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On the characterization of Jensen m-convex polynomials

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Abstract. The main objective of this research is to characterize all the real polynomial functions of degree less than the fourth which are Jensen m-convex on the set of non-negative real numbers. In the first section, it is established for that class of functions what conditions must satisfy a particular polynomial in order to be starshaped on the same set. Finally, both kinds of results are combined in order to find examples of either Jensen m-convex functions which are not starshaped or viceversa.

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1. Introduction

The study about the generalization of the classical concept of convexity of real functions began from its origin itself [2, 3, 9, 10]. In [11, p. 2] G. Toader reminds us that, according to T. Popoviciu [9], the class of convex functions was introduced by O. Stolz in 1893 while working on the study of derivatives, by considering the relation $f(x-h)-2f(x)+f(x+h)\geq 0$ or equivalently

$$\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\geq 0. \tag{1}$$

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Jensen, himself, was the first who studied them formally. The relation (1) is probably the first generalization regarding to convex functions and nowadays a function f satisfying it is called midconvex. Another type of generalized convexity for real functions is the well known m-convexity introduced in [11, Definition 0.3], which has been subsequently studied by the same author and colleagues in [8, 12, 13, 15] among others. An even more general kind of convexity is considered in [14], but limited to continuous functions, the Jensen m-convexity. In [5] some interesting results were found regarding this class of functions but restricted to be defined on intervals of the form $[0, b]$ where b is any positive real number, for instance it was proved that, for $m_1 \neq m_2$, the corresponding classes of Jensen m_1-convex functions and Jensen m_2-convex functions are different. Other results, related to this type of convexity or to some variations of it, can be found in [4, 6, 7].

Following a similar orientation to that presented in [15], we are headed to characterize the low-grade polynomials that are Jensen m-convex as the main objective of this research.

Let us start by recalling this couple of definitions for real functions whose domains contain the set $\mathbb{R}_{\geq 0} = [0, +\infty)$. It is customary to have domains that are convex sets or maybe something more general like m-convex sets. First, we establish the condition that must satisfy a function to be called starshaped. Then, we do the same to the term Jensen m-convex for $m \in (0, 1)$ which is a type of function much less known.

Definition 1. *([8, 12, 15]) A function $f : D \to \mathbb{R}$ $(D \supset [0, +\infty))$ is said to be starshaped on $[0, +\infty)$ if for any $x \in [0, +\infty)$ and $t \in [0, 1]$*

$$f(tx) \leq tf(x). \tag{2}$$

Definition 2. *([5, 14]) A function $f : D \to \mathbb{R}$ $(D \supset [0, +\infty))$ is said to be Jensen m-convex on $[0, +\infty)$ if for any $x, y \in [0, +\infty)$*

$$f\left(\frac{x + y}{c_m}\right) \leq \frac{f(x) + f(y)}{c_m} \tag{3}$$

where $c_m = 1 + \dfrac{1}{m}$, $m \in (0, 1)$.

Remark 3. *In [5] the functional inequality (3) was restricted to all x and y in $[0, b]$ in order to define a Jensen m-convex function on $[0, b]$ and the class of such functions was denoted by $J_m[b]$.*

From now on, D will be the set of all real numbers (i.e., $D = \mathbb{R}$) and the set of all functions $f : \mathbb{R} \to \mathbb{R}$ that are starshaped (Jensen m-convex) on $[0, +\infty)$ will be denoted by the symbol $SS[+\infty)$ $(J_m[+\infty))$, respectively.

In what follows all the functions considered are real polynomials of the form either $f_i(x) = a_i x^i + a_{i-1} x^{i-1} + \cdots + a_1 x + a_0$ $(a_i \neq 0)$ for $i = 0, 1, 2, 3$ or the zero polynomial $\mathit{0}$ (i.e., $\mathit{0}(x) = 0$ for all $x \in \mathbb{R}$). It is evident that $\mathit{0} \in SS[+\infty) \cap J_m[+\infty)$ for all $m \in (0, 1)$.

2. Starshaped polynomial functions of low degree

In [15, Lemma 2.1] all the real polynomial functions f of low degree that are starshaped on $[0, +\infty)$ and $f(0) = 0$ were characterized. In this short section we will extend that result to include those polynomials taking a nonzero value at zero but we will not include polynomials of degree four.

Well, let us begin by establishing the first result for $i = 0, 1, 2$.

Theorem 4. *The polynomial functions f_0, f_1 and f_2 with $a_0 \neq 0$ are starshaped on $[0, +\infty)$ if and only if their coefficients satisfy the following conditions:*

1. *For $i = 0$, $a_0 < 0$.*
2. *For $i = 1$, $a_1 \neq 0$, and $a_0 < 0$.*
3. *For $i = 2$, $a_2 > 0$, and $a_0 < 0$.*

Proof. The condition $a_0 < 0$ comes out from the following well known general fact: If $f \in SS[+\infty)$ then $f(0) \leq 0$. Then for $i = 0$, we are done. For $i = 1$, the inequality (2) becomes equivalent to $(1 - t)a_0 \leq 0$ for all $t \in [0, 1]$, so a_1 must be any nonzero real number. For $i = 2$, the same inequality equivalently turns into $(1 - t)(a_2 t x^2 - a_0) \geq 0$ for all $x \geq 0$ and for all $t \in [0, 1]$, so a_2 must be any positive real number.

For the remaining case, we proceed in a different way. Given f_3 we define a new function h_{f_3} as follows:

$$h_{f_3}(x) := xf_3'(x) - f_3(x) \quad \text{for all } x \in \mathbb{R}.$$

It is not difficult to check that $f_3 \in SS[+\infty)$ if and only if h_{f_3} is non-negative on $[0, +\infty)$. We will do an analysis of h_{f_3} by means of the First Derivative Test (FDT) to find conditions on the coefficients of f_3 to determine when $h_{f_3}(x) \geq 0$ for all $x \in [0, +\infty)$.

Theorem 5. *The polynomial function $f_3 \in SS[+\infty)$ if and only if its coefficients satisfy any of the following conditions:*

(1) $a_3 > 0$, $a_2 \geq 0$, *and* $a_0 < 0$.
(2) $a_3 > 0$, $a_2 < 0$, $a_0 < 0$, *and* $h_{f_3}(-\frac{a_2}{3a_3}) \geq 0$.

Proof. Since f_3 must be starshaped the first condition is $a_0 = f_3(0) < 0$. By doing some simple calculations we get that

$$h_{f_3}(x) := xf_3'(x) - f_3(x) = 2a_3x^3 + a_2x^2 - a_0.$$

Since h_{f_3} must be non-negative on $[0, +\infty)$, it must also satisfy the following formula $\lim_{x\to+\infty} h_{f_3}(x) = +\infty$, so $a_3 > 0$ is the second condition.

Let us continue the study of h_{f_3} through FDT. The first thing to do is the calculation of the derivative of h_{f_3} to determine its critical (only stationary) points. Well,

$$h_{f_3}'(x) = 6a_3x^2 + 2a_2x = 6a_3x\left(x + \frac{a_2}{3a_3}\right).$$

Then, the possible critical points are 0 and $-\frac{a_2}{3a_3}$.

As all the upcoming calculations are very simple we will make an account of all the results on the next table. Before doing that, we need to introduce some notation:

$A = (-\frac{a_2}{3a_3}, 0)$, $B = (-\infty, -\frac{a_2}{3a_3}) \cup (0, +\infty)$, $C = (-\infty, 0) \cup (0, +\infty)$, $D = (0, -\frac{a_2}{3a_3})$, and $E = (-\infty, 0) \cup (-\frac{a_2}{3a_3}, +\infty)$.

After checking all the results given in Table 1, we can resume them into the two cases shown in the statement of the theorem. Therefore, the proof has been completed.

Remark 6. *We estimate appropriate to clarify the identifier of each column of Table 1:*

(1) *The first three columns give conditions for the coefficients a_3, a_2, and a_0.*
(2) *The fourth contains the critical points of h_{f_3}.*
(3) *The fifth (sixth) column shows the region where h_{f_3} decreases (increases), respectively.*
(4) *The seventh (eighth) column shows the points where h_{f_3} has minima (maxima) extrema, respectively.*
(5) *The ninth shows the minimum of h_{f_3} on the interval $[0, +\infty)$.*
(6) *The last column gives extra conditions (if any) for making the original function f_3 starshaped on $[0, +\infty)$.*

a_3	a_2	a_0	c. p.	d.	i.	min	max	min on $\mathbb{R}_{\geq 0}$	$f_3 \in SS[+\infty)$.
$+$	$+$	$-$	$-\frac{a_2}{3a_3}, 0$	A	B	0	$-\frac{a_2}{3a_3}$	$-a_0$	✓
$+$	0	$-$	0		C			$-a_0$	✓
$+$	$-$	$-$	$0, -\frac{a_2}{3a_3}$	D	E	$-\frac{a_2}{3a_3}$	0	$h_{f_3}(-\frac{a_2}{3a_3})$	$h_{f_3}(-\frac{a_2}{3a_3}) \geq 0$

TABLE 1. Conditions imposed on the coefficients of f_3.

Remark 7. *The next two observations are important:*

(1) 0 *can not be root of h_{f_3}.*
(2) a_1 *can be any real number. The same affirmation counts for h_{f_2}.*

Example 1. *The polynomial function $f(x) = (x-1)^3$ is starshaped on $[0, +\infty)$.*

Well, in this case we have $a_3 = 1 > 0$, $a_2 = -3 < 0$, $a_0 = -1 < 0$ and $h_f\left(-\frac{a_2}{3a_3}\right) = h_f(1) = 0$. By part 2 from Theorem 5 $f \in SS[+\infty)$.

3. Jensen m-convex polynomial functions of low degree

Before classifying all the real polynomial functions of degree less than the fourth that are Jensen m-convex on $[0, +\infty)$ we recall some elementary properties assuming that $m \in (0, 1)$ [5].

(1) $c_m > 2$.
(2) $x^2 - 2mxy + y^2 \geq 0$ for all $x, y \in [0, +\infty)$.
(3) If $f \in J_m[+\infty)$ then $f(0) \leq 0$.

Well, here it is our first result on Jensen m-convex real polynomial functions.

Theorem 8. *The polynomial functions f_0, f_1 and f_2 are Jensen m-convex on $[0, +\infty)$ if and only if their coefficients satisfy the following conditions:*

(1) *For $i = 0$, $a_0 \leq 0$.*
(2) *For $i = 1$, $a_1 \neq 0$, and $a_0 \leq 0$.*
(3) *For $i = 2$, $a_2 > 0$, and $a_0 \leq 0$.*

Proof. The condition $a_0 \leq 0$ is true for $i = 0, 1, 2$ because of property 3 above indicated. Then for $i = 0$, we are done. For $i = 1$, the inequality (3) becomes equivalent to $(1 - \frac{2}{c_m})a_0 \leq 0$, so a_1 must be any nonzero real number. For $i = 2$, the same inequality equivalently turns into $a_2(x^2 - 2mxy + y^2) \geq -a_0(m+1)(c_m - 2)$ for all $x, y \geq 0$, so a_2 must be any positive real number.

For case $i = 3$, the first condition to be taken from now is $f_3(0) = a_0 \leq 0$. Then, we proceed separately as we did in the previous section. Likewise, we appeal to a procedure other than the simple application of the definition. In this situation we employ the auxiliary real function J_{f_3} of two real variables restricted to the first closed quadrant of the plane (i.e., $(\mathbb{R}_{\geq 0})^2$) defined as follows:

$$J_{f_3} : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$$
$$(x, y) \longmapsto \frac{f_3(x) + f_3(y)}{c_m} - f_3\left(\frac{x+y}{c_m}\right).$$

Assuming that $m \in (0, 1)$ is fixed we will apply the Second-Partials Test (SPT) [1, p. 943] in order to find the absolute minimum of the function J_{f_3} in the interior of $R = [0, +\infty) \times [0, +\infty)$. We need the first-order partial derivatives of J_{f_3}

$$\begin{cases} \dfrac{\partial J_{f_3}}{\partial x}(x, y) = \dfrac{1}{c_m}\left[f_3'(x) - f_3'\left(\dfrac{x+y}{c_m}\right)\right] \\[2ex] \dfrac{\partial J_{f_3}}{\partial y}(x, y) = \dfrac{1}{c_m}\left[f_3'(y) - f_3'\left(\dfrac{x+y}{c_m}\right)\right]. \end{cases} \tag{4}$$

By solving the appropriate system to find the critical points we get

$$f_3'(x) = f_3'(y)$$

which reduces to:

$$(3a_3(x+y) + 2a_2)(x - y) = 0. \tag{5}$$

Finally, we have to study the behavior of J_{f_3} on the boundary of R, ∂R. It means considering the one variable functions h_3 and v_3 defined by $h_3(x) := J_{f_3}(x, 0)$ for all $x \geq 0$ and $v_3(y) := J_{f_3}(0, y)$ for all $y \geq 0$. The second condition, $a_3 > 0$, comes out from the following simple fact:

$$\lim_{x \to +\infty} h_3(x) = \begin{cases} +\infty & \text{if } a_3 > 0 \\ -\infty & \text{if } a_3 < 0. \end{cases}$$

Now, we present a lemma in which we give conditions on the coefficients of f_3 that guarantee J_{f_3} has an absolute minimum on R.

Lemma 9. *The function J_{f_3} has an absolute minimum on the unbounded closed region R which is characterized by*

$$\min_{(x,y)\in R}\{J_{f_3}(x,y)\} = \begin{cases} \dfrac{(1-m)(8a_2^3m(1+m)-27a_3^2a_0(1+3m)^2)}{27a_3^2(1+m)(1+3m)^2} \\ \\ \dfrac{4a_2^3m(1+m)-27a_3^2a_0(1-m)(1+2m)^2}{27a_3^2(1+m)(1+2m)^2} \\ \\ -\dfrac{a_0(1-m)}{1+m} \end{cases} \tag{6}$$

if and only if the coefficients of f_3 satisfy the following conditions, respectively:

(1) $a_3 > 0$, $a_2 < 0$, $a_0 \le 0$, *and* $m \in (0, m_1]$ *where* $m_1 = \frac{\sqrt{33}-1}{16} \approx 0.296535$.
(2) $a_3 > 0$, $a_2 < 0$, $a_0 \le 0$, *and* $m \in (m_1, 1)$.
(3) $a_3 > 0$, $a_2 = 0$, *and* $a_0 \le 0$.

Proof. Equation (3.2) produces two kinds of critical points. Two on the line $y = x$ and two more points on the line $y = -x - \frac{2a_2}{3a_3}$. After many calculations performed we get the following stationary points:

$$O = (0,0) \in \partial R; \quad P_1 = \left(-\frac{2a_2(1+m)}{3a_3(1+3m)}, -\frac{2a_2(1+m)}{3a_3(1+3m)}\right)$$

and

$$P_2 = \left(-\frac{2a_2m}{3a_3(1+m)}, -\frac{2a_2}{3a_3(1+m)}\right); \; P_3 = \left(-\frac{2a_2}{3a_3(1+m)}, -\frac{2a_2m}{3a_3(1+m)}\right)$$

under the condition $a_2 < 0$ (because $a_3 > 0$) to ensure their location in the interior of R.

In order to decide the nature of each point, it is necessary to find the second-order partial derivatives of J_{f_3}. From (4)

$$\begin{cases} \dfrac{\partial^2 J_{f_3}}{\partial x^2}(x,y) = \dfrac{1}{c_m}\left[f_3''(x) - \dfrac{1}{c_m}f_3''\left(\dfrac{x+y}{c_m}\right)\right] \\ \\ \dfrac{\partial^2 J_{f_3}}{\partial y^2}(x,y) = \dfrac{1}{c_m}\left[f_3''(y) - \dfrac{1}{c_m}f_3''\left(\dfrac{x+y}{c_m}\right)\right] \\ \\ \dfrac{\partial^2 J_{f_3}}{\partial x \partial y}(x,y) = -\dfrac{1}{c_m^2}f_3''\left(\dfrac{x+y}{c_m}\right), \end{cases}$$

which yields Table 2 (O will be analyzed later on), where $A = \dfrac{\partial^2 J_{f_3}}{\partial x^2}(x_0,y_0)$, $B = \dfrac{\partial^2 J_{f_3}}{\partial y^2}(x_0,y_0)$, $C = \dfrac{\partial^2 J_{f_3}}{\partial x \partial y}(x_0,y_0)$, $H_m = \dfrac{2a_2m(1-m)}{1+m} < 0$, and $K_m = (1+m)(1+3m) > 0$.

By using SPT on each critical point in the interior of R, we get that J_{f_3} has a relative minimum at P_1 and the other two points are saddle points. So the minimum value attained by J_{f_3} in the interior of R is

$$J_{f_3}(P_1) = \frac{(1-m)(8a_2^3m(1+m)-27a_3^2a_0(1+3m)^2)}{27a_3^2(1+m)(1+3m)^2}. \tag{7}$$

Next step is finding the critical points of the restriction of J_{f_3} to ∂R. By doing so, we get the following two real one-variable functions:

(x_0, y_0)	A	B	C	$D = AB - C^2$
P_1	$-\dfrac{(1+2m)H_m}{K_m}$	$-\dfrac{(1+2m)H_m}{K_m}$	$-\dfrac{mH_m}{K_m}$	$\dfrac{H_m^2}{K_m}$
P_2	$\dfrac{H_m}{(1+m)^2}$	$-\dfrac{(1+2m)H_m}{(1+m)^2}$	$-\dfrac{mH_m}{(1+m)^2}$	$-\dfrac{H_m^2}{(1+m)^2}$
P_3	$-\dfrac{(1+2m)H_m}{(1+m)^2}$	$\dfrac{H_m}{(1+m)^2}$	$-\dfrac{mH_m}{(1+m)^2}$	$-\dfrac{H_m^2}{(1+m)^2}$

Table 2. Application of SPT.

$$\begin{cases} h_3(x) = \dfrac{f_3(x) + a_0}{c_m} - f_3\left(\dfrac{x}{c_m}\right), & x \in [0, +\infty) \\ \\ v_3(y) = \dfrac{a_0 + f_3(y)}{c_m} - f_3\left(\dfrac{y}{c_m}\right), & y \in [0, +\infty). \end{cases}$$

By employing basic calculus of one variable we obtain two critical points for h_3 and two more for v_3: $x_1 = 0$, $x_2 = -\frac{2a_2(1+m)}{3a_3(1+2m)}$ and $y_1 = 0$, $y_2 = -\frac{2a_2(1+m)}{3a_3(1+2m)}$, respectively. By calculating the second derivative of h_3 and v_3, respectively, we get

$$h_3''(x_1) = \frac{2ma_2}{(1+m)^2} \quad \text{and} \quad h_3''(x_2) = -\frac{2ma_2}{(1+m)^2}$$

and

$$v_3''(y_1) = \frac{2ma_2}{(1+m)^2} \quad \text{and} \quad v_3''(y_2) = -\frac{2ma_2}{(1+m)^2}.$$

By the Second Derivative Test:

(1) h_3 has a relative maximum (minimum) at $x_1(x_2)$, respectively.
(2) v_3 has a relative maximum (minimum) at $y_1(y_2)$, respectively.

So, we get three more points

$$O = (0,0); \quad Q_1 = \left(-\frac{2a_2(1+m)}{3a_3(1+2m)}, 0\right); \text{ and } Q_2 = \left(0, -\frac{2a_2(1+m)}{3a_3(1+2m)}\right)$$

and their images through J_{f_3} are:

$$J_{f_3}(O) = -\frac{a_0(1-m)}{1+m} \geq 0$$

and

$$J_{f_3}(Q_1) = J_{f_3}(Q_2) = \frac{4a_2^3 m(1+m) - 27a_3^2 a_0(1-m)(1+2m)^2}{27a_3^2(1+m)(1+2m)^2}. \tag{8}$$

Since a_2 is negative it is not hard to check the following inequality:

$$J_{f_3}(O) \geq J_{f_3}(Q_1). \tag{9}$$

Therefore, by relations (7) and (9), the minimum of J_{f_3} on $R = \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ is given by

$$\min_{(x,y)\in R}\{J_{f_3}(x,y)\} = \min\{J_{f_3}(P_1), J_{f_3}(Q_1)\}$$

It is also not very difficult to demonstrate algebraically (assuming $m \in (0,1)$) that $J_{f_3}(P_1) \leq J_{f_3}(Q_1)$ iff $8m^2+m-1 \leq 0$ iff $m \in (0, m_1]$ where $m_1 = \frac{\sqrt{33}-1}{16} \approx 0.296535$. So, the absolute minimum of J_{f_3} is either $J_{f_3}(P_1)$ and it is attained at the unique interior point P_1 when $m \in (0, m_1]$ or $J_{f_3}(Q_1)$ and it is attained at two points on the boundary, Q_1 and Q_2 (the first on the X axis and the second on the Y axis) when $m \in (m_1, 1)$.

On the other hand, if $a_2 = 0$ all the stationary points of J_{f_3} coincide with O (i.e., $P_1 = P_2 = P_3 = Q_1 = Q_2 = (0,0)$) and this point will be the point where J_{f_3} attains its absolute minimum ($a_3 > 0$). So, we have covered all the possible cases indicated in the statement of the lemma. Therefore, the proof has been completed.

So far, we were able to find conditions on the coefficients of f_3, which guarantee our auxiliary function J_{f_3} has an absolute minimum on R. Now, we are headed to uncover additional conditions that make the function J_{f_3} non-negative or equivalently f_3 to be a Jensen m-convex function on $[0, +\infty)$ that is one of the main objectives of this research. At the same time, we could try to find the greatest value of $m \in (0,1)$ for which $f_3 \in J_m[+\infty)$. Next theorem resumes our findings, but first we need to define another real (piecewise) function, which we will denote by g as follows:

$$g : (0,1) \longrightarrow \mathbb{R}$$

$$m \longmapsto \begin{cases} \dfrac{8m(1+m)}{27(1+3m)^2} & \text{if } m \in (0, m_1] \\ \\ \dfrac{4m(1+m)}{27(1-m)(1+2m)^2} & \text{if } m \in (m_1, 1), \end{cases}$$

being m_1 as before.

Theorem 10. *The polynomial function f_3 is Jensen m-convex on $[0, +\infty)$ if and only if its coefficients satisfy any of the following conditions:*

(1) $a_3 > 0$, $a_2 < 0$, $a_0 < 0$, *and* $a_3^2 a_2^{-3} a_0 \geq g(m)$.
(2) $a_3 > 0$, $a_2 = 0$, *and* $a_0 \leq 0$.

Proof. By formula (6), we know that the minimum value of J_{f_3} will be non-negative for the first case if and only if conditions given in part 1 from Lemma 9 are true excluding $a_0 = 0$ (otherwise, the minimum value of J_{f_3} would be always negative, according to equation (7)) and

$$8a_2^3 m(1+m) - 27a_3^2 a_0 (1+3m)^2 \geq 0, \tag{10}$$

and for the second case iff conditions shown in part 2 of the same lemma are true excluding $a_0 = 0$ (otherwise, the minimum value of J_{f_3} would be always negative, according to equation (8)) and

$$4a_2^3 m(1+m) - 27a_3^2 a_0 (1-m)(1+2m)^2 \geq 0, \tag{11}$$

and finally, for the last case no additional condition is necessary because

$$-\frac{a_0(1-m)}{1+m} \geq 0.$$

Clearly, inequalities (10) and (11) are equivalent to the sole condition $a_3^2 a_2^{-3} a_0 \geq g(m)$. Therefore, the proof has been completed.

The graph of the piecewise function g defined previously to Theorem 10 is shown in Figure 1. Each piece has been distinguished with a color. Observe that for both cases of Theorem 10 and following ideas from [8, Definition 2], it makes sense to define

$$m_{f_3} = \sup\{m \in (0,1) : f_3 \in J_m[+\infty)\}.$$

In fact, if we denote $c = a_3^2 a_2^{-3} a_0$ then m_{f_3} will be obtained by solving the equation $g(m) = c$ for m under the conditions given in part 1 of Theorem 10, and will be 1 if the conditions given in its part 2 are satisfied. Observe that the latter equation is solvable because the function g is an injective application from $(0,1)$ onto $(0, +\infty)$. So, m_{f_3} will be the unique root in $(0,1)$ of one of the next equations according to the case

$$(243c - 8)m^2 + (162c - 8)m + 27c = 0 \text{ if } a_3 > 0, a_2 < 0, a_0 < 0, c \in (0, g(m_1)] \tag{12}$$

$$108cm^3 + 4m^2 + (4 - 81c)m - 27c = 0 \text{ if } a_3 > 0, a_2 < 0, a_0 < 0, c \in (g(m_1), +\infty) \tag{13}$$

where $g(m_1) = \dfrac{5\sqrt{33}-27}{54} \approx 0.0319039$. Furthermore, as the corresponding discriminants from the quadratic equation (12) and the cubic equation (13) are:

$$\Delta_1 = 64(1-27c) > 0$$

for all $c \in (0, g(m_1)]$ and

$$\Delta_2 = 16(16 - 1944c + 59049c^2 - 1062882c^3) < 0$$

for all $c \in (g(m_1), +\infty)$, we can clear the value of m_{f_3} as follows:

$$m_{f_3} = \begin{cases} \dfrac{8-162c-\sqrt{\Delta_1}}{486c-16} & \text{if } c \in (0, g(m_1)] \\[2ex] -\dfrac{1}{81c} + \sqrt[3]{\dfrac{\sqrt{-\Delta_2}}{C_1} + C_2} + \dfrac{C_3}{\sqrt[3]{\dfrac{\sqrt{-\Delta_2}}{C_1} + C_2}} & \text{if } c \in (g(m_1), +\infty), \end{cases}$$

where

$$C_1 = 69984\sqrt{3}c^2, \quad C_2 = \frac{531441c^3 - 19683c^2 + 972c - 8}{4251528c^3},$$

and

$$C_3 = \frac{6561c^2 - 324c + 4}{26244c^2}.$$

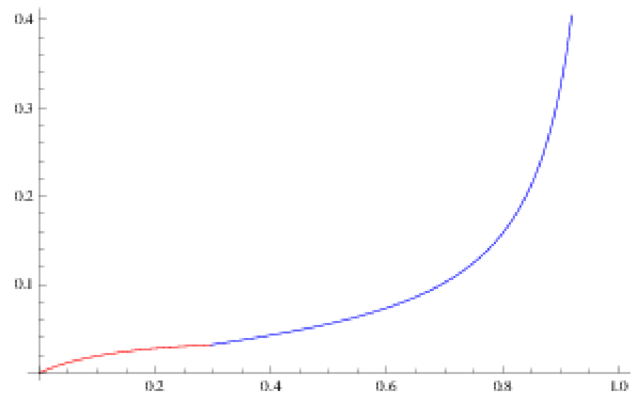

FIGURE 1. Graph of g

4. Examples and Counterexamples

Let us start by fixing some additional notation. The set of all real polynomial functions of degree less than or equal to i ($i = 0, 1, 2, 3$) and zero will be denoted as $P_i[x]$ (i.e., $P_i[x] = \{0\} \cup \{f \in \mathbb{R}[x] : deg(f) \leq i\}$).

Now observe that from Theorem 4, Theorem 8, and some comments given previously to [15, Lemma 2.1.] follows the equality

$$P_i[x] \cap SS[+\infty) = P_i[x] \cap J_m[+\infty) \quad (i = 0, 1, 2)$$

for all $m \in (0, 1)$.

On the other hand, for case $i = 3$ with an m fixed we have neither

$$P_3[x] \cap J_m[+\infty) \subset P_3[x] \cap SS[+\infty)$$

nor

$$P_3[x] \cap SS[+\infty) \subset P_3[x] \cap J_m[+\infty),$$

as the next two examples show.

Example 2. *The function* $f(x) = \frac{1}{2}x^3 - 4x^2 - 2 \in J_{m_f}[+\infty) - SS[+\infty)$. *Well, in this case we have* $a_3 = \frac{1}{2} > 0$, $a_2 = -4 < 0$, $a_0 = -2 < 0$ *and* $h_f\left(-\frac{a_2}{3a_3}\right) = h_f(\frac{8}{3}) = -\frac{202}{27} < 0$. *By part 2 from Theorem 5* $f \notin SS[+\infty)$.

Besides, by taking $c = a_3^2 a_2^{-3} a_0 = \frac{1}{128} = 0.0078125 \in (0, g(m_1)]$ *in equation 12 and solving for* m *we get* $m_f = \frac{-431+32\sqrt{202}}{781} \approx 0.0304807$. *So,* $f \in J_{m_f}[+\infty)$.

Example 3. *The function* $f(x) = x^3 + x^2 - 1 \in SS[+\infty) - J_m[+\infty)$ *for all* $m \in (0,1)$.

Well, in this case we have $a_3 = 1 > 0$, $a_2 = 1 > 0$, *and* $a_0 = -1 < 0$. *By part 1 from Theorem 5* $f \in SS[+\infty)$.

Besides, by Theorem 10 for all $m \in (0,1)$, f *is not Jensen* m*-convex on* $[0, +\infty)$.

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