,y)$ and $P(t,x)$ are defined as in Theorem 2.1.

Remark 2.1. *If we choose $w(t,s)=1$ for all $(t,s)\in[a,b]\times[c,d]$ in Corollary 2.1, then we have the following inequalities*

$$
\begin{aligned}
&\int_a^b\int_c^d |f_{12}(t,s)g(t,s)|\,dsdt \\
\leq\ & \frac{1}{4}\left[(b-a)\int_a^b\int_c^d Q(s,y)\,|f_{12}(t,s)|^2\,dsdt+(d-c)\int_a^b\int_c^d P(t,x)\,|f_{12}(t,s)|^2\,dsdt\right]^{\frac{1}{2}} \\
&\times\left[\int_a^b\int_c^d ((b-a)\,|y-s|+(d-c)\,|x-t|)\,|g_{12}(t,s)|^2\,dsdt\right]^{\frac{1}{2}} \\
\leq\ & \frac{1}{8}\left[(b-a)\int_a^b\int_c^d Q(s,y)\,|f_{12}(t,s)|^2\,dsdt+(d-c)\int_a^b\int_c^d P(t,x)\,|f_{12}(t,s)|^2\,dsdt\right. \\
&\left.+\int_a^b\int_c^d ((b-a)\,|y-s|+(d-c)\,|x-t|)\,|g_{12}(t,s)|^2\,dsdtt\right]
\end{aligned}
$$

which is proved by Budak and Sarıkaya in [2].

3. Concluding Remarks

In this study, we established the generalized Opial inequality for Riemann-Stieltjes integrals of functions with two independent variables. In further studies, one can obtain weighted version of the inequality given this paper.

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