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A-priori Estimates Near the Boundary for Solutions of a class of Degenerate Elliptic Problems in Besov-type Spaces

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Abstract. In this paper, we give a priori estimates near the boundary for solutions of a degenerate elliptic problems in the general Besov-type spaces $B^{s,\tau}_{p,q}$, containing as special cases: Goldberg space *bmo*, local Morrey-Campanato spaces $l^{2,\lambda}$ and the classical Hölder and Besov spaces $B^{s}_{p,q}$. This work extends the results of [13, 2, 15] from Hölder and Besov spaces to the general frame of $B^{s,\tau}_{p,q}$ spaces.
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1. Introduction, Definitions and Results

1.1. Introduction.

A priori estimates near the boundary for elliptic problems in Besov-type spaces was proved in [12]. The objective of this paper is to establish an a-priori estimates in Besov-type spaces $B^{s,\tau}_{p,q}$ for solutions of a degenerate elliptic boundary value problems. More precisely, we denote L the differential operator, which can be written in an appropriate local coordinates:

$$\begin{aligned} L &\equiv L(t,x';D_t,D_{x'}) \\ &= tD_t^2 + \sum_{j,k=1}^{n} a_{j,k}(t,x')tD_{x_j}D_{x_k} + \sum_{j=1}^{n} b_j(t,x')tD_{x_j}D_t + \sum_{j=1}^{n} c_j(t,x')D_{x_j} \\ &\quad + d(t,x')D_t + e(t,x'), \end{aligned} \tag{1}$$

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where $x = (t, x') = (t, x_1, x_2, \cdots, x_n) \in \mathbb{R}_+ \times \mathbb{R}^n$ and with complex-valued functions coefficients infinitely differentiable in $\overline{\mathbb{R}^{n+1}_+}$. The model of operators (1) is given by $M = t(D_t^2 + D_x^2) + \lambda D_t + \mu D_x$ where $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ and $\lambda,\ \mu \in \mathbb{C}$.

The operator L is a particular case of the following class of degenerate operators

$$\tilde{L} = \sum_{h=0}^{min(k,m)} \varphi^{k-h} P^{m-h}(x, D_x), \tag{2}$$

where $k \in \mathbb{N}$, $m \in \mathbb{N}^\star$, the function φ is C^∞ from $\mathbb{R}^{n+1}$ to $\mathbb{R}$, with $\Omega = \{x \in \mathbb{R}^{n+1}; \varphi(x) > 0\}$, $\Gamma = \{x \in \mathbb{R}^{n+1}; \varphi(x) = 0\}$ and $d\varphi \neq 0$ on Γ; $P^{m-h}(x, D_x)$ is a differential operator with smooth coefficients on $\overline{\Omega}$ and of order $\leq m - h$, $P^m(x, D_x)$ is a properly elliptic differential operator on $\overline{\Omega}$.

The operators (2) have been studied first in Sobolev spaces H^s by N. Shimakura [16] and by P. Bolley and J. Camus in [4]. Later on, the same class has been considered by C. Goulaouic and N. Shimakura [13], and by P. Bolley, J. Camus and G. Metivier [2] in Hölder spaces C^s. J. Rolland [15] has considered the same operators (2) in classical Besov spaces $B^s_{p,q}$ with $p = q$. In this work, we generalize the previous results to the more general frame of Besov-type spaces $B^{s,\tau}_{p,q}$, containing H^s, C^s and $B^s_{p,p}$ as special cases. Other degenerate operators have been considered in [6, 7, 10, 20].

As in [10], an application of these estimates can be given for the regularity of solutions of completely nonlinear boundary value problems, involving all the weighted derivatives in the operator (1).

The plan of the paper is as follows: In the first section, we give the definition of $B^{s,\tau}_{p,q}$ spaces, recall some properties of these spaces given in [8, 9, 17, 22] and we enunciate the main theorem. The second section contains some helpful lemmas needed in the proof. Then, the third section is devoted to the trace characterization for elements of the considered weighted spaces. Finally, in the fourth section, we give the proof of the main result (Theorem 1.1), which is based on one Peetre's method described in [2, 12]. It consists in doing a partial Fourier transform with respect to the tangential direction on the equation, and reducing the problem to an isomorphism theorem for an ordinary differential operator. By means of that, we estimate the "almost tangential" derivatives of the solutions. Next, we make use of an interpolation inequality to estimate the normal derivatives of the solution.

Notation: In this paper, for $\nu, J \in \mathbb{Z}$, we denote by B_J (resp. B'_J), the ball centred at $x_0 \in \mathbb{R}^{n+1}$ (resp. $x'_0 \in \mathbb{R}^n$) and with radius 2^{-J}, we set $B_J = \{x \in \mathbb{R}^{n+1} \ : \ |x - x_0| < 2^{-J}\}$ (resp. $B'_J = \{x' \in \mathbb{R}^n \ : \ |x' - x'_0| < 2^{-J}\}$, αB_J (resp. $\alpha B'_J$), $\alpha > 0$, is the ball centred at x_0 (resp. x'_0) and of radius $\alpha 2^{-J}$. Similarly, F_ν (resp. F'_ν) denotes the annulus centred at $x_0 \in \mathbb{R}^{n+1}$, (resp. $x'_0 \in \mathbb{R}^n$), such that $F_\nu = \{x \in \mathbb{R}^{n+1} \ : \ 2^\nu \leq |x - x_0| \leq 2^{\nu+1}\}$, (resp. $F'_\nu = \{x' \in \mathbb{R}^n \ : \ 2^\nu \leq |x' - x'_0| \leq 2^{\nu+1}\}$). We set $J^+ = max(J, 0)$ and we denote by $|\Omega|$ the measure of $\Omega \subset \mathbb{R}^{n+1}$. $C_0, C_1, C_M, C_N, C_K, C'_K, \varepsilon, \varepsilon_0, \varepsilon_1, \varepsilon_2, \cdots$, denote various real positive constants, which can be different at different places, and the symbol $\mathsf{X} \lesssim \mathsf{Y}$ means that there exists a positive constant C such that $\mathsf{X} \leq C\mathsf{Y}$. By $\mathcal{S}$ we denote the Schwartz space of all rapidly decreasing and infinitely differentiable functions on $\mathbb{R}^{n+1}$, and by $\mathcal{S}'$ his topological dual i.e., the collection of all complex-valued tempered distributions on $\mathbb{R}^{n+1}$.

1.2. Definitions.

To define the spaces, we will use a Littlewood-Paley partition of unity: Denote $x = (t, x') \in \mathbb{R} \times \mathbb{R}^n$, let $\varphi \in C_0^\infty(\mathbb{R})$ equals to 1 in $[-1, 1]$ and with support in $[-2, 2]$. For $j \in \mathbb{N}$ we set:

$$S_j = \varphi(2^{-j}|D_x|)\ ;\ S'_j = \varphi(2^{-j}|D_{x'}|)\ ;\ S''_j = \varphi(2^{-j}|D_t|),$$
$$S_{-1} = S'_{-1} = S''_{-1} = 0,$$

and

$$\Delta_j = S_j - S_{j-1}\ ;\ \Delta'_j = S'_j - S'_{j-1}\ ;\ \Delta''_j = S''_j - S''_{j-1}.$$

Definition 1.1 (Inhomogeneous version of Besov-type spaces)**.** *Let $s \in \mathbb{R}$, $\tau \geq 0$ and let $1 \leq p, q < +\infty$. The space $B^{s,\tau}_{p,q}(\mathbb{R}^{n+1})$ denotes the set of all tempered distributions $u \in \mathcal{S}'(\mathbb{R}^{n+1})$ such that*

$$||u||_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1})} \equiv \sup_{B_J} \frac{1}{|B_J|^\tau} \Big\{ \sum_{j \geq J^+} 2^{jsq} ||\Delta_j u||^q_{L^p(B_J)} \Big\}^{\frac{1}{q}} < +\infty,$$

where the supremum is taken over all $J \in \mathbb{Z}$ and all balls B_J.

Remark 1.1. *In the sense of equivalent norms, we have the following properties, see* [11, 22, 21]*:*

(1) $B^{s,0}_{p,q}(\mathbb{R}^{n+1}) = B^{s}_{p,q}(\mathbb{R}^{n+1})$ *the classical Besov spaces;*
(2) $B^{s,\tau}_{p,p}(\mathbb{R}^{n+1}) = F^{s,\tau}_{p,p}(\mathbb{R}^{n+1})$ *Triebel-Lizorkin type spaces;*
(3) $B^{s,1}_{p,p}(\mathbb{R}^{n+1}) = bmo^{s}_{p}(\mathbb{R}^{n+1})$ *Goldberg spaces;*
(4) $B^{0,\frac{\lambda}{2(n+1)}}_{2,2}(\mathbb{R}^{n+1}) = l^{2,\lambda}(\mathbb{R}^{n+1})$ *local Campanato spaces;*
(5) $B^{s,0}_{\infty,\infty}(\mathbb{R}^{n+1}) = F^{s,0}_{\infty,\infty}(\mathbb{R}^{n+1}) = C^{s}(\mathbb{R}^{n+1})$ *Hölder-Zygmund spaces, $s > 0$, $s \notin \mathbb{N}$.*
(6) $B^{s,\frac{1}{p}-\frac{1}{u}}_{p,\infty}(\mathbb{R}^{n+1}) = \mathcal{N}^{s}_{u,p,\infty}(\mathbb{R}^{n+1})$ *Besov-Morrey spaces, $0 < p \le u < +\infty$, $s \in \mathbb{R}$.*
(7) *Let $s \in \mathbb{R}$, $0 < p \le +\infty$ then $B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}) = B^{s,(n+1)(\tau-\frac{1}{p})}_{\infty,\infty}(\mathbb{R}^{n+1})$, for $0 < q < +\infty$ and $\frac{1}{p} < \tau < +\infty$, or $q = +\infty$ and $\frac{1}{p} \le \tau < +\infty$.*

Also, we collect elementary embeddings, see [9] *and* [8]*:*

(1) $B^{s_1,\tau}_{p,q}(\mathbb{R}^{n+1}) \hookrightarrow B^{s_2,\tau}_{p,q}(\mathbb{R}^{n+1})$, *if $s_2 \le s_1$;*
(2) $B^{s,\tau}_{p,q_0}(\mathbb{R}^{n+1}) \hookrightarrow B^{s,\tau}_{p,q_1}(\mathbb{R}^{n+1})$, *if $0 < q_0 \le q_1 < +\infty$ and $0 < p < +\infty$;*
(3) $B^{s+\frac{n+1}{p}-\frac{\tau(n+1)}{q},\tau}_{p,q}(\mathbb{R}^{n+1}) \hookrightarrow C^{s}(\mathbb{R}^{n+1})$;
(4) *Let $0 < t \le p < +\infty$, then $B^{s+\varepsilon}_{t,q}(\mathbb{R}^{n+1}) \hookrightarrow B^{s,\tau}_{p,q}(\mathbb{R}^{n+1})$, if and only if $\varepsilon \ge \frac{\tau(n+1)}{q} + \frac{n+1}{t} - \frac{n+1}{p}$.*

The weighted spaces we will need later in this work are listed below.

Definition 1.2. *We define the following spaces:*

(1) $W^{2,p}_{w}(\mathbb{R}) = \{u \in W^{1,p}(\mathbb{R}) \;:\; tD^2_t u,\, tD_t u \quad and \quad tu \in L^p(\mathbb{R})\}$, *with the associated norm:*

$$\| u \|_{W^{2,p}_{w}(\mathbb{R})} \equiv \Big\{ \| u \|^{p}_{L^p(\mathbb{R})} + \| D_t u \|^{p}_{L^p(\mathbb{R})} + \| tD^2_t u \|^{p}_{L^p(\mathbb{R})} + \| tD_t u \|^{p}_{L^p(\mathbb{R})} + \| tu \|^{p}_{L^p(\mathbb{R})} \Big\}^{\frac{1}{p}}.$$

(2) $W^{2,p}_{w}(\mathbb{R}; B^{s,\tau}_{p,q}(\mathbb{R}^n)) = \Big\{u \in L^p(\mathbb{R}; B^{s+1,\tau}_{p,q}(\mathbb{R}^n)) : tu \in L^p(\mathbb{R}; B^{s+2,\tau}_{p,q}(\mathbb{R}^n)), tD_t u \in L^p(\mathbb{R}; B^{s+1,\tau}_{p,q}(\mathbb{R}^n)) and D_t u, tD^2_t u \in L^p(\mathbb{R}; B^{s,\tau}_{p,q}(\mathbb{R}^n))\Big\}$, *the associated norm is:*

$$\| u \|_{W^{2,p}_{w}(\mathbb{R}; B^{s,\tau}_{p,q}(\mathbb{R}^n))} \equiv \Big\{ \| u \|^{q}_{L^p(\mathbb{R}; B^{s+1,\tau}_{p,q}(\mathbb{R}^n))} + \| D_t u \|^{q}_{L^p(\mathbb{R}; B^{s,\tau}_{p,q}(\mathbb{R}^n))} + \| tD^2_t u \|^{q}_{L^p(\mathbb{R}; B^{s,\tau}_{p,q}(\mathbb{R}^n))} + \| tD_t u \|^{q}_{L^p(\mathbb{R}; B^{s+1,\tau}_{p,q}(\mathbb{R}^n))} + \| tu \|^{q}_{L^p(\mathbb{R}; B^{s+2,\tau}_{p,q}(\mathbb{R}^n))} \Big\}^{\frac{1}{q}}.$$

(3) $B^{s+2,\tau}_{p,q,w}(\mathbb{R}^{n+1}) = \Big\{u \in B^{s+1,\tau}_{p,q}(\mathbb{R}^{n+1}) : tD^2_t u, tD_{x_j}D_{x_k}u \text{ and } tD_{x_j}D_t u \in B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}) \text{ for all } j,k = 1,\cdots,n\Big\}$, *the norm in this space is:*

$$\| u \|_{B^{s+2,\tau}_{p,q,w}(\mathbb{R}^{n+1})} \equiv \Big\{ \| u \|^{q}_{B^{s+1,\tau}_{p,q}(\mathbb{R}^{n+1})} + \| tD^2_t u \|^{q}_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1})} + \sum_{j,k=1}^{n} \| tD_{x_j}D_{x_k} u \|^{q}_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1})} + \sum_{j=1}^{n} \| tD_{x_j}D_t u \|^{q}_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1})} \Big\}^{\frac{1}{q}}.$$

The space $W^{2,p}_{w}(\mathbb{R}_+)$ $\big[resp.\, W^{2,p}_{w}(\mathbb{R}_+; B^{s,\tau}_{p,q}(\mathbb{R}^n))\big]$ denotes the set of restrictions to $\mathbb{R}_+$ of elements of $W^{2,p}_{w}(\mathbb{R})$ $\big[resp.\, W^{2,p}_{w}(\mathbb{R}; B^{s,\tau}_{p,q}(\mathbb{R}^n))\big]$; likewise, the space $B^{s+2,\tau}_{p,q,w}(\mathbb{R}^{n+1}_+)$ is the restriction to $\mathbb{R}^{n+1}_+ = \mathbb{R}_+ \times \mathbb{R}^n$ of elements of $B^{s+2,\tau}_{p,q,w}(\mathbb{R}^{n+1})$.

Remark 1.2. *We have the following embedding result when $s \ge 0$:*

$$B^{s+2,\tau}_{p,q,w}(\mathbb{R}^{n+1}) \hookrightarrow W^{2,p}_{w,loc}(\mathbb{R}; B^{s,\tau}_{p,q}(\mathbb{R}^n)).$$

1.3. Assumptions and main theorem.
We suppose that:

Assumption (H_1). *The operator $L(x; D_x)$ is elliptic for $t > 0$ in $\mathbb{R}^{n+1}_+$;*

Let L^0 be the "principal part" of the operator L defined as:

$$\begin{aligned} L^0 &= L^0(t, x'; D_t, D_{x'}) \\ &= tD_t^2 + \sum_{j,k=1}^{n} a_{j,k}(0, x')tD_{x_j}D_{x_k} + \sum_{j=1}^{n} b_j(0, x')tD_{x_j}D_t + \sum_{j=1}^{n} c_j(0, x')D_{x_j} \\ &\quad + d(0, x')D_t. \end{aligned} \tag{3}$$

Assumption (H_2). *For any $x' \in \mathbb{R}^n$ and any $\xi' \in \mathbb{R}^n\backslash\{0\}$, the polynomial in the variable τ:*

$$P(\tau) = \tau^2 + \sum_{j,k=1}^{n} a_{j,k}(0, x')\xi_j\xi_k + \sum_{j=1}^{n} b_j(0, x')\xi_j\tau + \sum_{j=1}^{n} c_j(0, x')\xi_j + d(0, x')\tau$$

have two imaginary roots $\tau^+(x', \xi')$ and $\tau^-(x', \xi')$ such that

$$\Im m(\tau^+(x', \xi')) > 0 \text{ and } \Im m(\tau^-(x', \xi')) < 0;$$

Assumption (H_3). *We assume that for any $x' \in \mathbb{R}^n$, $\Im m(d(0, x')) > max\big(1 + \frac{1}{p}, s + \frac{1}{p}\big)$, where $s \geq 0$ and $1 \leq p < +\infty$ are fixed.*

Also, we suppose that for any $x' \in \mathbb{R}^n$ and $\omega' \in \mathbb{R}^n\backslash\{0\}, |\omega'| = 1$, the problem:

$$\begin{cases} L^0(t, x'; D_t, \omega')u(t) = 0 \\ u(0) = 0 \\ u \in \mathcal{S}(\overline{\mathbb{R}}_+) \end{cases} \tag{4}$$

has only the solution $u \equiv 0$.

The main result of this paper is the following.

Theorem 1.1. *Let s, τ be two non-negative real numbers and let $1 \leq p, q < +\infty$. Under hypotheses (H_1), (H_2) and (H_3), for any compact set K in $\overline{\mathbb{R}^{n+1}_+}$, there exists a constant C_K such that for any $u \in B^{s+2,\tau}_{p,q,w}(\mathbb{R}^{n+1}_+)$ with support in K,*

$$\|u\|_{B^{s+2,\tau}_{p,q,w}(\mathbb{R}^{n+1}_+)} \leq C_K\Big\{\|Lu\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} + \|\gamma_0 u\|_{B^{s+1-\frac{1}{p},\tau}_{p,q}(\mathbb{R}^n)} + \|u\|_{B^{s-1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)}\Big\},$$

holds.

Taking $\tau = 0$ and $p = q$, we obtain the regularity in classical Besov spaces [15]. If $s = 0$, $\tau = \frac{\lambda}{2(n+1)}$, and $p = q = 2$, we get the regularity in local Morrey-Campanato spaces $l^{2,\lambda}$.

Example 1.1. *Let M be the following operator given in $\mathbb{R}_+ \times \mathbb{R}$ by*

$$M(t, x; D_t, D_x)u = t(D_t^2 + D_x^2)u + \lambda D_t u + \mu D_x u,$$

where $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, and $\lambda, \mu \in \mathbb{C}$. According to [1]*, if $\mu \neq \pm(2p + i\lambda)$, $p \in \mathbb{N}^*$, and $\Im m(\lambda) > max(1 + \frac{1}{p}, s + \frac{1}{p})$, and if u is a solution of the problem*

$$\begin{cases} u \in B^{s,\tau}_{p,q}(\mathbb{R}^2_+) \\ Mu = f \in B^{s,\tau}_{p,q}(\mathbb{R}^2_+) \\ \gamma_0 u = g \in B^{s+1-\frac{1}{p},\tau}_{p,q}(\mathbb{R}), \end{cases}$$

then $u \in B^{s+2,\tau}_{p,q,w,loc}(\mathbb{R}^2_+)$.

2. Preliminary Lemmas

In this section, we shall recall basic lemmas needed in the proof of the theorem.

Lemma 2.1 ([9]). *Let $1 \leq p < +\infty$ and let $A < 0$. If $(a_{j\nu})_{j,\nu}$ is a sequence of positive real numbers satisfying $(a_{j\nu})_j \in \ell^p$ for any $\nu \geq 1$, then*

$$\sum_{j\geq 1}\Big(\sum_{\nu\geq 1} 2^{\nu A} a_{j\nu}\Big)^p \lesssim \sup_{\nu\geq 1}\sum_{j\geq 1} a_{j\nu}^p,$$

holds.

Lemma 2.2 ([9]). *Let $1 \leq p < +\infty$. For any integer $M > 0$, there exists a constant $C_M > 0$ such that for any ball B_J, any $l \in \mathbb{Z}$ and $u \in L^p(\mathbb{R}^{n+1})$,*

$$\|A_l u\|_{L^p(B_J)} \leq C_M\Big\{\|u\|_{L^p(2B_J)} + \sum_{\nu\geq -J+1} 2^{-(\nu+l)M}\|u\|_{L^p(F_\nu)}\Big\},$$

holds for $A_l = \Delta_l, \Delta'_l, \Delta''_l, S_l, S'_l, S''_l$.

Lemma 2.3. *Let s and τ be two real numbers such that $\tau \geq 0$, for any $\varepsilon > 0$ and $u \in W^{2,p}_w(\mathbb{R}; B^{s,\tau}_{p,q}(\mathbb{R}^n))$, we have the following inequalities:*

(1) $\|u\|_{L^p(\mathbb{R};B^{s+1,\tau}_{p,q}(\mathbb{R}^n))} \lesssim \varepsilon\|tD_t^2 u\|_{L^p(\mathbb{R};B^{s,\tau}_{p,q}(\mathbb{R}^n))} + \frac{1}{\varepsilon}\|tu\|_{L^p(\mathbb{R};B^{s+2,\tau}_{p,q}(\mathbb{R}^n))}$,

(2) $\|D_t u\|_{L^p(\mathbb{R};B^{s,\tau}_{p,q}(\mathbb{R}^n))} \lesssim \varepsilon\|tD_t^2 u\|_{L^p(\mathbb{R};B^{s,\tau}_{p,q}(\mathbb{R}^n))} + \frac{1}{\varepsilon}\|tu\|_{L^p(\mathbb{R};B^{s+2,\tau}_{p,q}(\mathbb{R}^n))}$,

(3) $\|tD_t u\|_{L^p(\mathbb{R};B^{s+1,\tau}_{p,q}(\mathbb{R}^n))} \lesssim \varepsilon\|tD_t^2 u\|_{L^p(\mathbb{R};B^{s,\tau}_{p,q}(\mathbb{R}^n))} + \frac{1}{\varepsilon}\|tu\|_{L^p(\mathbb{R};B^{s+2,\tau}_{p,q}(\mathbb{R}^n))}$.

Proof. It is classical according to [18], that if $tD_t^2 u \in L^p(\mathbb{R})$ and $tu \in L^p(\mathbb{R})$, then $u \in L^p(\mathbb{R})$, and

$$\|u\|_{L^p(\mathbb{R})} \lesssim \|tD_t^2 u\|_{L^p(\mathbb{R})} + \|tu\|_{L^p(\mathbb{R})},$$

applying this inequality to $u(\lambda t)$ with $\lambda > 0$, we obtain

$$\|u\|_{L^p(\mathbb{R})} \lesssim \lambda\|tD_t^2 u\|_{L^p(\mathbb{R})} + \frac{1}{\lambda}\|tu\|_{L^p(\mathbb{R})}.$$

Now, by replacing λ with $\varepsilon\lambda$, $\varepsilon > 0$, we get

$$\|u\|_{L^p(\mathbb{R})} \lesssim \varepsilon\lambda\|tD_t^2 u\|_{L^p(\mathbb{R})} + \frac{1}{\varepsilon\lambda}\|tu\|_{L^p(\mathbb{R})}. \tag{5}$$

Let $u \in W^{2,p}_w(\mathbb{R}; B^{s,\tau}_{p,q}(\mathbb{R}^n))$, we apply (5) to $t \mapsto u_j(t,x') = \Delta'_j u(t,x')$, $j \in \mathbb{N}$ and $x' \in B'_J$. Taking $\lambda = 2^{-j}$ and integrating with respect to $x' \in B'_J$, we obtain

$$\|\Delta'_j u\|_{L^p(B'_J\times\mathbb{R})} \lesssim \varepsilon 2^{-j}\|\Delta'_j tD_t^2 u\|_{L^p(B'_J\times\mathbb{R})} + \frac{1}{\varepsilon}2^j\|\Delta'_j tu\|_{L^p(B'_J\times\mathbb{R})},$$

multiplying by $\frac{2^{j(s+1)}}{|B'_J|^\tau}$

$$\frac{1}{|B'_J|^\tau}2^{j(s+1)}\|\Delta'_j u\|_{L^p(B'_J\times\mathbb{R})} \lesssim \frac{\varepsilon}{|B'_J|^\tau}2^{js}\|\Delta'_j tD_t^2 u\|_{L^p(B'_J\times\mathbb{R})} + \frac{1}{\varepsilon|B'_J|^\tau}2^{j(s+2)}\|\Delta'_j tu\|_{L^p(B'_J\times\mathbb{R})},$$

we take the $l^q - norm$ on each side of the inequality and we sum over $j \geq J^+$, we obtain

$$\frac{1}{|B'_J|^\tau}\Big(\sum_{j\geq J^+} 2^{j(s+1)q}\|\Delta'_j u\|^q_{L^p(B'_J\times\mathbb{R})}\Big)^{\frac{1}{q}} \lesssim \frac{\varepsilon}{|B'_J|^\tau}\Big(\sum_{j\geq J^+} 2^{jsq}\|\Delta'_j tD_t^2 u\|^q_{L^p(B'_J\times\mathbb{R})}\Big)^{\frac{1}{q}}$$

$$+\frac{1}{\varepsilon|B'_J|^\tau}\left(\sum_{j\geq J^+}2^{j(s+2)q}\|\Delta'_j tu\|^q_{L^p(B'_J\times\mathbb{R})}\right)^{\frac{1}{q}}.$$

We deduce the first inequality (1).

The second and the third inequalities are obtained analogously starting respectively from

$$\|D_t u\|_{L^p(\mathbb{R})}\lesssim\|tD_t^2u\|_{L^p(\mathbb{R})}+\|tu\|_{L^p(\mathbb{R})},$$

and

$$\|tD_t u\|_{L^p(\mathbb{R})}\lesssim\|tD_t^2u\|_{L^p(\mathbb{R})}+\|tu\|_{L^p(\mathbb{R})}.$$

The following three lemmas are proved in [12] for $p=q$. In the same way we obtain them for $p\neq q$.

Lemma 2.4 ([12])**.** *Let τ be a positive real number, $1\leq p,q<+\infty$, m an integer ≥ 1 and let s be a real $<m$. If $u\in L^p(\mathbb{R};B^{s,\tau}_{p,q}(\mathbb{R}^n))$ such that $D^m_t u\in L^p(\mathbb{R};B^{s-m,\tau}_{p,q}(\mathbb{R}^n))$ then $u\in B^{s,\tau}_{p,q}(\mathbb{R}^{n+1})$ and*

$$\|u\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1})}\lesssim\|D^m_t u\|_{L^p(\mathbb{R};B^{s-m,\tau}_{p,q}(\mathbb{R}^n))}+\|u\|_{L^p(\mathbb{R};B^{s,\tau}_{p,q}(\mathbb{R}^n))},$$

holds.

Lemma 2.5 ([12])**.** *There exists $C_0>0$ such that for any $\varphi\in\mathcal{S}(\mathbb{R}^{n+1})$, there exists $C_1>0$ satisfying for any $u\in B^{s,\tau}_{p,q}(\mathbb{R}^{n+1})$ $\left[resp.\ L^p(\mathbb{R};B^{s,\tau}_{p,q}(\mathbb{R}^n))\right]$*

$$\|\varphi u\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1})}\leq C_0\|\varphi\|_{L^\infty(\mathbb{R}^{n+1})}\|u\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1})}+C_1\|u\|_{B^{s-1,\tau}_{p,q}(\mathbb{R}^{n+1})},$$

$$\Big[resp.\ \|\varphi u\|_{L^p(\mathbb{R},B^{s,\tau}_{p,q}(\mathbb{R}^n))}\leq C_0\|\varphi\|_{L^\infty(\mathbb{R}^{n+1})}\|u\|_{L^p(\mathbb{R};B^{s,\tau}_{p,q}(\mathbb{R}^n))}+C_1\|u\|_{L^p(\mathbb{R};B^{s-1,\tau}_{p,q}(\mathbb{R}^n))}\Big].$$

Lemma 2.6 ([12])**.** *Let $s_1\leq s_2<s_3$ be three real numbers, $\tau\geq 0$ and let $1\leq p,q<+\infty$. For any $\varepsilon>0$ and $u\in B^{s_3,\tau}_{p,q}(\mathbb{R}^{n+1})$ $\left[resp.\ L^p(\mathbb{R};B^{s_3,\tau}_{p,q}(\mathbb{R}^n))\right]$, we get*

$$\|u\|_{B^{s_2,\tau}_{p,q}(\mathbb{R}^{n+1})}\lesssim\varepsilon\|u\|_{B^{s_3,\tau}_{p,q}(\mathbb{R}^{n+1})}+\varepsilon^{-\frac{s_2-s_1}{s_3-s_2}}\|u\|_{B^{s_1,\tau}_{p,q}(\mathbb{R}^{n+1})},$$

$$\Big[resp.\ \|u\|_{L^p(\mathbb{R};B^{s_2,\tau}_{p,q}(\mathbb{R}^n))}\lesssim\varepsilon\|u\|_{L^p(\mathbb{R};B^{s_3,\tau}_{p,q}(\mathbb{R}^n))}+\varepsilon^{-\frac{s_2-s_1}{s_3-s_2}}\|u\|_{L^p(\mathbb{R};B^{s_1,\tau}_{p,q}(\mathbb{R}^n))}\Big].$$

3. Trace of Elements of $B^{s+2,\tau}_{p,q,w}(\mathbb{R}^{n+1}_+)$ and $W^{2,p}_w(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))$

Theorem 3.1. *Let s, τ be two real numbers such that $\tau\geq 0$ and let $1\leq p,q<+\infty$. For $u\in W^{2,p}_{w,loc}(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))$, the series $\sum_{j\geq 0}\Delta'_j u(0,\cdot)$ converges in $\mathcal{S}'(\mathbb{R}^n)$ and defines an element $\gamma_0 u$ belonging to $B^{s+1-\frac{1}{p},\tau}_{p,q}(\mathbb{R}^n)$. In addition, the mapping $u\mapsto\gamma_0 u$ is continuous and surjective from $W^{2,p}_{w,loc}(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))$ to $B^{s+1-\frac{1}{p},\tau}_{p,q}(\mathbb{R}^n)$. Also, there exists an extension operator R_0 from $B^{s+1-\frac{1}{p},\tau}_{p,q}(\mathbb{R}^n)$ to $W^{2,p}_w(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))$ such that*

$$\gamma_0 oR_0=Id_{B^{s+1-\frac{1}{p},\tau}_{p,q}(\mathbb{R}^n)}.$$

In particular, if $s\geq 0$, the operator γ_0 is bounded and surjective from $B^{s+2,\tau}_{p,q,w}(\mathbb{R}^{n+1}_+)$ onto $B^{s+1-\frac{1}{p},\tau}_{p,q}(\mathbb{R}^n)$.

Proof. Since $W^{2,p}_{w,loc}(\mathbb{R}_+) \subset W^{1,p}_{loc}(\mathbb{R}_+) \hookrightarrow C^0_{loc}(\mathbb{R}_+)$, for any $\varphi \in C^\infty_0(\mathbb{R}_+)$ and any $v \in W^{2,p}_w(\mathbb{R}_+)$, we have

$$|v(0)| \lesssim \|\varphi v\|_{W^{2,p}_w(\mathbb{R}_+)},$$

then

$$\begin{aligned}|v(0)| \lesssim & \|\varphi v\|_{L^p(\mathbb{R}_+)} + \|D_t\varphi v\|_{L^p(\mathbb{R}_+)} + \|tD_t\varphi v\|_{L^p(\mathbb{R}_+)} + \|tD_t^2\varphi v\|_{L^p(\mathbb{R}_+)} \\ & + \|t\varphi v\|_{L^p(\mathbb{R}_+)},\end{aligned}$$

we change $v(t)$ into $v(\lambda t)$ and $\varphi(t)$ into $\varphi(\lambda t)$

$$\begin{aligned}|v(0)| \lesssim & \lambda^{-\frac{1}{p}}\|\varphi v\|_{L^p(\mathbb{R}_+)} + \lambda^{\frac{(p-1)}{p}}\|D_t\varphi v\|_{L^p(\mathbb{R}_+)} + \lambda^{-\frac{1}{p}}\|tD_t\varphi v\|_{L^p(\mathbb{R}_+)} \\ & + \lambda^{\frac{(p-1)}{p}}\|tD_t^2\varphi v\|_{L^p(\mathbb{R}_+)} + \lambda^{\frac{(p+1)}{p}}\|t\varphi v\|_{L^p(\mathbb{R}_+)}.\end{aligned} \tag{6}$$

Let $u \in W^{2,p}_w(\mathbb{R}_+; B^{s,\tau}_{p,q}(\mathbb{R}^n))$, for $j \in \mathbb{N}$, we set $u_j(t,x') = \Delta'_j u(t,x') \in W^{2,p}_{w,loc}(\mathbb{R}_+; C^\infty(\mathbb{R}^n) \cap L^p(\mathbb{R}^n))$, so we can apply the inequality (6) to u_j. By choosing $\lambda = 2^{-j}$, integrating over a ball B'_J and multiplying by $\frac{2^{j(s+1-\frac{1}{p})}}{|B'_J|^\tau}$, we get

$$\begin{aligned}&\frac{2^{j(s+1-\frac{1}{p})}}{|B'_J|^\tau}\|\Delta'_j u(0,\cdot)\|_{L^p(B'_J)} \\ &\quad\lesssim \frac{2^{j(s+1)}}{|B'_J|^\tau}\|\varphi\Delta'_j u\|_{L^p(\mathbb{R}_+\times B'_J)} + \frac{2^{js}}{|B'_J|^\tau}\|D_t\varphi\Delta'_j u\|_{L^p(\mathbb{R}_+\times B'_J)} \\ &\qquad + \frac{2^{j(s+1)}}{|B'_J|^\tau}\|tD_t\varphi\Delta'_j u\|_{L^p(\mathbb{R}_+\times B'_J)} + \frac{2^{js}}{|B'_J|^\tau}\|tD_t^2\varphi\Delta'_j u\|_{L^p(\mathbb{R}_+\times B'_J)} \\ &\qquad + \frac{2^{j(s+2)}}{|B'_J|^\tau}\|t\varphi\Delta'_j u\|_{L^p(\mathbb{R}_+\times B'_J)},\end{aligned}$$

since $\varphi\Delta'_j u = \Delta'_j(\varphi u)$, we take the $l^q - norm$ on both sides to get

$$\|\gamma_0 u\|_{B^{s+1-\frac{1}{p},\tau}_{p,q}(\mathbb{R}^n)} \lesssim \|u\|_{W^{2,p}_{w,loc}(\mathbb{R}_+; B^{s,\tau}_{p,q}(\mathbb{R}^n))}.$$

This confirms that the series $\sum_{j\geq 0}\Delta'_j u(0,\cdot)$ converges in $\mathcal{S}'(\mathbb{R}^n)$ and the operator γ_0 is continuous from $W^{2,p}_{w,loc}(\mathbb{R}_+; B^{s,\tau}_{p,q}(\mathbb{R}^n))$ to $B^{s+1-\frac{1}{p},\tau}_{p,q}(\mathbb{R}^n)$.

Now let $u \in B^{s+1-\frac{1}{p},\tau}_{p,q}(\mathbb{R}^n)$. Let $\varphi \in C^\infty_0(\mathbb{R})$ such that $\varphi \equiv 1$ in a neighbourhood of 0. For $j \in \mathbb{N}$ we set:

$$R_0 u(t,x') = \sum_{j\geq 0}\varphi(2^j t)\Delta'_j u(0,x'). \tag{7}$$

We'll prove that the series converges in $\mathcal{S}'(\mathbb{R}^{n+1})$ and its sum belongs to $W^{2,p}_w(\mathbb{R}_+; B^{s,\tau}_{p,q}(\mathbb{R}^n))$ with

$$\|R_0 u\|_{W^{2,p}_w(\mathbb{R}_+; B^{s,\tau}_{p,q}(\mathbb{R}^n))} \lesssim \|u\|_{B^{s+1-\frac{1}{p},\tau}_{p,q}(\mathbb{R}^n)}.$$

We prove the inequality for each term of the norm $\|R_0 u\|_{W^{2,p}_w(\mathbb{R}_+; B^{s,\tau}_{p,q}(\mathbb{R}^n))}$. First, we apply the operator Δ'_k with $k \in \mathbb{N}$ to (7)

$$\Delta'_k R_0 u(t,x') = \sum_{j=k-1}^{k+1}\varphi(2^j t)\Delta'_k\Delta'_j u(0,x'),$$

then

$$|\Delta'_k R_0 u(t,x')| \leq \sum_{j\sim k}|\varphi(2^j t)\Delta'_k\Delta'_j u(0,x')|,$$

we deduce

$$\int_{\mathbb{R}_+} |\Delta'_k R_0 u(t,x')|^p dt \le \int_{\mathbb{R}_+} \Big(\sum_{j\sim k} |\varphi(2^j t)\Delta'_k \Delta'_j u(0,x')| \Big)^p dt,$$

then

$$\|\Delta'_k R_0 u(\cdot,x')\|_{L^p(\mathbb{R}_+)} \lesssim 2^{\frac{-k}{p}} \sum_{j\sim k} |\Delta'_k \Delta'_j u(0,x')|,$$

we integrate with respect to x' over a ball B'_J to obtain

$$\|\Delta'_k R_0 u\|_{L^p(\mathbb{R}_+\times B'_J)} \lesssim 2^{\frac{-k}{p}} \sum_{j\sim k} \|\Delta'_k \Delta'_j u\|_{L^p(B'_J)},$$

now Lemma 2.2 gives

$$\begin{aligned}\|\Delta'_k R_0 u\|_{L^p(\mathbb{R}_+\times B'_J)} \le & C_M \Big\{ 2^{\frac{-k}{p}} \|\Delta'_k u\|_{L^p(2B'_J)} \\ & + 2^{\frac{-k}{p}} \sum_{\nu \ge -J+1} 2^{-(\nu+k)M} \|\Delta'_k u\|_{L^p(F'_\nu)} \Big\},\end{aligned}$$

multiply by $\frac{2^{k(s+1)}}{|B'_J|^\tau}$ and take the $l^q - norm$

$$\begin{aligned}&\|R_0 u\|_{L^p(\mathbb{R}_+;B^{s+1,\tau}_{p,q}(\mathbb{R}^n))} \\ &\le C_M \Big\{ \frac{1}{|B'_J|^\tau} \Big(\sum_{k\ge J^+} 2^{kq(s+1-\frac{1}{p})} \|\Delta'_k u\|^q_{L^p(2B'_J)} \Big)^{\frac{1}{q}} \\ &\quad + \frac{1}{|B'_J|^\tau} \Big[\sum_{k\ge J^+} 2^{kq(s+1-\frac{1}{p})} \Big(\sum_{\nu\ge -J+1} 2^{-(\nu+k)M} \|\Delta'_k u\|_{L^p(F'_\nu)} \Big)^q \Big]^{\frac{1}{q}} \Big\}.\end{aligned} \tag{8}$$

For the second term $\|tR_0 u\|_{L^p(\mathbb{R}_+;B^{s+2,\tau}_{p,q}(\mathbb{R}^n))}$, using the same technique, we obtain

$$\int_{\mathbb{R}_+} |\Delta'_k t R_0 u(t,x')|^p dt \le \int_{\mathbb{R}_+} \Big(\sum_{j\sim k} |t\varphi(2^j t)\Delta'_k \Delta'_j u(0,x')| \Big)^p dt,$$

then

$$\begin{aligned}&\|tR_0 u\|_{L^p(\mathbb{R}_+;B^{s+2,\tau}_{p,q}(\mathbb{R}^n))} \\ &\le C_M \Big\{ \frac{1}{|B'_J|^\tau} \Big(\sum_{k\ge J^+} 2^{kq(s+1-\frac{1}{p})} \|\Delta'_k u\|^q_{L^p(2B'_J)} \Big)^{\frac{1}{q}} \\ &\quad + \frac{1}{|B'_J|^\tau} \Big[\sum_{k\ge J^+} 2^{kq(s+1-\frac{1}{p})} \Big(\sum_{\nu\ge -J+1} 2^{-(\nu+k)M} \|\Delta'_k u\|_{L^p(F'_\nu)} \Big)^q \Big]^{\frac{1}{q}} \Big\}.\end{aligned} \tag{9}$$

Similarly, for the third term $\|D_t R_0 u\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))}$, we have

$$\int_{\mathbb{R}_+} |\Delta'_k D_t R_0 u(t,x')|^p dt \le \int_{\mathbb{R}_+} \Big(\sum_{j\sim k} |2^j (D_t\varphi)(2^j t)\Delta'_k \Delta'_j u(0,x')| \Big)^p dt,$$

then

$$\begin{aligned}&\|D_t R_0 u\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))} \\ &\le C_M \Big\{ \frac{1}{|B'_J|^\tau} \Big(\sum_{k\ge J^+} 2^{kq(s+1-\frac{1}{p})} \|\Delta'_k u\|^q_{L^p(2B'_J)} \Big)^{\frac{1}{q}}\end{aligned} \tag{10}$$

$$+ \frac{1}{|B'_J|^\tau}\Big[\sum_{k\geq J^+} 2^{kq(s+1-\frac{1}{p})}\Big(\sum_{\nu\geq -J+1} 2^{-(\nu+k)M}\|\Delta'_k u\|_{L^p(F'_\nu)}\Big)^q\Big]^{\frac{1}{q}}\Big\}.$$

Afterwards, for the fourth term $\|tD_t^2R_0u\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))}$, we have

$$\int_{\mathbb{R}_+}|\Delta'_k tD_t^2R_0u(t,x')|^p dt \leq \int_{\mathbb{R}_+}\Big(\sum_{j\sim k}2^j|2^jt(D_t^2\varphi)(2^jt)\Delta'_k\Delta'_j u(0,x')|\Big)^p dt,$$

then

$$\begin{aligned}\|tD_t^2R_0u\|&_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))} \\ &\leq C_M\Big\{\frac{1}{|B'_J|^\tau}\Big(\sum_{k\geq J^+} 2^{kq(s+1-\frac{1}{p})}\|\Delta'_k u\|^q_{L^p(2B'_J)}\Big)^{\frac{1}{q}} \\ &\quad + \frac{1}{|B'_J|^\tau}\Big[\sum_{k\geq J^+} 2^{kq(s+1-\frac{1}{p})}\Big(\sum_{\nu\geq -J+1} 2^{-(\nu+k)M}\|\Delta'_k u\|_{L^p(F'_\nu)}\Big)^q\Big]^{\frac{1}{q}}\Big\}.\end{aligned} \tag{11}$$

Finally, to estimate the last term $\|tD_tR_0u\|_{L^p(\mathbb{R}_+;B^{s+1,\tau}_{p,q}(\mathbb{R}^n))}$, we write

$$\int_{\mathbb{R}_+}|\Delta'_k tD_tR_0u(t,x')|^p dt \leq \int_{\mathbb{R}_+}\Big(\sum_{j\sim k}|2^jt(D_t\varphi)(2^jt)\Delta'_k\Delta'_j u(0,x')|\Big)^p dt,$$

then

$$\begin{aligned}\|tD_tR_0u\|&_{L^p(\mathbb{R}_+;B^{s+1,\tau}_{p,q}(\mathbb{R}^n))} \\ &\leq C_M\Big\{\frac{1}{|B'_J|^\tau}\Big(\sum_{k\geq J^+} 2^{kq(s+1-\frac{1}{p})}\|\Delta'_k u\|^q_{L^p(2B'_J)}\Big)^{\frac{1}{q}} \\ &\quad + \frac{1}{|B'_J|^\tau}\Big[\sum_{k\geq J^+} 2^{kq(s+1-\frac{1}{p})}\Big(\sum_{\nu\geq -J+1} 2^{-(\nu+k)M}\|\Delta'_k u\|_{L^p(F'_\nu)}\Big)^q\Big]^{\frac{1}{q}}\Big\}.\end{aligned} \tag{12}$$

From inequalities (8)–(12), we deduce

$$\begin{aligned}\|R_0u\|&_{W^{2,p}_w(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))} \\ &\leq C_M\Big\{\frac{1}{|B'_J|^\tau}\Big(\sum_{k\geq J^+} 2^{kq(s+1-\frac{1}{p})}\|\Delta'_k u\|^q_{L^p(2B'_J)}\Big)^{\frac{1}{q}} \\ &\quad + \frac{1}{|B'_J|^\tau}\Big[\sum_{k\geq J^+} 2^{kq(s+1-\frac{1}{p})}\Big(\sum_{\nu\geq -J+1} 2^{-(\nu+k)M}\|\Delta'_k u\|_{L^p(F'_\nu)}\Big)^q\Big]^{\frac{1}{q}}\Big\},\end{aligned}$$

thus

$$\|R_0u\|_{W^{2,p}_w(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))} \leq C_M\{I_1+I_2\},$$

where

$$I_1 = \frac{1}{|B'_J|^\tau}\Big(\sum_{k\geq J^+} 2^{kq(s+1-\frac{1}{p})}\|\Delta'_k u\|^q_{L^p(2B'_J)}\Big)^{\frac{1}{q}} \leq \|u\|_{B^{s+1-\frac{1}{p},\tau}_{p,q}(\mathbb{R}^n)}.$$

For I_2, we set $\mu = \nu + J$ and $|F'_{\mu-J}| \sim 2^{n(\mu-J)}$, then

$$I_2^q \leq \frac{1}{|B'_J|^{\tau q}}\sum_{k\geq J^+}\Big(\sum_{\mu\geq 1}\frac{2^{n\tau(\mu-J)}}{|F'_{\mu-J}|^\tau}\times 2^{-(\mu-J+k)M}\times 2^{k(s+1-\frac{1}{p})}\|\Delta'_k u\|_{L^p(F'_{\mu-J})}\Big)^q$$

$$\leq \sum_{k\geq J^+} 2^{(J-k)Mq}\Big(\sum_{\mu\geq 1} 2^{\mu(n\tau-M)} \times \frac{2^{k(s+1-\frac{1}{p})}}{|F'_{\mu-J}|^{\tau}}\|\Delta'_k u\|_{L^p(F'_{\mu-J})}\Big)^q,$$

applying Lemma 2.1 for M sufficiently large

$$\begin{aligned} I_2^q &\lesssim \sup_{\mu\geq 1}\frac{1}{|F'_{\mu-J}|^{\tau q}}\sum_{k\geq J^+} 2^{kq(s+1-\frac{1}{p})}\|\Delta'_k u\|^q_{L^p(F'_{\mu-J})} \\ &\lesssim \sup_{\mu\geq 1}\frac{1}{|F'_{\mu-J}|^{\tau q}}\sum_{k\geq (J-\mu+1)^+} 2^{kq(s+1-\frac{1}{p})}\|\Delta'_k u\|^q_{L^p(F'_{\mu-J})}, \end{aligned}$$

finally, we deduce

$$\|R_0 u\|_{W^{2,p}_w(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))} \lesssim \|u\|_{B^{s+1-\frac{1}{p},\tau}_{p,q}(\mathbb{R}^n)}.$$

Consequently, R_0 is bounded from $B^{s+1-\frac{1}{p},\tau}_{p,q}(\mathbb{R}^n)$ to $W^{2,p}_w(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))$ with

$$\gamma_0 o R_0 = id_{B^{s+1-\frac{1}{p},\tau}_{p,q}(\mathbb{R}^n)}.$$

If $s\geq 0$, the space $B^{s+2,\tau}_{p,q,w}(\mathbb{R}^{n+1}_+)$ is continuously embedded in the space $W^{2,p}_{w,loc}(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))$. The proof of Theorem 3.1 is complete.

4. Proof of theorem 1.1

Proof. To prove the theorem, we should first start by proving the following estimate.

Proposition 4.1. *Let s, τ be two non-negative real numbers and let $1\leq p,q<+\infty$. Under hypotheses (H_1), (H_2) and (H_3), for any compact set K of $\overline{\mathbb{R}^{n+1}_+}$, there exists a constant $C_K>0$ such that for any $u\in W^{2,p}_w(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))$ with supp $u\subset K$, we get:*

$$\begin{aligned} \|u\|_{W^{2,p}_w(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))} \leq & C_K\Big\{\|Lu\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))} + \|\gamma_0 u\|_{B^{s+1-\frac{1}{p},\tau}_{p,q}(\mathbb{R}^n)} \\ & + \|u\|_{L^p(\mathbb{R}_+;B^{s-1,\tau}_{p,q}(\mathbb{R}^n))}\Big\}. \end{aligned}$$

Proof. The idea of the proof is the same as in [2, 10, 12]. We use the Korn technique by decomposing the operator L as follows

$$L = L^0 + L^1,$$

where

$$\begin{aligned} L^0(t,x';D_t,D_{x'}) =& tD_t^2 + \sum_{j,k=1}^n a_{j,k}(0,0)tD_{x_j}D_{x_k} + \sum_{j=1}^n b_j(0,0)tD_{x_j}D_t \\ & + \sum_{j=1}^n c_j(0,0)D_{x_j} + d(0,0)D_t, \end{aligned}$$

with coefficients in $\mathbb{C}$, and

$$\begin{aligned} L^1(t,x';D_t,D_{x'}) = & \sum_{j,k=1}^n \Big(a_{j,k}(t,x')-a_{j,k}(0,0)\Big)tD_{x_j}D_{x_k} \\ & + \sum_{j=1}^n \Big(b_j(t,x')-b_j(0,0)\Big)tD_{x_j}D_t \\ & + \sum_{j=1}^n \Big(c_j(t,x')-c_j(0,0)\Big)D_{x_j} \end{aligned}$$

$$+\Big(d(t,x')-d(0,0)\Big)D_t+e(t,x').$$

This is the freezing technique of the coefficients of L at the point 0. It allows us to obtain a homogeneous operator L^0 with constant coefficients, that we first estimate by applying a partial Fourier transformation on x'. Next, we estimate the different terms of the operator L^1.

As in [2], under hypotheses $(H_1),(H_2)$ and (H_3), for every $\xi'\in\mathbb{R}^n\setminus\{0\}$, the operator $(L^0(t,0;D_t,\xi'),\gamma_0)$ is invertible from $W^{2,p}_w(\mathbb{R}_+)$ onto $L^p(\mathbb{R}_+)\times\mathbb{C}$ and if $K_{\xi'}$ denotes its inverse, then the mapping $\xi'\mapsto K_{\xi'}$ is C^∞ from $\mathbb{R}^n\setminus\{0\}$ to $\mathcal{L}\big(L^p(\mathbb{R}_+)\times\mathbb{C};W^{2,p}_w(\mathbb{R}_+)\big)$ and for any multi-index α', there exists $C_{\alpha'}>0$ such that for any ξ' with $\frac{1}{2}\le|\xi'|\le 2$ and any $(f,g)\in L^p(\mathbb{R}_+)\times\mathbb{C}$

$$\|D^{\alpha'}_{\xi'}K_{\xi'}(f,g)\|_{W^{2,p}_w(\mathbb{R}_+)}\le C_{\alpha'}\|(f,g)\|_{L^p(\mathbb{R}_+)\times\mathbb{C}}. \tag{13}$$

First, we prove that for any $N\ge 1$ sufficiently large and for any ball B'_J of $\mathbb{R}^n$, the following estimate

$$\begin{aligned}\|u\|_{L^p(B'_J;W^{2,p}_w(\mathbb{R}_+))}\lesssim&\|L^0u\|_{L^p(2B'_J;L^p(\mathbb{R}_+))}+\|\gamma_0u\|_{L^p(2B'_J)}\\&+|B'_J|^{\frac{1}{p}}\sum_{\nu\ge -J+1}2^{-2\nu N}|F'_\nu|^{1-\frac{1}{p}}\Big(\|L^0u\|_{L^p(F'_\nu;L^p(\mathbb{R}_+))}\\&+\|\gamma_0u\|_{L^p(F'_\nu)}\Big),\end{aligned}\tag{14}$$

holds for any $u\in\mathcal{S}(\mathbb{R}^n;W^{2,p}_w(\mathbb{R}_+))$ with tangential spectrum belonging to the annulus $\frac{1}{2}\le|\xi'|\le 2$.

We apply the operator $(L^0(t,0;D_t,\xi'),\gamma_0(\xi'))$ to the relation

$$\widehat{u}(\cdot,\xi')=\int_{y'\in\mathbb{R}^n}e^{-iy'\cdot\xi'}u(\cdot,y')dy',$$

to get the system

$$\left\{\begin{array}{ccc}L^0(t,0;D_t,\xi')\widehat{u}(\cdot,\xi')&=&\widehat{L^0u}(\cdot,\xi')=\int e^{-iy'\cdot\xi'}L^0u(\cdot,y')dy'\\ \gamma_0\widehat{u}(\xi')&=&\widehat{\gamma_0u}(\xi')\ =\int e^{-iy'\cdot\xi'}\gamma_0u(y')dy',\end{array}\right.\tag{15}$$

apply $K_{\xi'}$ to that system, we obtain

$$\widehat{u}(\cdot,\xi')=\int e^{-iy'\cdot\xi'}K_{\xi'}\big(L^0u(\cdot,y'),\gamma_0u(y')\big)dy'.$$

Let $\phi(\xi')\in C^\infty_0(\mathbb{R}^n)$ equals to 1 on $\frac{1}{2}\le|\xi'|\le 2$ and its support belongs to an annulus. Then

$$\begin{aligned}u(\cdot,x')&=\int e^{ix'\cdot\xi'}\phi(\xi')\widehat{u}(\cdot,\xi')\frac{d\xi'}{(2\pi)^n}\\&=\int\int e^{i(x'-y')\cdot\xi'}\Big\{\phi(\xi')K_{\xi'}\big(L^0u(\cdot,y'),\gamma_0u(y')\big)\Big\}dy'\frac{d\xi'}{(2\pi)^n}\\&=\int\int\frac{e^{i(x'-y')\cdot\xi'}}{\big(1+|x'-y'|^2\big)^N}\big(I-\Delta_{\xi'}\big)^N\\&\quad\times\Big\{\phi(\xi')K_{\xi'}\big(L^0u(\cdot,y'),\gamma_0u(y')\big)\Big\}dy'\frac{d\xi'}{(2\pi)^n},\end{aligned}$$

then

$$\begin{aligned}&\|u(\cdot,x')\|_{W^{2,p}_w(\mathbb{R}_+)}\\&\le C_N\int_{y'\in\mathbb{R}^n}\frac{1}{1+|x'-y'|^{2N}}\\&\quad\times\left\|\int_{\frac{1}{2}\le|\xi'|\le 2}D^{\alpha'}_{\xi'}\Big\{\phi(\xi')K_{\xi'}\big(L^0u(\cdot,y'),\gamma_0u(y')\big)\Big\}\frac{d\xi'}{(2\pi)^n}\right\|_{W^{2,p}_w(\mathbb{R}_+)}dy',\end{aligned}$$

applying the inequality (13), we deduce

$$\|u(\cdot,x')\|_{W^{2,p}_w(\mathbb{R}_+)} \le C_N \int_{y'\in\mathbb{R}^n} \frac{1}{1+|x'-y'|^{2N}} \Big\|\big(L^0u(\cdot,y'),\gamma_0u(y')\big)\Big\|_{L^p(\mathbb{R}_+)\times\mathbb{C}} dy',$$

integrating with respect to x' over B'_J, we obtain

$$\|u\|_{L^p(B'_J;W^{2,p}_w(\mathbb{R}_+))} \le C_N \Big\{ \int_{B'_J} \Big(\int_{y'\in\mathbb{R}^n} \frac{1}{1+|x'-y'|^{2N}} \Big\|\big(L^0u(\cdot,y'),\gamma_0u(y')\big)\Big\|_{L^p(\mathbb{R}_+)\times\mathbb{C}} dy' \Big)^p dx' \Big\}^{\frac{1}{p}}.$$

Now, we decompose $\mathbb{R}^n$

$$\mathbb{R}^n = 2B'_J \cup \bigcup_{\nu\ge -J+1} F'_\nu.$$

Then

$$\begin{aligned}\|u\|_{L^p(B'_J;W^{2,p}_w(\mathbb{R}_+))} &\lesssim \Big\{ \int_{B'_J} \Big(\int_{y'\in\mathbb{R}^n} \frac{1}{1+|x'-y'|^{2N}} \\ &\quad \times \chi_{2B'_J}(y')\Big\|\big(L^0u(\cdot,y'),\gamma_0u(y')\big)\Big\|_{L^p(\mathbb{R}_+)\times\mathbb{C}} dy' \Big)^p dx' \Big\}^{\frac{1}{p}} \\ &\quad + \Big\{ \int_{B'_J} \Big(\sum_{\nu\ge -J+1} \int_{y'\in F'_\nu} \frac{1}{1+|x'-y'|^{2N}} \\ &\quad \times \Big\|\big(L^0u(\cdot,y'),\gamma_0u(y')\big)\Big\|_{L^p(\mathbb{R}_+)\times\mathbb{C}} dy' \Big)^p dx' \Big\}^{\frac{1}{p}} \\ &\lesssim I'_1 + I'_2.\end{aligned}$$

The first term I'_1 is an $L^p - norm$ of a convolution product between a function of $L^1(\mathbb{R}^n)$ (for N large) and a function of $L^p(\mathbb{R}^n)$. Young's inequality yields

$$I'_1 \le C_N \big\{ \|L^0u\|_{L^p(2B'_J;L^p(\mathbb{R}_+))} + \|\gamma_0 u\|_{L^p(2B'_J)} \big\}.$$

For I'_2, since $x'\in B'_J$ and $y'\in F'_\nu$, we have $|x'-y'|\sim 2^\nu$. Then

$$I'_2 \lesssim |B'_J|^{\frac{1}{p}} \sum_{\nu\ge -J+1} 2^{-2\nu N} \int_{y'\in F'_\nu} \Big\|\big(L^0u(\cdot,y'),\gamma_0u(y')\big)\Big\|_{L^p(\mathbb{R}_+)\times\mathbb{C}} dy',$$

Hölder's inequality yields

$$I'_2 \lesssim |B'_J|^{\frac{1}{p}} \sum_{\nu\ge -J+1} 2^{-2\nu N} |F'_\nu|^{1-\frac{1}{p}} \Big(\int_{y'\in F'_\nu} \Big\|\big(L^0u(\cdot,y'),\gamma_0u(y')\big)\Big\|^p_{L^p(\mathbb{R}_+)\times\mathbb{C}} dy' \Big)^{\frac{1}{p}}.$$

Let $u\in W^{2,p}_w(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))$ with *supp* $u\subset K$, where K is a compact set of $\overline{\mathbb{R}^{n+1}_+}$, for $j\in\mathbb{N}$, we set $u_j(t,x') = \Delta'_j u(2^{-j}t, 2^{-j}x')$, then $u_j\in\mathcal{S}(\mathbb{R}^n;W^{2,p}_w(\mathbb{R}_+))$ for $j\ge 1$, with tangential spectrum belonging to the annulus $\{\frac{1}{2}\le |\xi'|\le 2\}$. Since

$$\begin{cases} (L^0u)_j &= 2^jL^0u_j \\ (\gamma_0u)_j &= \gamma_0u_j, \end{cases}$$

we have

$$\|u\|_{L^p(B'_J;W^{2,p}_w(\mathbb{R}_+))} \lesssim \|L^0u\|_{L^p(2B'_J;L^p(\mathbb{R}_+))} + \|\gamma_0u\|_{L^p(2B'_J)} \tag{16}$$

$$+|B'_J|^{\frac{1}{p}} \sum_{\nu\geq -J+1} 2^{-2\nu N}|F'_\nu|^{1-\frac{1}{p}}\left(\int_{y'\in F'_\nu}\left\|\left(L^0u(\cdot,y'),\gamma_0u(y')\right)\right\|^p_{L^p(\mathbb{R}_+)\times\mathbb{C}}dy'\right)^{\frac{1}{p}},$$

applying u_j to the inequality (16) for $j\geq 1$, we obtain

$$\begin{aligned}
\|u_j\|^q_{L^p(B'_J;W^{2,p}_w(\mathbb{R}_+))} &\lesssim \|L^0u_j\|^q_{L^p(2B'_J;L^p(\mathbb{R}_+))}+\|\gamma_0u_j\|^q_{L^p(2B'_J)}\\
&\quad+|B'_J|^{\frac{q}{p}}\Big\{\sum_{\nu\geq -J+1}2^{-2\nu N}|F'_\nu|^{1-\frac{1}{p}}\Big(\|L^0u_j\|_{L^p(F'_\nu;L^p(\mathbb{R}_+))}\\
&\quad+\|\gamma_0u_j\|_{L^p(F'_\nu)}\Big)\Big\}^q\\
&\lesssim 2^{-jq}\|(L^0u)_j\|^q_{L^p(2B'_J;L^p(\mathbb{R}_+))}+\|(\gamma_0u)_j\|^q_{L^p(2B'_J)}\\
&\quad+|B'_J|^{\frac{q}{p}}\Big\{\sum_{\nu\geq -J+1}2^{-2\nu N}|F'_\nu|^{1-\frac{1}{p}}\Big(2^{-j}\|(L^0u)_j\|_{L^p(F'_\nu;L^p(\mathbb{R}_+))}\\
&\quad+\|(\gamma_0u)_j\|_{L^p(F'_\nu)}\Big)\Big\}^q.
\end{aligned}$$

Now, replacing u_j with $\Delta'_ju(2^{-j}t,2^{-j}x')$ and multiplying by 2^{jqs}, we obtain

$$\begin{aligned}
&2^{jq(s+1)}\|\Delta'_ju\|^q_{L^p(\mathbb{R}_+\times 2^{-j}B'_J)}+2^{jqs}\|tD_t^2\Delta'_ju\|^q_{L^p(\mathbb{R}_+\times 2^{-j}B'_J)}\\
&+2^{jq(s+1)}\|tD_t\Delta'_ju\|^q_{L^p(\mathbb{R}_+\times 2^{-j}B'_J)}+2^{jqs}\|D_t\Delta'_ju\|^q_{L^p(\mathbb{R}_+\times 2^{-j}B'_J)}\\
&+2^{jq(s+2)}\|t\Delta'_ju\|^q_{L^p(\mathbb{R}_+\times 2^{-j}B'_J)}\\
&\lesssim 2^{jqs}\|\Delta'_jL^0u\|^q_{L^p(\mathbb{R}_+\times 2^{-j+1}B'_J)}+2^{jq(s+1-\frac{1}{p})}\|\Delta'_j\gamma_0u\|^q_{L^p(2^{-j+1}B'_J)}\\
&\quad+|B'_J|^{\frac{q}{p}}\Big\{\sum_{\nu\geq -J+1}2^{-2\nu N}|F'_\nu|^{1-\frac{1}{p}}\Big(2^{js}\|\Delta'_jL^0u\|_{L^p(\mathbb{R}_+\times 2^{-j}F'_\nu)}\\
&\quad+2^{j(s+1-\frac{1}{p})}\|\Delta'_j\gamma_0u\|_{L^p(2^{-j}F'_\nu)}\Big)\Big\}^q\\
&\lesssim I''_1+I''_2,
\end{aligned}\tag{17}$$

I''_1 represents both the first and the second term on the right-hand side of the above inequality, while I''_2 represents the last. We set $K=J+j$ and $\mu=\nu-j$, then

$$I''_1=2^{jqs}\|\Delta'_jL^0u\|^q_{L^p(\mathbb{R}_+\times 2B'_K)}+2^{jq(s+1-\frac{1}{p})}\|\Delta'_j\gamma_0u\|^q_{L^p(2B'_K)}.\tag{18}$$

On the other side, since $|F'_\nu|^{1-\frac{1}{p}}\sim 2^{n\nu(1-\frac{1}{p})}$

$$\begin{aligned}
I''_2\leq&|B'_J|^{\frac{q}{p}}\Big\{2^{j(n(1-\frac{1}{p})-2N)}\sum_{\mu\geq -K+1}2^{\mu(n(1-\frac{1}{p})-2N)}\Big(2^{js}\|\Delta'_jL^0u\|_{L^p(\mathbb{R}_+\times F'_\mu)}\\
&+2^{j(s+1-\frac{1}{p})}\|\Delta'_j\gamma_0u\|_{L^p(F'_\mu)}\Big)\Big\}^q,
\end{aligned}$$

let us set $\mu'=\mu+K$,

$$\begin{aligned}
I''_2\leq&|B'_J|^{\frac{q}{p}}2^{(j-K)q(n(1-\frac{1}{p})-2N)}2^{-Kn\tau q}\Big\{\sum_{\mu'\geq 1}2^{\mu'(n(1-\frac{1}{p})+n\tau-2N)}\\
&\times\Big(\frac{2^{js}}{|F'_{\mu'-K}|^\tau}\|\Delta'_jL^0u\|_{L^p(\mathbb{R}_+\times F'_{\mu'-K})}+\frac{2^{j(s+1-\frac{1}{p})}}{|F'_{\mu'-K}|^\tau}\|\Delta'_j\gamma_0u\|_{L^p(F'_{\mu'-K})}\Big)\Big\}^q,
\end{aligned}\tag{19}$$

considering inequalities (17)–(19), multiplying by $\frac{1}{|B'_K|^{\tau q}}$ and summing over $j \geq max(K,1)$, we obtain

$$\begin{aligned}
&\sum_{j\geq max(K,1)} \frac{2^{jq(s+1)}}{|B'_K|^{\tau q}} \|\Delta'_j u\|^q_{L^p(\mathbb{R}_+\times B'_K)} + \sum_{j\geq max(K,1)} \frac{2^{jqs}}{|B'_K|^{\tau q}} \|\Delta'_j t D_t^2 u\|^q_{L^p(\mathbb{R}_+\times B'_K)} \\
&+ \sum_{j\geq max(K,1)} \frac{2^{jq(s+1)}}{|B'_K|^{\tau q}} \|\Delta'_j t D_t u\|^q_{L^p(\mathbb{R}_+\times B'_K)} + \sum_{j\geq max(K,1)} \frac{2^{jqs}}{|B'_K|^{\tau q}} \|\Delta'_j D_t u\|^q_{L^p(\mathbb{R}_+\times B'_K)} \\
&+ \sum_{j\geq max(K,1)} \frac{2^{jq(s+2)}}{|B'_K|^{\tau q}} \|\Delta'_j t u\|^q_{L^p(\mathbb{R}_+\times B'_K)} \\
&\lesssim \sum_{j\geq max(K,1)} \frac{2^{jqs}}{|B'_K|^{\tau q}} \|\Delta'_j L^0 u\|^q_{L^p(\mathbb{R}_+\times 2B'_K)} + \sum_{j\geq max(K,1)} \frac{2^{jq(s+1-\frac{1}{p})}}{|B'_K|^{\tau q}} \\
&\quad \times \|\Delta'_j \gamma_0 u\|^q_{L^p(2B'_K)} + \frac{1}{|B'_K|^{\tau q}} \sum_{j\geq max(K,1)} 2^{(j-K)(n-2N)q} 2^{-Kn\tau q} \\
&\quad \times \Big\{ \sum_{\mu'\geq 1} 2^{\mu'(n(1-\frac{1}{p})+n\tau-2N)} \Big(\frac{2^{js}}{|F'_{\mu'-K}|^{\tau}} \|\Delta'_j L^0 u\|_{L^p(\mathbb{R}_+\times F'_{\mu'-K})} \\
&\quad + \frac{2^{j(s+1-\frac{1}{p})}}{|F'_{\mu'-K}|^{\tau}} \|\Delta'_j \gamma_0 u\|_{L^p(F'_{\mu'-K})} \Big) \Big\}^q,
\end{aligned}$$

since $K^+ \leq max(K,1)$, Lemma 2.1 gives

$$\begin{aligned}
&\sum_{j\geq max(K,1)} \frac{2^{jq(s+1)}}{|B'_K|^{\tau q}} \|\Delta'_j u\|^q_{L^p(\mathbb{R}_+\times B'_K)} + \sum_{j\geq max(K,1)} \frac{2^{jqs}}{|B'_K|^{\tau q}} \|\Delta'_j t D_t^2 u\|^q_{L^p(\mathbb{R}_+\times B'_K)} \\
&+ \sum_{j\geq max(K,1)} \frac{2^{jq(s+1)}}{|B'_K|^{\tau q}} \|\Delta'_j t D_t u\|^q_{L^p(\mathbb{R}_+\times B'_K)} + \sum_{j\geq max(K,1)} \frac{2^{jqs}}{|B'_K|^{\tau q}} \|\Delta'_j D_t u\|^q_{L^p(\mathbb{R}_+\times B'_K)} \\
&+ \sum_{j\geq max(K,1)} \frac{2^{jq(s+2)}}{|B'_K|^{\tau q}} \|\Delta'_j t u\|^q_{L^p(\mathbb{R}_+\times B'_K)} \\
&\lesssim \|L^0 u\|^q_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))} + \|\gamma_0 u\|^q_{B^{s+1-\frac{1}{p},\tau}_{p,q}(\mathbb{R}^n)} + \sup_{\mu'\geq 1} \frac{1}{|F'_{\mu'-K}|^{\tau q}} \\
&\quad \times \sum_{j\geq K^+} \Big(2^{jqs} \|\Delta'_j L^0 u\|^q_{L^p(\mathbb{R}_+\times F'_{\mu'-K})} + 2^{jq(s+1-\frac{1}{p})} \|\Delta'_j \gamma_0 u\|^q_{L^p(F'_{\mu'-K})} \Big),
\end{aligned}$$

we add the terms associated to $j = 0$ and we replace the condition on the right-hand side of the inequality above $j \geq K^+$ with $j \geq (K - \mu' + 1)^+$, we obtain

$$\|u\|^q_{W^{2,p}_w(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))} \leq C_K \Big\{ \|L^0 u\|^q_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))} + \|\gamma_0 u\|^q_{B^{s+1-\frac{1}{p},\tau}_{p,q}(\mathbb{R}^n)} \Big\} + R_0, \tag{20}$$

where

$$\begin{aligned}
R_0 =& \frac{1}{|B'_K|^{\tau q}} \|\Delta'_0 D_t u\|^q_{L^p(\mathbb{R}_+\times B'_K)} + \frac{1}{|B'_K|^{\tau q}} \|\Delta'_0 t D_t u\|^q_{L^p(\mathbb{R}_+\times B'_K)} \\
&+ \frac{1}{|B'_K|^{\tau q}} \|\Delta'_0 u\|^q_{L^p(\mathbb{R}_+\times B'_K)} + \frac{1}{|B'_K|^{\tau q}} \|\Delta'_0 t u\|^q_{L^p(\mathbb{R}_+\times B'_K)} \\
&+ \frac{1}{|B'_K|^{\tau q}} \|\Delta'_0 t D_t^2 u\|^q_{L^p(\mathbb{R}_+\times B'_K)}.
\end{aligned}$$

Now, we'll estimate the remaining term R_0. From Lemma 2.3 and since $supp\, u \subset K$, we can estimate the first and the second term

$$\begin{aligned}
&\frac{1}{|B'_K|^{\tau q}}\|\Delta'_0 D_t u\|^q_{L^p(\mathbb{R}_+\times B'_K)}\\
&\lesssim \varepsilon \frac{1}{|B'_K|^{\tau q}}\|\Delta'_0 tD_t^2 u\|^q_{L^p(\mathbb{R}_+\times B'_K)} + \varepsilon^{-1}\frac{1}{|B'_K|^{\tau q}}\|\Delta'_0 tu\|^q_{L^p(\mathbb{R}_+\times B'_K)}\\
&\lesssim \varepsilon \frac{1}{|B'_K|^{\tau q}}\|\Delta'_0 tD_t^2 u\|^q_{L^p(\mathbb{R}_+\times B'_K)} + C_K\|u\|^q_{L^p(\mathbb{R}_+;B^{s-1,\tau}_{p,q}(\mathbb{R}^n))},
\end{aligned}\tag{21}$$

$$\begin{aligned}
&\frac{1}{|B'_K|^{\tau q}}\|\Delta'_0 tD_t u\|^q_{L^p(\mathbb{R}_+\times B'_K)}\\
&\lesssim \varepsilon \frac{1}{|B'_K|^{\tau q}}\|\Delta'_0 tD_t^2 u\|^q_{L^p(\mathbb{R}_+\times B'_K)} + \varepsilon^{-1}\frac{1}{|B'_K|^{\tau q}}\|\Delta'_0 tu\|^q_{L^p(\mathbb{R}_+\times B'_K)}\\
&\lesssim \varepsilon \frac{1}{|B'_K|^{\tau q}}\|\Delta'_0 tD_t^2 u\|^q_{L^p(\mathbb{R}_+\times B'_K)} + C_K\|u\|^q_{L^p(\mathbb{R}_+;B^{s-1,\tau}_{p,q}(\mathbb{R}^n))}.
\end{aligned}\tag{22}$$

For the third term of R_0, we have

$$\frac{1}{|B'_K|^{\tau q}}\|\Delta'_0 u\|^q_{L^p(\mathbb{R}_+\times B'_K)} \lesssim \|u\|^q_{L^p(\mathbb{R}_+;B^{s-1,\tau}_{p,q}(\mathbb{R}^n))}.\tag{23}$$

Since $supp\, u \subset K$, we deduce the estimate of the fourth term of R_0

$$\frac{1}{|B'_K|^{\tau q}}\|\Delta'_0 tu\|^q_{L^p(\mathbb{R}_+\times B'_K)} \le C_K\|u\|^q_{L^p(\mathbb{R}_+;B^{s-1,\tau}_{p,q}(\mathbb{R}^n))}.\tag{24}$$

From (21)–(24), we deduce

$$R_0 \lesssim C_K\|u\|^q_{L^p(\mathbb{R}_+;B^{s-1,\tau}_{p,q}(\mathbb{R}^n))} + \frac{1}{|B'_K|^{\tau q}}\|\Delta'_0 tD_t^2 u\|^q_{L^p(\mathbb{R}_+\times B'_K)}.\tag{25}$$

Finally, we estimate the last term. From the equation we have

$$\begin{aligned}
tD_t^2 u =& L^0(t,x';D_t,D_{x'})u - \Big(\sum_{j,k=1}^n a_{j,k}(0,0)tD_{x_j}D_{x_k}u + \sum_{j=1}^n b_j(0,0)tD_{x_j}D_t u\\
&+\sum_{j=1}^n c_j(0,0)D_{x_j}u + d(0,0)D_t u\Big),
\end{aligned}$$

then

$$\begin{aligned}
&\|\Delta'_0 tD_t^2 u\|^q_{L^p(\mathbb{R}_+\times B'_K)}\\
&\lesssim \|\Delta'_0 L^0 u\|^q_{L^p(\mathbb{R}_+\times B'_K)} + \sum_{j,k=1}^n \|a_{j,k}(0,0)\Delta'_0 tD_{x_j}D_{x_k}u\|^q_{L^p(\mathbb{R}_+\times B'_K)}\\
&\quad+\sum_{j=1}^n \|b_j(0,0)\Delta'_0 tD_{x_j}D_t u\|^q_{L^p(\mathbb{R}_+\times B'_K)} + \sum_{j=1}^n \|c_j(0,0)\Delta'_0 D_{x_j}u\|^q_{L^p(\mathbb{R}_+\times B'_K)}\\
&\quad+\|d(0,0)\Delta'_0 D_t u\|^q_{L^p(\mathbb{R}_+\times B'_K)},
\end{aligned}$$

thus

$$\begin{aligned}
&\frac{1}{|B'_K|^{\tau q}}\|\Delta'_0 tD_t^2 u\|^q_{L^p(\mathbb{R}_+\times B'_K)}\\
&\lesssim \|L^0 u\|^q_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))} + \sum_{j,k=1}^n \|tD_{x_j}D_{x_k}u\|^q_{L^p(\mathbb{R}_+;B^{s-3,\tau}_{p,q}(\mathbb{R}^n))}
\end{aligned}$$

$$+\sum_{j=1}^{n}\|tD_{x_j}D_tu\|^q_{L^p(\mathbb{R}_+;B^{s-3,\tau}_{p,q}(\mathbb{R}^n))}+\sum_{j=1}^{n}\|D_{x_j}u\|^q_{L^p(\mathbb{R}_+;B^{s-2,\tau}_{p,q}(\mathbb{R}^n))}$$
$$+\|D_tu\|^q_{L^p(\mathbb{R}_+;B^{s-3,\tau}_{p,q}(\mathbb{R}^n))}.$$

Since the operator $D_{x'}$ maps continuously from $L^p(\mathbb{R}_+;B^{s+1,\tau}_{p,q}(\mathbb{R}^n))$ to $L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))$,

$$\sum_{j,k=1}^{n}\|tD_{x_j}D_{x_k}u\|^q_{L^p(\mathbb{R}_+;B^{s-3,\tau}_{p,q}(\mathbb{R}^n))}\lesssim\|tu\|^q_{L^p(\mathbb{R}_+;B^{s-1,\tau}_{p,q}(\mathbb{R}^n))},$$

however, $supp\,u\subset K$, then

$$\sum_{j,k=1}^{n}\|tD_{x_j}D_{x_k}u\|^q_{L^p(\mathbb{R}_+;B^{s-3,\tau}_{p,q}(\mathbb{R}^n))}\le C_K\|u\|^q_{L^p(\mathbb{R}_+;B^{s-1,\tau}_{p,q}(\mathbb{R}^n))}. \tag{26}$$

Similarly

$$\sum_{j=1}^{n}\|D_{x_j}u\|^q_{L^p(\mathbb{R}_+;B^{s-2,\tau}_{p,q}(\mathbb{R}^n))}\lesssim\|u\|^q_{L^p(\mathbb{R}_+;B^{s-1,\tau}_{p,q}(\mathbb{R}^n))}. \tag{27}$$

Also

$$\sum_{j=1}^{n}\|tD_{x_j}D_tu\|^q_{L^p(\mathbb{R}_+;B^{s-3,\tau}_{p,q}(\mathbb{R}^n))}\lesssim\|tD_tu\|^q_{L^p(\mathbb{R}_+;B^{s-2,\tau}_{p,q}(\mathbb{R}^n))},$$

from Lemma 2.3, we have

$$\begin{aligned}\sum_{j=1}^{n}&\|tD_{x_j}D_tu\|^q_{L^p(\mathbb{R}_+;B^{s-3,\tau}_{p,q}(\mathbb{R}^n))}\\&\lesssim\varepsilon\|tD_t^2u\|^q_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))}+\frac{C_K}{\varepsilon}\|u\|^q_{L^p(\mathbb{R}_+;B^{s-1,\tau}_{p,q}(\mathbb{R}^n))}.\end{aligned} \tag{28}$$

Lemma 2.3 yields

$$\|D_tu\|^q_{L^p(\mathbb{R}_+;B^{s-3,\tau}_{p,q}(\mathbb{R}^n))}\lesssim\varepsilon\|tD_t^2u\|^q_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))}+\frac{C_K}{\varepsilon}\|u\|^q_{L^p(\mathbb{R}_+;B^{s-1,\tau}_{p,q}(\mathbb{R}^n))}, \tag{29}$$

From (26)–(29), we deduce

$$\begin{aligned}\frac{1}{|B'_K|^{\tau q}}\|\Delta'_0tD_t^2u\|^q_{L^p(\mathbb{R}_+\times B'_K)}\lesssim&C_K\Big\{\|L^0u\|^q_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))}+\|u\|^q_{L^p(\mathbb{R}_+;B^{s-1,\tau}_{p,q}(\mathbb{R}^n))}\Big\}\\&+\varepsilon\|tD_t^2u\|^q_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))}.\end{aligned} \tag{30}$$

For $\varepsilon>0$ small enough, inequalities (20), (25) and (30) imply the Proposition 4.1 for the operator L^0.

The proof of the proposition for the operator $L=L^0+L^1$, is essentially based on Lemma 2.5. Indeed, we must estimate the terms of the operator L^1:

$$\begin{aligned}L^1(t,x';D_t,D_{x'})u=&\sum_{j,k=1}^{n}\Big(a_{j,k}(t,x')-a_{j,k}(0,0)\Big)tD_{x_j}D_{x_k}u+\sum_{j=1}^{n}\Big(b_j(t,x')\\&-b_j(0,0)\Big)tD_{x_j}D_tu+\sum_{j=1}^{n}\Big(c_j(t,x')-c_j(0,0)\Big)D_{x_j}u+\Big(d(t,x)\\&-d(0,0)\Big)D_tu+e(t,x')u.\end{aligned} \tag{31}$$

We assume that the support of u is included in the half-ball of center $(0,0)$ with a small enough radius ε_0. For the first term, according to Lemma 2.5, there exists C_0, $C_1>0$ such that

$$\|\big(a_{j,k}(t,x')-a_{j,k}(0,0)\big)tD_{x_j}D_{x_k}u\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))}$$

$$\begin{aligned}&\leq C_0\|a_{j,k}(t,x')-a_{j,k}(0,0)\|_{L^\infty(\mathbb{R}^{n+1}_+)}\|tD_{x_j}D_{x_k}u\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))}\\&\quad+C_1\|tD_{x_j}D_{x_k}u\|_{L^p(\mathbb{R}_+;B^{s-1,\tau}_{p,q}(\mathbb{R}^n))},\end{aligned}$$

since the support of u is included in the half-ball of center $(0,0)$ with a small enough radius ε_0

$$\begin{aligned}&\|\big(a_{j,k}(t,x')-a_{j,k}(0,0)\big)tD_{x_j}D_{x_k}u\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))}\\&\qquad\leq C_0\varepsilon_0\|tu\|_{L^p(\mathbb{R}_+;B^{s+2,\tau}_{p,q}(\mathbb{R}^n))}+C_1\|tu\|_{L^p(\mathbb{R}_+;B^{s+1,\tau}_{p,q}(\mathbb{R}^n))},\end{aligned}$$

making use of Lemma 2.6, we obtain

$$\begin{aligned}&\|\big(a_{j,k}(t,x')-a_{j,k}(0,0)\big)tD_{x_j}D_{x_k}u\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))}\\&\qquad\leq C_0\varepsilon_0\|tu\|_{L^p(\mathbb{R}_+;B^{s+2,\tau}_{p,q}(\mathbb{R}^n))}+C_1\varepsilon_1\|tu\|_{L^p(\mathbb{R}_+;B^{s+2,\tau}_{p,q}(\mathbb{R}^n))}\\&\qquad\quad+C_K\varepsilon_1^{-2}\|u\|_{L^p(\mathbb{R}_+;B^{s-1,\tau}_{p,q}(\mathbb{R}^n))},\end{aligned}\tag{32}$$

for $1\leq j,k\leq n$. In order to estimate the second term of (31), we first apply Lemma 2.5 to obtain

$$\begin{aligned}&\|\big(b_j(t,x')-b_j(0,0)\big)tD_{x_j}D_tu\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))}\\&\qquad\leq C_0\|b_j(t,x')-b_j(0,0)\|_{L^\infty(\mathbb{R}^{n+1}_+)}\|tD_tu\|_{L^p(\mathbb{R}_+;B^{s+1,\tau}_{p,q}(\mathbb{R}^n))}\\&\qquad\quad+C_1\|tD_tu\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))}\\&\qquad\leq C_0\varepsilon_0\|tD_tu\|_{L^p(\mathbb{R}_+;B^{s+1,\tau}_{p,q}(\mathbb{R}^n))}+C_1\|tD_tu\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))},\end{aligned}$$

then, applying Lemma 2.6 to the second term of the right-hand side of the inequality, we obtain

$$\begin{aligned}&\|\big(b_j(t,x')-b_j(0,0)\big)tD_{x_j}D_tu\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))}\\&\qquad\leq C_0\varepsilon_0\|tD_tu\|_{L^p(\mathbb{R}_+;B^{s+1,\tau}_{p,q}(\mathbb{R}^n))}+C_1\varepsilon_1\|tD_tu\|_{L^p(\mathbb{R}_+;B^{s+1,\tau}_{p,q}(\mathbb{R}^n))}\\&\qquad\quad+C_1\varepsilon_1^{-2}\|tD_tu\|_{L^p(\mathbb{R}_+;B^{s-2,\tau}_{p,q}(\mathbb{R}^n))},\end{aligned}$$

thus, by means of Lemma 2.3, we deduce

$$\begin{aligned}&\|\big(b_j(t,x')-b_j(0,0)\big)tD_{x_j}D_tu\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))}\\&\qquad\lesssim C_0\varepsilon_0\|tD_tu\|_{L^p(\mathbb{R}_+;B^{s+1,\tau}_{p,q}(\mathbb{R}^n))}+C_1\varepsilon_1\|tD_tu\|_{L^p(\mathbb{R}_+;B^{s+1,\tau}_{p,q}(\mathbb{R}^n))}\\&\qquad\quad+\varepsilon_2\|tD_t^2u\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))}+C_{K,\varepsilon_2}\|u\|_{L^p(\mathbb{R}_+;B^{s-1,\tau}_{p,q}(\mathbb{R}^n))},\end{aligned}\tag{33}$$

for $1\leq j\leq n$. For the third term of (31), Lemma 2.5 gives

$$\begin{aligned}&\|\big(c_j(t,x')-c_j(0,0)\big)D_{x_j}u\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))}\\&\qquad\leq C_0\|c_j(t,x')-c_j(0,0)\|_{L^\infty(\mathbb{R}^{n+1}_+)}\|u\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))}\\&\qquad\quad+C_1\|u\|_{L^p(\mathbb{R}_+;B^{s-1,\tau}_{p,q}(\mathbb{R}^n))}\\&\qquad\leq C_0\varepsilon_0\|u\|_{L^p(\mathbb{R}_+;B^{s+1,\tau}_{p,q}(\mathbb{R}^n))}+C_1\|u\|_{L^p(\mathbb{R}_+;B^{s-1,\tau}_{p,q}(\mathbb{R}^n))},\end{aligned}\tag{34}$$

for $1\leq j\leq n$. For the fourth term of (31), we apply Lemma 2.5

$$\begin{aligned}&\|\big(d(t,x')-d(0,0)\big)D_tu\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))}\\&\qquad\leq C_0\|d(t,x')-d(0,0)\|_{L^\infty(\mathbb{R}^{n+1}_+)}\|D_tu\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))}\\&\qquad\quad+C_1\|D_tu\|_{L^p(\mathbb{R}_+;B^{s-1,\tau}_{p,q}(\mathbb{R}^n))}\\&\qquad\leq C_0\varepsilon_0\|D_tu\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))}+C_1\|D_tu\|_{L^p(\mathbb{R}_+;B^{s-1,\tau}_{p,q}(\mathbb{R}^n))},\end{aligned}$$

next, Lemma 2.6 gives

$$\begin{aligned}&\|\big(d(t,x')-d(0,0)\big)D_tu\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))}\\&\qquad\leq C_0\varepsilon_0\|D_tu\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))}+C_1\varepsilon_1\|D_tu\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))}\\&\qquad\quad+C_1\varepsilon_1^{-2}\|D_tu\|_{L^p(\mathbb{R}_+;B^{s-3,\tau}_{p,q}(\mathbb{R}^n))},\end{aligned}$$

applying Lemma 2.3 to the last term of the right-hand side of the above estimate, we obtain

$$\begin{aligned}
\|\big(d(t,x')-d(0,0)\big)D_t u\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))} &\\
\lesssim C_0\varepsilon_0\|D_t u\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))} &+ C_1\varepsilon_1\|D_t u\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))}\\
+\varepsilon_2\|tD_t^2 u\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))} &+ C_{K,\varepsilon_2}\|u\|_{L^p(\mathbb{R}_+;B^{s-1,\tau}_{p,q}(\mathbb{R}^n))}.
\end{aligned} \tag{35}$$

Finally, for the last term of (31), Lemma 2.5 gives

$$\begin{aligned}
\|e(t,x')u\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))} \leq & C_0\|e(t,x')-e(0,0)\|_{L^\infty(\mathbb{R}^{n+1}_+)}\|u\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))}\\
&+ C_1\|u\|_{L^p(\mathbb{R}_+;B^{s-1,\tau}_{p,q}(\mathbb{R}^n))}\\
\leq & C_0\varepsilon_0\|u\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))} + C_1\|u\|_{L^p(\mathbb{R}_+;B^{s-1,\tau}_{p,q}(\mathbb{R}^n))}
\end{aligned} \tag{36}$$

Inequalities (32)–(36), complete the proof for the operator $L = L^0 + L^1$, then the proof of the Proposition 4.1 for u with a small enough support around $(0,0)$ and with a suitable $\varepsilon_0, \varepsilon_1, \varepsilon_2 > 0$. In the same way, the estimate is proved around the point $(0,x_0)$ of K. Otherwise, the assumption (H_1) yields the same estimation in the neighbourhood of the point (t_0,x_0) with $t_0 \neq 0$ of K. Finally, the general a-priori estimate is obtained by the use of a partition of unity.

To complete the proof of Theorem 1.1, we need the following lemma.

Lemma 4.1. *Let s, τ be two non-negative real numbers and $1 \leq p,q < +\infty$. For any compact K of $\overline{\mathbb{R}^{n+1}_+}$, there exists a constant $C_K > 0$ such that for any $u \in B^{s+2,\tau}_{p,q,w}(\mathbb{R}^{n+1}_+)$ with supp $u \subset K$*

$$\|u\|_{B^{s+2,\tau}_{p,q,w}(\mathbb{R}^{n+1}_+)} \leq C_K\Big\{\|Lu\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} + \|u\|_{W^{2,p}_w(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))} + \|u\|_{B^{s-1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)}\Big\},$$

holds.

Proof. At first, we prove the lemma for the case $0 \leq s < 1$, to be able to make use of Lemma 2.4. Next, we prove it by induction for all $s \geq 0$.

- **Case** $0 \leq s < 1$**:**

The proof is based essentially on Lemma 2.4. We do the same previous decomposition: $L = L^0 + L^1$, and we prove the lemma first for L^0. Hence, we estimate the different terms of the norm of u in $B^{s+2,\tau}_{p,q,w}(\mathbb{R}^{n+1}_+)$.

Lemma 2.4 yields the following inequalities

$$\begin{aligned}
\|u\|_{B^{s+1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} \equiv & \sum_{i=1}^n \|D_{x_i}u\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} + \|D_t u\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)}\\
\lesssim & \sum_{i=1}^n \Big[\|D_{x_i}D_t u\|_{L^p(\mathbb{R}_+;B^{s-1,\tau}_{p,q}(\mathbb{R}^n)} + \|D_{x_i}u\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n)}\Big]\\
& + \|D_t u\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)}\\
\lesssim & \|u\|_{W^{2,p}_w(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))} + \|D_t u\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)}.
\end{aligned} \tag{37}$$

$$\begin{aligned}
&\|tD_{x_j}D_{x_k}u\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)}\\
&\quad\lesssim \|D_t(tD_{x_j}D_{x_k})u\|_{L^p(\mathbb{R}_+;B^{s-1,\tau}_{p,q}(\mathbb{R}^n))} + \|tD_{x_j}D_{x_k}u\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))}\\
&\quad\lesssim \|D_{x_j}D_{x_k}u\|_{L^p(\mathbb{R}_+;B^{s-1,\tau}_{p,q}(\mathbb{R}^n))} + \|tD_{x_j}D_{x_k}D_t u\|_{L^p(\mathbb{R}_+;B^{s-1,\tau}_{p,q}(\mathbb{R}^n))}\\
&\quad\quad + \|tD_{x_j}D_{x_k}u\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))}\\
&\quad\lesssim \|u\|_{W^{2,p}_w(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))},
\end{aligned} \tag{38}$$

for $1 \leq j,k \leq n$. And

$$\begin{aligned}
&\|tD_{x_j}D_t u\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)}\\
&\quad\lesssim \|D_t(tD_{x_j}D_t)u\|_{L^p(\mathbb{R}_+;B^{s-1,\tau}_{p,q}(\mathbb{R}^n))} + \|tD_{x_j}D_t u\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))}
\end{aligned}$$

$$\begin{aligned}
&\lesssim \|D_{x_j} D_t u\|_{L^p(\mathbb{R}_+;B^{s-1,\tau}_{p,q}(\mathbb{R}^n))} + \|tD_{x_j} D_t^2 u\|_{L^p(\mathbb{R}_+;B^{s-1,\tau}_{p,q}(\mathbb{R}^n))} \\
&\quad + \|tD_{x_j} D_t u\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))} \\
&\lesssim \|u\|_{W^{2,p}_w(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))},
\end{aligned} \tag{39}$$

for $1 \le j \le n$. It remains now to estimate the following terms: $\|D_t u\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)}$ and $\|tD_t^2 u\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)}$. For this, we decompose $B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)$ as follows

$$B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+) = L^p_{loc}(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n)) \cap L^p_{loc}(\mathbb{R}^n;B^{s,\tau}_{p,q}(\mathbb{R}_+)),$$

then, since the support of u is included in a compact set K, we have: $u \in B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)$, if and only if $u \in L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))$ and $u \in L^p(\mathbb{R}^n;B^{s,\tau}_{p,q}(\mathbb{R}_+))$. When $u \in L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))$, we obtain immediately

$$\|D_t u\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))} + \|tD_t^2 u\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))} \lesssim \|u\|_{W^{2,p}_w(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))}. \tag{40}$$

When $u \in L^p(\mathbb{R}^n;B^{s,\tau}_{p,q}(\mathbb{R}_+))$, returning to the ordinary differential equation, since the operator $\big(L^0(0,t;e_1,D_t),\gamma_0\big)$ is invertible from $V^{s,\tau}_{p,q}(\mathbb{R}_+)$ to $B^{s,\tau}_{p,q}(\mathbb{R}_+)\times\mathbb{C}$ such that $V^{s,\tau}_{p,q}(\mathbb{R}_+) = \{v \in B^{s+1,\tau}_{p,q}(\mathbb{R}_+) \,:\, tv \in B^{s+2,\tau}_{p,q}(\mathbb{R}_+), tD_t v \in B^{s+1,\tau}_{p,q}(\mathbb{R}_+)$ and $D_t v, tD_t^2 v \in B^{s,\tau}_{p,q}(\mathbb{R}_+)\}$, then for any $v \in V^{s,\tau}_{p,q}(\mathbb{R}_+)$, we have

$$\|v\|_{V^{s,\tau}_{p,q}(\mathbb{R}_+)} \lesssim \|L^0(0,t;e_1,D_t)v\|_{B^{s,\tau}_{p,q}(\mathbb{R}_+)} + \|v(0)\|_{\mathbb{C}}, \tag{41}$$

with $e_1 = (1,0,\cdots,0) \in \mathbb{R}^n$. We write

$$\begin{aligned}
L^0(0,t;e_1,D_t)u =& L^0 u + a_{1,1}(0,0)tu + b_1(0,0)tD_t u + c_1(0,0)u \\
&- \Big(\sum_{j,k=1}^n a_{j,k}(0,0)tD_{x_j}D_{x_k}u + \sum_{j=1}^n b_j(0,0)tD_{x_j}D_t u \\
&+ \sum_{j=1}^n c_j(0,0)D_{x_j}u\Big),
\end{aligned}$$

according to (41), we deduce

$$\begin{aligned}
&\|D_t u\|_{L^p(\mathbb{R}^n;B^{s,\tau}_{p,q}(\mathbb{R}_+))} + \|tD_t^2 u\|_{L^p(\mathbb{R}^n;B^{s,\tau}_{p,q}(\mathbb{R}_+))} \\
&\lesssim \|L^0 u\|_{L^p(\mathbb{R}^n;B^{s,\tau}_{p,q}(\mathbb{R}_+))} + \|tu\|_{L^p(\mathbb{R}^n;B^{s,\tau}_{p,q}(\mathbb{R}_+))} + \|tD_t u\|_{L^p(\mathbb{R}^n;B^{s,\tau}_{p,q}(\mathbb{R}_+))} \\
&\quad + \|u\|_{L^p(\mathbb{R}^n;B^{s,\tau}_{p,q}(\mathbb{R}_+))} + \sum_{j,k=1}^n \|tD_{x_j}D_{x_k}u\|_{L^p(\mathbb{R}^n;B^{s,\tau}_{p,q}(\mathbb{R}_+))} \\
&\quad + \sum_{j=1}^n \|tD_{x_j}D_t u\|_{L^p(\mathbb{R}^n;B^{s,\tau}_{p,q}(\mathbb{R}_+))} + \sum_{j=1}^n \|D_{x_j}u\|_{L^p(\mathbb{R}^n;B^{s,\tau}_{p,q}(\mathbb{R}_+))},
\end{aligned}$$

thus

$$\begin{aligned}
&\|D_t u\|_{L^p(\mathbb{R}^n;B^{s,\tau}_{p,q}(\mathbb{R}_+))} + \|tD_t^2 u\|_{L^p(\mathbb{R}^n;B^{s,\tau}_{p,q}(\mathbb{R}_+))} \\
&\lesssim \|L^0 u\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} + \|tu\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} + \|tD_t u\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} + \|u\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} \\
&\quad + \sum_{j,k=1}^n \|tD_{x_j}D_{x_k}u\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} + \sum_{j=1}^n \|tD_{x_j}D_t u\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} \\
&\quad + \sum_{j=1}^n \|D_{x_j}u\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)}.
\end{aligned}$$

Thanks to Lemma 2.4

$$\begin{aligned}
\|tu\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} &\lesssim \|D_t(tu)\|_{L^p(\mathbb{R}_+;B^{s-1,\tau}_{p,q}(\mathbb{R}^n))} + \|tu\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))} \\
&\lesssim \|u\|_{L^p(\mathbb{R}_+;B^{s-1,\tau}_{p,q}(\mathbb{R}^n))} + \|tD_t u\|_{L^p(\mathbb{R}_+;B^{s-1,\tau}_{p,q}(\mathbb{R}^n))}
\end{aligned}$$

$$\begin{aligned}&+\|tu\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))}\\ &\lesssim \|u\|_{W^{2,p}_w(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))},\end{aligned}\tag{42}$$

$$\begin{aligned}\|tD_tu\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} &\lesssim \|D_t(tD_tu)\|_{L^p(\mathbb{R}_+;B^{s-1,\tau}_{p,q}(\mathbb{R}^n))}+\|tD_tu\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))}\\ &\lesssim \|D_tu\|_{L^p(\mathbb{R}_+;B^{s-1,\tau}_{p,q}(\mathbb{R}^n))}+\|tD_t^2u\|_{L^p(\mathbb{R}_+;B^{s-1,\tau}_{p,q}(\mathbb{R}^n))}\\ &\quad+\|tD_tu\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))}\\ &\lesssim \|u\|_{W^{2,p}_w(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))},\end{aligned}\tag{43}$$

$$\begin{aligned}\|u\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} &\lesssim \|D_tu\|_{L^p(\mathbb{R}_+;B^{s-1,\tau}_{p,q}(\mathbb{R}^n))}+\|u\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))}\\ &\lesssim \|u\|_{W^{2,p}_w(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))},\end{aligned}\tag{44}$$

and

$$\begin{aligned}\|D_{x_j}u\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} &\lesssim \|D_tD_{x_j}u\|_{L^p(\mathbb{R}_+;B^{s-1,\tau}_{p,q}(\mathbb{R}^n))}+\|D_{x_j}u\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))}\\ &\lesssim \|u\|_{W^{2,p}_w(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))}.\end{aligned}\tag{45}$$

(42)–(45), together with (38) and (39) imply

$$\begin{aligned}\|D_tu\|_{L^p(\mathbb{R}^n;B^{s,\tau}_{p,q}(\mathbb{R}_+))}&+\|tD_t^2u\|_{L^p(\mathbb{R}^n;B^{s,\tau}_{p,q}(\mathbb{R}_+))}\\ &\lesssim \|L^0u\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)}+\|u\|_{W^{2,p}_w(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))}.\end{aligned}\tag{46}$$

By (40) and (46)

$$\|D_tu\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)}+\|tD_t^2u\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)}\lesssim \|L^0u\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)}+\|u\|_{W^{2,p}_w(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))}.\tag{47}$$

Considering estimates (37)–(39) and (47), we deduce the Lemma 4.1 for the operator L^0 and for $0\le s<1$.

Now, for $L=L^0+L^1$, we estimate the terms of L^1 (31) in $B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)$. Applying Lemma 2.5 and since $suppu\subset K$, we have

$$\|\big(a_{j,k}(t,x')-a_{j,k}(0,0)\big)tD_{x_j}D_{x_k}u\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)}\le C_K\|tD_{x_j}D_{x_k}u\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)},$$

inequality (38) yields

$$\|\big(a_{j,k}(t,x')-a_{j,k}(0,0)\big)tD_{x_j}D_{x_k}u\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)}\le C_K\|u\|_{W^{2,p}_w(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))},\tag{48}$$

for $1\le j,k\le n$.

$$\|\big(b_j(t,x')-b_j(0,0)\big)tD_{x_j}D_tu\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)}\le C_K\|tD_{x_j}D_tu\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)}$$

by inequality (39)

$$\|\big(b_j(t,x')-b_j(0,0)\big)tD_{x_j}D_tu\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)}\le C_K\|u\|_{W^{2,p}_w(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))},\tag{49}$$

for $1\le j\le n$. Inequality (45) implies

$$\begin{aligned}\|\big(c_j(t,x')-c_j(0,0)\big)D_{x_j}u\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} &\le C_K\|D_{x_j}u\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)}\\ &\le C_K\|u\|_{W^{2,p}_w(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))},\end{aligned}\tag{50}$$

for $1\le j\le n$. Afterwards, we assume that the support of u is included in the half-ball of center (0,0) with a small enough radius ε. Lemma 2.5 implies that there are a constants $C_1,C_2>0$ such that

$$\|\big(d(t,x')-d(0,0)\big)D_tu\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)}\le C_0\varepsilon\|D_tu\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)}+C_1\|D_tu\|_{B^{s-1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)},$$

then, Lemma 2.6 yields

$$\begin{aligned}&\|\big(d(t,x')-d(0,0)\big)D_tu\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)}\\ &\qquad\le C_0\varepsilon\|D_tu\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)}+C_1\varepsilon'\|D_tu\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)}+\frac{C_1}{\varepsilon'}\|D_tu\|_{B^{s-2,\tau}_{p,q}(\mathbb{R}^{n+1}_+)}\end{aligned}$$

$$\lesssim C_0\varepsilon\|D_t u\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} + C_1\varepsilon'\|D_t u\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} + C_{1,\varepsilon'}\|u\|_{B^{s-1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)}. \tag{51}$$

Finally, for the last term, Lemma 2.4 gives

$$\|e(t,x')u\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} \leq C_K\|u\|_{W^{2,p}_w(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))}. \tag{52}$$

The inequalities from (48) to (52) implies the Lemma 4.1 for u with a small enough support around the origin (0,0). As previously, the general a-priori estimate holds true by the use of a partition of unity.

- **Case $s \geq 0$:**

We set $s = r+\sigma$, such that r is a non-negative integer and $\sigma \in [0,1[$. We proceed by induction on r. This is true for $r = 0$ (i.e. $0 \leq s < 1$). We assume that the estimate is true for any r, then for any s

$$\|u\|_{B^{s+2,\tau}_{p,q,w}(\mathbb{R}^{n+1}_+)} \leq C_K\Big\{\|Lu\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} + \|u\|_{W^{2,p}_w(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))}\Big\},$$

and we prove it for $r+1$ (i.e. $s+1$), thus

$$\|u\|_{B^{s+3,\tau}_{p,q,w}(\mathbb{R}^{n+1}_+)} \leq C_K\Big\{\|Lu\|_{B^{s+1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} + \|u\|_{W^{2,p}_w(\mathbb{R}_+,B^{s+1,\tau}_{p,q}(\mathbb{R}^n))}\Big\},$$

for $u \in B^{s+3,\tau}_{p,q,w}(\mathbb{R}^{n+1}_+)$ and with $supp\, u \subset K$, such that K is a compact of $\overline{\mathbb{R}^{n+1}_+}$. We notice that $u \in B^{s+3,\tau}_{p,q,w}(\mathbb{R}^{n+1}_+)$, if and only if $u \in B^{s+2,\tau}_{p,q,w}(\mathbb{R}^{n+1}_+)$, $D_{x_i}u \in B^{s+2,\tau}_{p,q,w}(\mathbb{R}^{n+1}_+)$ for $1 \leq i \leq n$ and $D_t u, tD_t^2 u \in B^{s+1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)$, then

$$\begin{aligned}\|u\|_{B^{s+3,\tau}_{p,q,w}(\mathbb{R}^{n+1}_+)} \leq & C_K\Big\{\|u\|_{B^{s+2,\tau}_{p,q,w}(\mathbb{R}^{n+1}_+)} + \|D_{x_i}u\|_{B^{s+2,\tau}_{p,q,w}(\mathbb{R}^{n+1}_+)} \\ & + \|D_t u\|_{B^{s+1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} + \|tD_t^2 u\|_{B^{s+1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)}\Big\}.\end{aligned} \tag{53}$$

By applying the induction hypothesis, we have for all $i = 1,\cdots,n$:

$$\begin{aligned}&\|D_{x_i}u\|_{B^{s+2,\tau}_{p,q,w}(\mathbb{R}^{n+1}_+)} \\ &\leq C_K\Big\{\|L(D_{x_i}u)\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} + \|D_{x_i}u\|_{W^{s,p}_{weight}(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))}\Big\},\end{aligned} \tag{54}$$

we write

$$D_{x_i}Lu = [L;D_{x_i}]u + L(D_{x_i}u),$$

such that

$$\begin{aligned}[L;D_{x_i}]u = &\sum_{j,k=1}^n (D_{x_i}a_{j,k})(t,x')tD_{x_j}D_{x_k}u + \sum_{j=1}^n (D_{x_i}b_j)(t,x')tD_{x_j}D_t u \\ &+ \sum_{j=1}^n (D_{x_i}c_j)(t,x')D_{x_j}u + (D_{x_i}d)(t,x')D_t u + (D_{x_i}e)(t,x')u,\end{aligned}$$

then

$$L(D_{x_i}u) = D_{x_i}Lu - [L;D_{x_i}]u,$$

we apply the norm in $B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)$ to obtain

$$\|L(D_{x_i}u)\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} \leq \|Lu\|_{B^{s+1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} + \|[L;D_{x_i}]u\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)},$$

using Lemma 2.5 to estimate the terms of $\|[L;D_{x_i}]u\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)}$, we obtain

$$\|L(D_{x_i}u)\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} \leq C_K\Big\{\|Lu\|_{B^{s+1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} + \|u\|_{B^{s+2,\tau}_{p,q,w}(\mathbb{R}^{n+1}_+)}\Big\},$$

replacing the first term in the right-hand side of (54) by the above inequality, we deduce

$$\begin{aligned}&\|D_{x_i}u\|_{B^{s+2,\tau}_{p,q,w}(\mathbb{R}^{n+1}_+)} \\ &\leq C_K\Big\{\|Lu\|_{B^{s+1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} + \|u\|_{B^{s+2,\tau}_{p,q,w}(\mathbb{R}^{n+1}_+)} + \|u\|_{W^{2,p}_w(\mathbb{R}_+;B^{s+1,\tau}_{p,q}(\mathbb{R}^n))}\Big\}.\end{aligned} \tag{55}$$

Now, we estimate $\|D_t u\|_{B^{s+1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)}$ and $\|tD_t^2 u\|_{B^{s+1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)}$. In the same way as in the case $s \in [0,1[$, since $L = L^0 + L^1$, we estimate the two terms first by considering the operator L^0. We set

$$B^{s+1,\tau}_{p,q}(\mathbb{R}^{n+1}_+) = L^p_{loc}(\mathbb{R}_+; B^{s+1,\tau}_{p,q}(\mathbb{R}^n)) \cap L^p_{loc}(\mathbb{R}^n; B^{s+1,\tau}_{p,q}(\mathbb{R}_+)),$$

we recall that the support of u is included in a compact set K then $u \in B^{s+1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)$ if and only if $u \in L^p(\mathbb{R}_+; B^{s+1,\tau}_{p,q}(\mathbb{R}^n))$ and $u \in L^p(\mathbb{R}^n; B^{s+1,\tau}_{p,q}(\mathbb{R}_+))$. When $u \in L^p(\mathbb{R}_+; B^{s+1,\tau}_{p,q}(\mathbb{R}^n))$, we have

$$\|D_t u\|_{L^p(\mathbb{R}_+; B^{s+1,\tau}_{p,q}(\mathbb{R}^n))} + \|tD_t^2 u\|_{L^p(\mathbb{R}_+; B^{s+1,\tau}_{p,q}(\mathbb{R}^n))} \leq \|u\|_{W^{2,p}_w(\mathbb{R}_+; B^{s+1,\tau}_{p,q}(\mathbb{R}^n))}. \tag{56}$$

If $u \in L^p(\mathbb{R}^n; B^{s+1,\tau}_{p,q}(\mathbb{R}_+))$, we return again to the ordinary differential equation, since the operator $\big(L^0(0,t;e_1,D_t),\gamma_0\big)$ is invertible from $V^{s+1,\tau}_{p,q}(\mathbb{R}_+)$ to $B^{s+1,\tau}_{p,q}(\mathbb{R}_+)\times\mathbb{C}$ such that $V^{s+1,\tau}_{p,q}(\mathbb{R}_+) = \{v \in B^{s+2,\tau}_{p,q}(\mathbb{R}_+) : tv \in B^{s+3,\tau}_{p,q}(\mathbb{R}_+), tD_t v \in B^{s+2,\tau}_{p,q}(\mathbb{R}_+)$ and $D_t v, tD_t^2 v \in B^{s+1,\tau}_{p,q}(\mathbb{R}_+)\}$, then for any $v \in V^{s+1,\tau}_{p,q}(\mathbb{R}_+)$, we have

$$\|v\|_{V^{s+1,\tau}_{p,q}(\mathbb{R}_+)} \lesssim \|L^0(0,t;e_1,D_t)v\|_{B^{s+1,\tau}_{p,q}(\mathbb{R}_+)} + \|v(0)\|_{\mathbb{C}}, \tag{57}$$

with $e_1 = (1,0,\cdots,0) \in \mathbb{R}^n$.

$$\begin{aligned} L^0(0,t;e_1,D_t)u =& L^0 u + a_{1,1}(0,0)tu + b_1(0,0)tD_t u + c_1(0,0)u \\ & - \Big(\sum_{j,k=1}^n a_{j,k}(0,0)tD_{x_j}D_{x_k}u + \sum_{j=1}^n b_j(0,0)tD_{x_j}D_t u \\ & + \sum_{j=1}^n c_j(0,0)D_{x_j}u\Big), \end{aligned}$$

then, according to (57), we obtain

$$\begin{aligned} &\|D_t u\|_{L^p(\mathbb{R}^n; B^{s+1,\tau}_{p,q}(\mathbb{R}_+))} + \|tD_t^2 u\|_{L^p(\mathbb{R}^n; B^{s+1,\tau}_{p,q}(\mathbb{R}_+))} \\ &\lesssim \|L^0 u\|_{L^p(\mathbb{R}^n; B^{s+1,\tau}_{p,q}(\mathbb{R}_+))} + \|tu\|_{L^p(\mathbb{R}^n; B^{s+1,\tau}_{p,q}(\mathbb{R}_+))} + \|tD_t u\|_{L^p(\mathbb{R}^n; B^{s+1,\tau}_{p,q}(\mathbb{R}_+))} \\ &\quad + \|u\|_{L^p(\mathbb{R}^n; B^{s+1,\tau}_{p,q}(\mathbb{R}_+))} + \sum_{j,k=1}^n \|tD_{x_j}D_{x_k}u\|_{L^p(\mathbb{R}^n; B^{s+1,\tau}_{p,q}(\mathbb{R}_+))} \\ &\quad + \sum_{j=1}^n \|tD_{x_j}D_t u\|_{L^p(\mathbb{R}^n; B^{s+1,\tau}_{p,q}(\mathbb{R}_+))} + \sum_{j=1}^n \|D_{x_j}u\|_{L^p(\mathbb{R}^n; B^{s+1,\tau}_{p,q}(\mathbb{R}_+))}, \end{aligned}$$

then

$$\begin{aligned} &\|D_t u\|_{L^p(\mathbb{R}^n; B^{s+1,\tau}_{p,q}(\mathbb{R}_+))} + \|tD_t^2 u\|_{L^p(\mathbb{R}^n; B^{s+1,\tau}_{p,q}(\mathbb{R}_+))} \\ &\lesssim \|L^0 u\|_{B^{s+1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} + \|tu\|_{B^{s+1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} + \|tD_t u\|_{B^{s+1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} + \|u\|_{B^{s+1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} \\ &\quad + \sum_{j,k=1}^n \|tD_{x_j}D_{x_k}u\|_{B^{s+1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} + \sum_{j=1}^n \|tD_{x_j}D_t u\|_{B^{s+1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} \\ &\quad + \sum_{j=1}^n \|D_{x_j}u\|_{B^{s+1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)}, \end{aligned}$$

we can easily see that

$$\begin{aligned} &\|D_t u\|_{L^p(\mathbb{R}^n; B^{s+1,\tau}_{p,q}(\mathbb{R}_+))} + \|tD_t^2 u\|_{L^p(\mathbb{R}^n; B^{s+1,\tau}_{p,q}(\mathbb{R}_+))} \\ &\qquad \lesssim \|L^0 u\|_{B^{s+1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} + \|u\|_{B^{s+2,\tau}_{p,q,w}(\mathbb{R}^{n+1}_+)} + \|D_{x_i}u\|_{B^{s+2,\tau}_{p,q,w}(\mathbb{R}^{n+1}_+)}. \end{aligned} \tag{58}$$

Now, for L^1, Lemma 2.5 yields the following estimate

$$\begin{aligned} \|L^1 u\|_{B^{s+1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} \leq& C_K\{\|D_{x_i}u\|_{B^{s+2,\tau}_{p,q,w}(\mathbb{R}^{n+1}_+)} + \|u\|_{B^{s+2,\tau}_{p,q,w}(\mathbb{R}^{n+1}_+)}\} \\ & + \|\big(d(t,x') - d(0,0)\big)D_t u\|_{B^{s+1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)}, \end{aligned}$$

as previously, assuming that the support is included in the half-ball of center (0,0) and radius ε small enough, and making use of Lemma 2.5, we get

$$\begin{aligned}\|L^1u\|_{B^{s+1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} \lesssim & C_K\{\|D_{x_i}u\|_{B^{s+2,\tau}_{p,q,w}(\mathbb{R}^{n+1}_+)} + \|u\|_{B^{s+2,\tau}_{p,q,w}(\mathbb{R}^{n+1}_+)}\} \\ & + \varepsilon\|D_t u\|_{B^{s+1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)},\end{aligned} \tag{59}$$

(58) and (59) imply the estimate for the operator L

$$\begin{aligned}&\|D_t u\|_{L^p(\mathbb{R}^n;B^{s+1,\tau}_{p,q}(\mathbb{R}_+))} + \|tD_t^2 u\|_{L^p(\mathbb{R}^n;B^{s+1,\tau}_{p,q}(\mathbb{R}_+))} \\ &\quad \lesssim C_K\{\|Lu\|_{B^{s+1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} + \|u\|_{B^{s+2,\tau}_{p,q,w}(\mathbb{R}^{n+1}_+)} + \|D_{x_i}u\|_{B^{s+2,\tau}_{p,q,w}(\mathbb{R}^{n+1}_+)}\} \\ &\quad\quad + \varepsilon\|D_t u\|_{B^{s+1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)}.\end{aligned} \tag{60}$$

Both (56) and (60) give

$$\begin{aligned}&\|D_t u\|_{B^{s+1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} + \|tD_t^2 u\|_{B^{s+1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} \\ &\le C_K\{\|Lu\|_{B^{s+1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} + \|u\|_{W^{2,p}_w(\mathbb{R}_+;B^{s+1,\tau}_{p,q}(\mathbb{R}^n))} + \|u\|_{B^{s+2,\tau}_{p,q,w}(\mathbb{R}^{n+1}_+)} \\ &\quad + \|D_{x_i}u\|_{B^{s+2,\tau}_{p,q,w}(\mathbb{R}^{n+1}_+)}\}.\end{aligned} \tag{61}$$

Similarly to the previous case $0 \le s < 1$, the general a-priori estimates is obtained by the use of a partition of unity. Now, substituting inequalities (53), (55) and (61) as well as using the induction hypothesis, we deduce the estimate for the $r+1$ order, then the Lemma 4.1 for any $s \ge 0$.

Finally the Proposition 4.1 with the Lemma 4.1 and the following estimate

$$\begin{aligned}\|u\|_{L^p(\mathbb{R}_+;B^{s-1,\tau}_{p,q}(\mathbb{R}^n))} &\lesssim \|u\|_{L^p(\mathbb{R}_+;B^{s,\tau}_{p,q}(\mathbb{R}^n))} \\ &\lesssim \|u\|_{B^{s,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} \\ &\lesssim \varepsilon\|u\|_{B^{s+1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)} + \frac{1}{\varepsilon}\|u\|_{B^{s-1,\tau}_{p,q}(\mathbb{R}^{n+1}_+)},\end{aligned}$$

imply the Theorem 1.1.

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