

Infinitely Many Solutions of the Neumann Problem for Elliptic systems in Anisotropic Variable Exponent Sobolev Spaces

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ABSTRACT. In this paper, we prove the existence of infinitely many solutions for the following system

$$\begin{cases} \sum_{i=1}^N \frac{\partial}{\partial x_i} a_i \left(\cdot, \frac{\partial u}{\partial x_i} \right) + h(\cdot) |u|^{p_0(\cdot)-2} u = \theta(\cdot) f(u, v) & \text{in } \Omega \\ \sum_{i=1}^N \frac{\partial}{\partial x_i} b_i \left(\cdot, \frac{\partial v}{\partial x_i} \right) + k(\cdot) |v|^{p_0(\cdot)-2} v = \theta(\cdot) g(u, v) & \text{in } \Omega \\ \sum_{i=1}^N a_i \left(\cdot, \frac{\partial u}{\partial x_i} \right) \gamma_i = \sum_{i=1}^N b_i \left(\cdot, \frac{\partial v}{\partial x_i} \right) \gamma_i = 0 & \text{on } \partial\Omega, \end{cases}$$

by applying a critical point variational principle obtained by Ricceri as a consequence of a more general variational principle and the theory of the anisotropic variable exponent Sobolev spaces.

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with boundary $\partial\Omega$ of class C^1 , and let $\vec{\gamma}$ be the outward unit normal to $\partial\Omega$.

In these last years, the anisotropic variable exponent Sobolev space $W^{1,\vec{p}(\cdot)}(\Omega)$ have captured the attention of many researchers and a new operator has been introduced in the

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literature, namely

$$\Delta_{\vec{p}(x)} u = \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)-2} \frac{\partial u}{\partial x_i} \right). \quad (1)$$

It's clear that this $\vec{p}(x)$ -Laplace operator is a generalization of the $p(\cdot)$ -Laplace operator. For some existing results of strongly nonlinear elliptic equations in the anisotropic variable exponent Sobolev spaces, see references [2, 10, 19].

In the present paper, we consider a problem involving a more general type of operator, that is,

$$\sum_{i=1}^N \frac{\partial}{\partial x_i} a_i \left(x, \frac{\partial u}{\partial x_i} \right), \quad (2)$$

the applications $a_i : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ are Carathéodory functions satisfying some hypothesis conditions for $i=1, \dots, N$ (see Section 3). The operator defined above is of $\vec{p}(\cdot)$ -Laplacian type when we take $a_i(x, s) = |s|^{p_i(x)-2}s$ for all $x \in \Omega$, $s \in \mathbb{R}$ and all $i = 1, \dots, N$.

Notice that the general operator given by (2) can admit degenerate and singular points. It is no surprising that there exist previous works treating problems with same operator, to give some examples, we refer the reader to [8, 12, 13] where the authors were concerned with Dirichlet problems. Our work is focused on a Neumann problem, we refer the reader to [1, 9].

In this paper, we are interested in the following problem

$$(\mathcal{P}) \begin{cases} \sum_{i=1}^N \frac{\partial}{\partial x_i} a_i \left(\cdot, \frac{\partial u}{\partial x_i} \right) + h(\cdot) |u|^{p_0(\cdot)-2} u = \theta(\cdot) f(u, v) & \text{in } \Omega \\ \sum_{i=1}^N \frac{\partial}{\partial x_i} b_i \left(\cdot, \frac{\partial v}{\partial x_i} \right) + k(\cdot) |v|^{p_0(\cdot)-2} v = \theta(\cdot) g(u, v) & \text{in } \Omega \\ \sum_{i=1}^N a_i \left(\cdot, \frac{\partial u}{\partial x_i} \right) \gamma_i = \sum_{i=1}^N b_i \left(\cdot, \frac{\partial v}{\partial x_i} \right) \gamma_i = 0 & \text{on } \partial\Omega. \end{cases}$$

More precisely we are interested to the existence of infinitely many weak solutions to such a problem.

In the context of Dirichlet boundary conditions J. Vlin have studied the existence of non trivial solutions of (\mathcal{P}) in the isotropic variable exponent Sobolev space, see [18]. In the same case M. Bendahmane and F. Mokhtari have studied in [5] the problem (\mathcal{P}) in the isotropic variable Sobolev space where the second term was a measure data. In the same framework, D. S. Moschetto in [15] have studied the problem (\mathcal{P}) in the particular case of homogeneous Neumann condition and $\sum_{i=1}^N \frac{\partial}{\partial x_i} a_i \left(\cdot, \frac{\partial u}{\partial x_i} \right) = \Delta_{\vec{p}(\cdot)}(u)$ and $\sum_{i=1}^N \frac{\partial}{\partial x_i} b_i \left(\cdot, \frac{\partial v}{\partial x_i} \right) = \Delta_{\vec{p}(\cdot)}(v)$.

Even though the problem (\mathcal{P}) has been studied by some other authors (see e.g. [3, 6, 7, 14, 17, 20]) the hypotheses we use in this paper are totally different and so are our results.

The aim of the present paper is to generalize the results of [1, 5, 15, 18] in the anisotropic cases.

The main difficulties with this kind of problems are the framework of anisotropic Sobolev spaces and the fact that we have Neumann boundary conditions that make some difficulties in the application of theorem 1.1.

The following theorem plays an important role in this paper.

Theorem 1.1. (See [16], Theorem 2.5). *Let X be a reflexive real Banach space, and let $\Phi, \Psi : X \rightarrow \mathbb{R}$ be two sequentially weakly lower semicontinuous and Gâteaux differentiable functionals. Also, assume that Ψ is (strongly) continuous and satisfies $\lim_{\|u\| \rightarrow +\infty} \Psi(u) = +\infty$. For each $\rho > \inf_X \Psi$, put*

$$\varphi(\rho) = \inf_{u \in \Psi^{-1}((-\infty, \rho))} \frac{\Phi(u) - \inf_{v \in \overline{\Psi^{-1}((-\infty, \rho))}^w} \Phi(v)}{\rho - \Psi(u)}, \quad (3)$$

where $\overline{\Psi^{-1}((-\infty, \rho))}^w$ is the closure of $\Psi^{-1}((-\infty, \rho))$ in the weak topology. Furthermore, set

$$\gamma = \liminf_{\rho \rightarrow +\infty} \varphi(\rho) \quad (4)$$

and

$$\delta = \liminf_{\rho \rightarrow (\inf_X \Psi)^+} \varphi(\rho). \quad (5)$$

then, the following conclusions hold:

(a) For each $\rho > \inf_X \Psi$ and each $t > \varphi(\rho)$, the functional $\Phi + t\Psi$ has a critical point which lies in $\Psi^{-1}((-\infty, \rho))$.

(b) If $\gamma < +\infty$, then, for each $t > \gamma$, the following alternative holds: either $\Phi + t\Psi$ has a global minimum, that is, it exists a sequence $\{u_n\}$ of critical points of $\Phi + t\Psi$ such that $\lim_{n \rightarrow \infty} \Psi(u_n) = +\infty$.

(c) If $\delta < +\infty$, then, for each $t > \delta$, the following alternative holds: either there exists a global minimum of Ψ which is a local minimum of $\Phi + t\Psi$, or there exists a sequence of pairwise distinct critical points of $\Phi + t\Psi$ which weakly converges to a global minimum of Ψ .

This paper is organized as follows: In Section 2, we present some preliminary knowledge on the anisotropic Sobolev spaces with variable exponent. We introduce in Section 3 some assumptions for which our problem has solutions. In Section 4, we prove the existence of infinitely many weak solutions for our Neumann elliptic problem and we give a conclusion and some perspectives.

2. Preliminaries

Let Ω be an open bounded subset of \mathbb{R}^N ($N \geq 1$), we define

$$\mathcal{C}_+(\overline{\Omega}) = \{ \text{measurable function } z(\cdot) : \overline{\Omega} \mapsto \mathbb{R} \text{ such that } 1 < z^- \leq z^+ < \infty \},$$

where

$$z^- = \text{ess inf} \{ z(x) / x \in \Omega \} \quad \text{and} \quad z^+ = \text{ess sup} \{ z(x) / x \in \Omega \}.$$

We define the Lebesgue space with variable exponent $L^{z(\cdot)}(\Omega)$ as the set of all measurable functions $u : \Omega \mapsto \mathbb{R}$ for which the convex modular

$$\rho_{z(\cdot)}(u) := \int_{\Omega} |u|^{z(x)} dx,$$

is finite, then

$$\|u\|_{z(\cdot)} = \inf \left\{ \lambda > 0 : \rho_{z(\cdot)}(u/\lambda) \leq 1 \right\},$$

defines a norm in $L^{z(\cdot)}(\Omega)$, called the Luxemburg norm.

The space $(L^{z(\cdot)}(\Omega), \|\cdot\|_{z(\cdot)})$ is a separable and reflexive Banach space. Moreover, the space $L^{z(\cdot)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{z'(\cdot)}(\Omega)$,

where $\frac{1}{z(x)} + \frac{1}{z'(x)} = 1$.

Finally, we have the Hölder type inequality:

$$\left| \int_{\Omega} uv \, dx \right| \leq \left(\frac{1}{z^-} + \frac{1}{(z')^-} \right) \|u\|_{z(\cdot)} \|v\|_{z'(\cdot)} \quad (6)$$

for all $u \in L^{z(\cdot)}(\Omega)$ and $v \in L^{z'(\cdot)}(\Omega)$.

An important role in manipulating the generalized Lebesgue spaces is played by the modular $\rho_{z(\cdot)}$ of the space $L^{z(\cdot)}(\Omega)$, we have the following result.

Proposition 2.1. (See [11]). *If $u \in L^{z(\cdot)}(\Omega)$, then the following properties hold true:*

- (i): $\|u\|_{z(\cdot)} > 1 \Rightarrow \|u\|_{z(\cdot)}^{z^-} < \rho_{z(\cdot)}(u) < \|u\|_{z(\cdot)}^{z^+}$,
- (ii): $\|u\|_{z(\cdot)} < 1 \Rightarrow \|u\|_{z(\cdot)}^{p^+} < \rho_{z(\cdot)}(u) < \|u\|_{z(\cdot)}^{z^-}$.

The Sobolev spaces with variable exponent is defined by

$$W^{1,z(\cdot)}(\Omega) = \{u \in L^{z(\cdot)}(\Omega) \quad \text{and} \quad |\nabla u| \in L^{z(\cdot)}(\Omega)\},$$

equipped with the following norm

$$\|u\|_{1,z(\cdot)} = \|u\|_{z(\cdot)} + \|\nabla u\|_{z(\cdot)}.$$

The space $(W^{1,z(\cdot)}(\Omega), \|\cdot\|_{1,z(\cdot)})$ is a separable and reflexive Banach space.

Now, we present the anisotropic variable exponent Sobolev space used for the study of the main problem.

Let $z_0(\cdot), z_1(\cdot), \dots, z_N(\cdot)$ be $N + 1$ variable exponents in $\mathcal{C}_+(\overline{\Omega})$. We denote

$$\vec{z}(\cdot) = \{z_0(\cdot), \dots, z_N(\cdot)\}, \quad D^0 u = u \quad \text{and} \quad D^i u = \frac{\partial u}{\partial x_i} \quad \text{for } i = 1, \dots, N.$$

We define

$$\underline{z} = \min \{z_i^-, \quad i = 0, 1, \dots, N\} \quad \text{then} \quad \underline{z} > 1, \quad (7)$$

and

$$\bar{z} = \max \{z_i^+, \quad i = 0, 1, \dots, N\}. \quad (8)$$

The anisotropic Sobolev space with variable exponent $W^{1,\vec{z}(\cdot)}(\Omega)$ is defined as follows

$$W^{1,\vec{z}(\cdot)}(\Omega) = \{u \in L^{z_0(\cdot)}(\Omega) \quad \text{and} \quad D^i u \in L^{z_i(\cdot)}(\Omega) \quad \text{for } i = 1, 2, \dots, N\},$$

endowed with the norm

$$\|u\|_{1,\vec{z}(\cdot)} = \sum_{i=0}^N \|D^i u\|_{z_i(\cdot)}. \quad (9)$$

The space $(W^{1,\vec{z}(\cdot)}(\Omega), \|\cdot\|_{1,\vec{z}(\cdot)})$ is separable and reflexive Banach space (cf. [4, 12]).

3. Essential assumptions

Let $p_0(\cdot), p_1(\cdot), \dots, p_N(\cdot)$ and $q_0(\cdot), q_1(\cdot), \dots, q_N(\cdot)$ be $2(N + 1)$ variable exponents in $\mathcal{C}_+(\overline{\Omega})$. We assume that

$$\underline{p} > N \quad \text{and} \quad \underline{q} > N. \quad (10)$$

Proposition 3.1. *Since $W^{1,\vec{p}(\cdot)}(\Omega)$ (respectively $W^{1,\vec{q}(\cdot)}(\Omega)$) is continuously embedded in $W^{1,p}(\Omega)$ (respectively $W^{1,q}(\Omega)$), and since $W^{1,p}(\Omega)$ and $W^{1,q}(\Omega)$ are compactly embedded in $C^0(\overline{\Omega})$ (the space of continuous functions), thus the spaces $W^{1,\vec{p}(\cdot)}(\Omega)$ and $W^{1,\vec{q}(\cdot)}(\Omega)$ are compactly embedded in $C^0(\overline{\Omega})$.*

Then we can set

$$C_1 = \sup_{u \in W^{1,\vec{p}(\cdot)}(\Omega) \setminus \{0\}} \frac{\|u\|_\infty}{\|u\|_{1,\vec{p}(\cdot)}}. \quad (11)$$

$$C_2 = \sup_{u \in W^{1,\vec{q}(\cdot)}(\Omega) \setminus \{0\}} \frac{\|u\|_\infty}{\|u\|_{1,\vec{q}(\cdot)}}. \quad (12)$$

The applications $a_i, b_i : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ are Carathéodory functions which satisfy the following assumptions :

(H₁): The growth condition :

$$|a_i(x, s)| \leq \eta_i(c_i(x) + |s|^{p_i(x)-1}) \quad \text{for } i = 1, \dots, N,$$

$$|b_i(x, s)| \leq \delta_i(d_i(x) + |s|^{q_i(x)-1}) \quad \text{for } i = 1, \dots, N,$$

where $c_i(\cdot)$ (respectively $d_i(\cdot)$) is a nonnegative function in $L^{p'_i(\cdot)}(\Omega)$ (respectively in $L^{q'_i(\cdot)}(\Omega)$), η_i and δ_i are positives constants.

(H₂): The coercivity condition : There exist four constants $\alpha_1, \beta_1, \alpha_2, \beta_2 > 0$ such that

$$\alpha_1 |s|^{p_i(x)} \leq a_i(x, s)s \leq \beta_1 A_i(x, s),$$

$$\alpha_2 |s|^{q_i(x)} \leq b_i(x, s)s \leq \beta_2 B_i(x, s),$$

where the functions $A_i, B_i : \Omega \times \mathbb{R} \mapsto \mathbb{R}$, are defined by

$$A_i(x, s) = \int_0^s a_i(x, t) dt \quad \text{and} \quad B_i(x, s) = \int_0^s b_i(x, t) dt.$$

(H₃): The monotonicity condition :

$$(a_i(x, s) - a_i(x, t))(s - t) > 0, \text{ for all } x \in \Omega, \forall s, t \in \mathbb{R} \text{ with } t \neq s.$$

$$(b_i(x, s) - b_i(x, t))(s - t) > 0, \text{ for all } x \in \Omega, \forall s, t \in \mathbb{R} \text{ with } t \neq s.$$

(H₄): $h(\cdot), k(\cdot) \in L^\infty(\Omega)$ and $\theta(\cdot) \in \mathcal{C}(\overline{\Omega})$ such that there exist $h_0, k_0 > 0$ such that

$$\text{ess inf}_{x \in \Omega} h(x) > h_0 \quad \text{and} \quad \text{ess inf}_{x \in \Omega} k(x) > k_0.$$

(H₅): $f, g \in \mathcal{C}(\mathbb{R}^2)$ such that the differential form $f(u, v)du + g(u, v)dv$ be exact.

Remark 3.1. *(H₅) implies that exists $H : \mathbb{R}^2 \mapsto \mathbb{R}$ be the integral of the differential form $f(u, v)du + g(u, v)dv$ such that $H(0, 0) = 0$.*

Let X be the Cartesian product between $W^{1,\vec{p}(\cdot)}(\Omega)$ and $W^{1,\vec{q}(\cdot)}(\Omega)$ Sobolev spaces with the norm $\|(u, v)\|_X = \sqrt{\|u\|_{1,\vec{p}(\cdot)}^2 + \|v\|_{1,\vec{q}(\cdot)}^2}$ or another equivalent to it.

We introduce the functionals $\Psi(\cdot, \cdot), \Phi(\cdot, \cdot) : W^{1,\vec{p}(\cdot)}(\Omega) \times W^{1,\vec{q}(\cdot)}(\Omega) \mapsto \mathbb{R}$ by

$$\Psi(u, v) = \sum_{i=1}^N \int_{\Omega} A_i(x, \frac{\partial u}{\partial x_i}) dx + \sum_{i=1}^N \int_{\Omega} B_i(x, \frac{\partial v}{\partial x_i}) dx$$

$$+ \int_{\Omega} \frac{h(x)}{p_0(x)} |u|^{p_0(x)} dx + \int_{\Omega} \frac{k(x)}{q_0(x)} |v|^{q_0(x)} dx, \quad (13)$$

and

$$\Phi(u, v) = - \int_{\Omega} F(x, u(x), v(x)) dx. \quad (14)$$

where $F : \bar{\Omega} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be defined as $F(x, u, v) = \theta(x)H(u, v)$.

Lemma 3.1. (see [9], [12]) *The functionals $\Psi(\cdot, \cdot)$ and $\Phi(\cdot, \cdot)$ are well defined on X . In addition, $\Psi(\cdot, \cdot)$ and $\Phi(\cdot, \cdot)$ is of class $\mathcal{C}^1(X, \mathbb{R})$ and*

$$\begin{aligned} \Psi'(u, v)(w, \phi) &= \sum_{i=1}^N \int_{\Omega} a_i \left(x, \frac{\partial u}{\partial x_i} \right) \frac{\partial w}{\partial x_i} dx + \sum_{i=1}^N \int_{\Omega} b_i \left(x, \frac{\partial v}{\partial x_i} \right) \frac{\partial \phi}{\partial x_i} dx \\ &+ \int_{\Omega} h(x) |u|^{p_0(x)-2} u w dx + \int_{\Omega} k(x) |v|^{q_0(x)-2} v \phi dx, \end{aligned} \quad (15)$$

and

$$\Phi'(u, v)(w, \phi) = - \int_{\Omega} \theta(x) \left(\frac{\partial H}{\partial u}(u(x), v(x)) w(x) + \frac{\partial H}{\partial v}(u(x), v(x)) \phi(x) \right) dx. \quad (16)$$

$\forall (u, v)(w, \phi) \in X$.

Lemma 3.2. (see [9]) *Under the hypothesis (H_1) – (H_5) and (10) the functionals $\Psi(\cdot, \cdot)$ and $\Phi(\cdot, \cdot)$ are weakly lower semicontinuous.*

Lemma 3.3. *Under the hypothesis (H_1) – (H_5) the functional $\Psi(\cdot, \cdot)$ is ceorcrive, that is,*

$$\Psi(u, v) \longrightarrow +\infty \quad \text{as } \|(u, v)\|_X \longrightarrow +\infty \quad \text{for } (u, v) \in X.$$

Proof Let $(u, v) \in X$, one has

$$\begin{aligned} \Psi(u, v) &= \sum_{i=1}^N \int_{\Omega} A_i \left(x, \frac{\partial u}{\partial x_i} \right) dx + \sum_{i=1}^N \int_{\Omega} B_i \left(x, \frac{\partial v}{\partial x_i} \right) dx \\ &+ \int_{\Omega} \frac{h(x)}{p_0(x)} |u|^{p_0(x)} dx + \int_{\Omega} \frac{k(x)}{q_0(x)} |v|^{q_0(x)} dx, \end{aligned} \quad (17)$$

then by using (H_2) and (H_4) , we get

$$\begin{aligned} \Psi(u, v) &\geq \sum_{i=1}^N \int_{\Omega} \frac{\alpha_1}{\beta_1} \left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)} dx + \int_{\Omega} \frac{h_0}{p_0^+} |u|^{p_0(x)} dx \\ &+ \sum_{i=1}^N \int_{\Omega} \frac{\alpha_2}{\beta_2} \left| \frac{\partial v}{\partial x_i} \right|^{q_i(x)} dx + \int_{\Omega} \frac{k_0}{q_0^+} |v|^{q_0(x)} dx \\ &\geq \min \left(\frac{\alpha_1}{\beta_1}, \frac{h_0}{p_0^+} \right) \left[\sum_{i=1}^N \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)} dx + \int_{\Omega} |u|^{p_0(x)} dx \right] \\ &+ \min \left(\frac{\alpha_2}{\beta_2}, \frac{k_0}{q_0^+} \right) \left[\sum_{i=1}^N \int_{\Omega} \left| \frac{\partial v}{\partial x_i} \right|^{q_i(x)} dx + \int_{\Omega} |v|^{q_0(x)} dx \right] \\ &\geq \min \left(\frac{\alpha_1}{\beta_1}, \frac{h_0}{p_0^+} \right) \left[\frac{1}{N^{p-1}} \left(\sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{p_i(\cdot)} \right)^p + \|u\|_{p_0(\cdot)}^p - N - 1 \right] \end{aligned}$$

$$\begin{aligned}
& + \min\left(\frac{\alpha_2}{\beta_2}, \frac{k_0}{q_0^+}\right) \left[\frac{1}{N^{\underline{q}-1}} \left(\sum_{i=1}^N \left\| \frac{\partial v}{\partial x_i} \right\|_{q_i(\cdot)} \right)^{\underline{q}} + \|u\|_{q_0(\cdot)}^{\underline{q}} - N - 1 \right] \\
& \geq \min\left(\frac{\alpha_1}{\beta_1}, \frac{h_0}{p_0^+}\right) \left[\frac{1}{(2N)^{\underline{p}-1}} \left(\sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{p_i(\cdot)} + \|u\|_{p_0(\cdot)} \right)^{\underline{p}} - N - 1 \right] \\
& + \min\left(\frac{\alpha_2}{\beta_2}, \frac{k_0}{q_0^+}\right) \left[\frac{1}{(2N)^{\underline{q}-1}} \left(\sum_{i=1}^N \left\| \frac{\partial v}{\partial x_i} \right\|_{q_i(\cdot)} + \|u\|_{q_0(\cdot)} \right)^{\underline{q}} - N - 1 \right] \\
& = \frac{\min\left(\frac{\alpha_1}{\beta_1}, \frac{h_0}{p_0^+}\right)}{(2N)^{\underline{p}-1}} \|u\|_{1,\vec{p}(\cdot)}^{\underline{p}} + \frac{\min\left(\frac{\alpha_2}{\beta_2}, \frac{k_0}{q_0^+}\right)}{(2N)^{\underline{q}-1}} \|u\|_{1,\vec{q}(\cdot)}^{\underline{q}} - K_2 \\
& \geq K_1 \left(\|u\|_{1,\vec{q}(\cdot)}^{\underline{p}} + \|v\|_{1,\vec{q}(\cdot)}^{\underline{q}} \right) - K_2 \\
& \geq K_1 \| (u, v) \|_X - K_2,
\end{aligned}$$

where $K_1, K_2 > 0$ constants.

Thus, if $\|(u, v)\|_X \rightarrow +\infty$ then $\Psi(u, v) \rightarrow +\infty$. □

Now, we set $\eta_1 = \left(\frac{C_1}{h_0 \text{meas}(\Omega)} \right)^{\bar{p}}$ and $\eta_2 = \left(\frac{C_2}{k_0 \text{meas}(\Omega)} \right)^{\bar{q}}$,

$$\alpha = \min \left\{ \frac{1}{\bar{p}\eta_1^{\bar{p}}}, \frac{1}{\bar{p}\eta_1^{\bar{p}}} \right\},$$

and

$$\beta = \min \left\{ \frac{1}{\bar{q}\eta_2^{\bar{q}}}, \frac{1}{\bar{q}\eta_2^{\bar{q}}} \right\}.$$

The sets $A(r), B(r)$, $r > 0$, below satisfied, play an important role in our exposition

$$A(r) = \{(\xi, \eta) \in \mathbb{R}^2 \text{ such that } \alpha F_{\vec{p}(\cdot)}(\xi) + \beta F_{\vec{q}(\cdot)}(\eta) \leq r\}$$

and

$$B(r) = \left\{ (\xi, \eta) \in \mathbb{R}^2 \text{ such that } \frac{1}{\underline{p}} \int_{\Omega} h(x) dx D_{\vec{p}(\cdot)}(\xi) + \frac{1}{\underline{q}} \int_{\Omega} k(x) dx D_{\vec{q}(\cdot)}(\eta) \leq r \right\}$$

where $D_{\vec{r}(\cdot)}(t) = \max(|t|^{\bar{r}}, |t|^{\underline{r}})$ and $F_{\vec{r}(\cdot)}(t) = \min(|t|^{\bar{r}}, |t|^{\underline{r}})$ with $\vec{r}(\cdot) \in \{\vec{p}(\cdot), \vec{q}(\cdot)\}$ and $t \in \{\xi, \eta\}$.

Lemma 3.4. *For all $r > 0$, we have*

$$B(r) \subset A(r).$$

Proof We observe that, by the definition of constants C_1 and C_2 , we have

$$\|u\|_{\infty} \leq C_1 \|u\|_{1,\vec{p}(\cdot)}, \forall u \in W^{1,\vec{p}(\cdot)}(\Omega)$$

and

$$\|v\|_{\infty} \leq C_2 \|v\|_{1,\vec{q}(\cdot)}, \forall v \in W^{1,\vec{q}(\cdot)}(\Omega).$$

For $u \equiv v \equiv 1$, we get

$$1 \leq \eta_1^{\bar{p}} h_0 \text{meas}(\Omega) \leq \eta_1^{\bar{p}} \int_{\Omega} h(x) dx,$$

and

$$1 \leq \eta_2^{\bar{q}} k_0 \text{meas}(\Omega) \leq \int_{\Omega} \eta_2^{\bar{q}} k(x) dx.$$

Thus, we obtain

$$\alpha \leq \frac{1}{\bar{p}\eta_1^{\bar{p}}} \leq \frac{1}{\bar{p}} \int_{\Omega} h(x) dx, \quad \text{and } \beta \leq \frac{1}{\bar{q}\eta_2^{\bar{q}}} \leq \frac{1}{\bar{q}} \int_{\Omega} k(x) dx.$$

Hence

$$\alpha \leq \frac{1}{\underline{p}} \int_{\Omega} h(x) dx, \quad \text{and } \beta \leq \frac{1}{\underline{q}} \int_{\Omega} k(x) dx.$$

Since

$$F_{\bar{p}(\cdot)}(t) \leq D_{\bar{p}(\cdot)}(t), \quad \text{and } F_{\bar{q}(\cdot)}(t) \leq D_{\bar{q}(\cdot)}(t), \forall t \in \mathbb{R}.$$

Thus, the inequality

$$\alpha F_{\bar{p}(\cdot)}(\xi) + \beta F_{\bar{q}(\cdot)}(\eta) \leq \frac{1}{\underline{p}} \int_{\Omega} h(x) dx D_{\bar{p}(\cdot)}(\xi) + \frac{1}{\underline{q}} \int_{\Omega} k(x) dx D_{\bar{q}(\cdot)}(\eta).$$

holds for every $(\xi, \eta) \in \mathbb{R}^2$ and therefore the inclusion

$$B(r) \subset A(r), \forall r > 0$$

holds. \square

4. Main results

Definition 4.1. We say that $(u, v) \in X$ a weak solution to the problem (\mathcal{P}) if for all $(w, \phi) \in X$, we have

$$\begin{aligned} & \int_{\Omega} \left[\sum_{i=1}^N a_i \left(x, \frac{\partial u}{\partial x_i}(x) \right) \frac{\partial w}{\partial x_i}(x) + h(x) |u(x)|^{p_0(x)-2} u(x) w(x) \right] dx \\ & + \int_{\Omega} \left[\sum_{i=1}^N b_i \left(x, \frac{\partial v}{\partial x_i}(x) \right) \frac{\partial \phi}{\partial x_i}(x) + k(x) |v(x)|^{q_0(x)-2} v(x) \phi(x) \right] dx \\ & = \int_{\Omega} \theta(x) \left[f(u(x), v(x)) w(x) + g(u(x), v(x)) \phi(x) \right] dx, \end{aligned}$$

the weak solutions of (\mathcal{P}) are precisely critical points of $\Psi + \Phi$.

One of our main results is the following theorem

Theorem 4.1. Suppose that $\Psi(\cdot, \cdot)$ and $\Phi(\cdot, \cdot)$ are as in (13) and (14) and (H_1) – (H_5) and (10) hold true.

- (a): If there exist $\rho_0 > 0$, $(\xi_0, \eta_0) \in \mathbb{R}^2$ with $(\xi_0, \eta_0) \in \text{Int}(B(\rho_0))$ ($\text{Int}(B)$ is the interior of B) and $\max_{A(\rho_0)} H(\xi, \eta) = H(\xi_0, \eta_0)$. Then, Problem (\mathcal{P}) admits a weak solution $(u, v) \in X$ such that $\Psi(u, v) < \rho_0$.
- (b): If there exist a sequences, $(\rho_n)_n \subset \mathbb{R}^+$ with $\rho_n \rightarrow \infty$ as $n \rightarrow +\infty$ and $(\xi_n)_n, (\eta_n)_n \subset \mathbb{R}$ such that $(\xi_n, \eta_n) \in \text{Int}(B(\rho_n))$ and $\max_{A(\rho_n)} H(\xi, \eta) = H(\xi_n, \eta_n)$, $\forall n > 0$ and if

$$\limsup_{(\xi, \eta) \rightarrow +\infty} \frac{H(\xi, \eta) \int_{\Omega} \theta(x) dx}{D_{\bar{p}(\cdot)}(\xi) \int_{\Omega} h(x) dx + D_{\bar{q}(\cdot)}(\eta) \int_{\Omega} k(x) dx} > \max \left(\frac{1}{\underline{p}}, \frac{1}{\underline{q}} \right).$$

Then, the problem (\mathcal{P}) admits an unbounded sequence of weak solutions in X .

(c): If there exist a sequences, $(\rho_n)_n \subset \mathbb{R}^+$ with $\rho_n \rightarrow 0$ as $n \rightarrow +\infty$ and $(\xi_n)_n, (\eta_n)_n \subset \mathbb{R}$ such that $(\xi_n, \eta_n) \in \text{Int}(B(\rho_n))$ and $\max_{A(\rho_n)} H(\xi, \eta) = H(\xi_n, \eta_n)$, $\forall n > 0$ and if

$$\limsup_{(\xi, \eta) \rightarrow (0,0)} \frac{H(\xi, \eta) \int_{\Omega} \theta(x) dx}{D_{\bar{p}(\cdot)}(\xi) \int_{\Omega} h(x) dx + D_{\bar{q}(\cdot)}(\eta) \int_{\Omega} k(x) dx} > \max\left(\frac{1}{\underline{p}}, \frac{1}{\underline{q}}\right).$$

Then the problem (\mathcal{P}) admits a sequence of non zero weak solutions which strongly converges to (u, v) in X .

Proof

Step 1: Proof of assertion (a). We apply the part a of theorem 1.1 for showing that $\varphi(\rho_0) = 0$ (here φ is the function defined in the theorem 1.1 and $t = 1$ is assumed).

First, we observe that $\forall (u, v) \in \Psi^{-1}(]-\infty, \rho_0[)$, one has

$$\begin{aligned} 0 \leq \varphi(\rho_0) &= \inf_{\Psi^{-1}(]-\infty, \rho_0[)} \frac{\Phi(u, v) - \frac{\inf}{(\Psi^{-1}(]-\infty, \rho_0[))^w} \Phi(u, v)}{\rho_0 - \Psi(u, v)} \\ &\leq \frac{\Phi(u, v) - \frac{\inf}{\Psi^{-1}(]-\infty, \rho_0[))^w} \Phi(u, v)}{\rho_0 - \Psi(u, v)}. \end{aligned} \quad (18)$$

Let $u_0(x) = \xi_0$, $v_0(x) = \eta_0$, $\forall x \in \Omega$, then $\nabla u_n = \nabla v_0 = 0$ and Since $(\xi_0, \eta_0) \in \text{Int}(B(\rho_0))$, one has

$$\begin{aligned} \Psi(u_0, v_0) &= \int_{\Omega} \left[\frac{1}{p_0(x)} h(x) |\xi_0|^{p_0(x)} + \frac{1}{q_0(x)} k(x) |\eta_0|^{q_0(x)} \right] dx \\ &\leq \frac{1}{\underline{p}} \int_{\Omega} h(x) dx D_{\bar{p}(\cdot)}(\xi_0) + \frac{1}{\underline{q}} \int_{\Omega} k(x) dx D_{\bar{q}(\cdot)}(\eta_0) < \rho_0. \end{aligned}$$

Then, for almost every $x \in \bar{\Omega}$ and $\forall (u, v) \in \overline{\Psi^{-1}(]-\infty, \rho_0[)^w}$, one has

$$\alpha F_{\bar{p}(\cdot)}(u(x)) + \beta F_{\bar{q}(\cdot)}(v(x)) \leq \Psi(u, v) \leq \rho_0. \quad (19)$$

The first inequality in (19) is obtained by the proposition 2.1, while the second inequality in (19) follows from the fact that $\overline{\Psi^{-1}(]-\infty, \rho_0[)^w} = \Psi^{-1}(]-\infty, \rho_0])$.

Thu, since $(u(x), v(x)) \in A(\rho_0)$ and $H(u(x), v(x)) \leq H(\xi_0, \eta_0)$, $\forall x \in \bar{\Omega}$.

Hence $-\Phi(u, v) \leq -\Phi(u_0, v_0) \forall (u, v) \in \overline{\Psi^{-1}(]-\infty, \rho_0[)^w}$. because,

$$-\Phi(u_0, v_0) = \sup_{\overline{\Psi^{-1}(]-\infty, \rho_0[)^w}} (-\Phi(u, v)) = - \frac{\inf}{\Psi^{-1}(]-\infty, \rho_0[)_w} \Phi(u, v),$$

and since $\Phi(u_0, v_0) < \rho_0$, it follows that

$$\Phi(u_0, v_0) - \frac{\inf}{\Psi^{-1}(]-\infty, \rho_0[)^w} \Phi(u, v) = \Phi(u_0, v_0) - \Phi(u_0, v_0) = 0.$$

Then, by choosing $(u, v) = (u_0, v_0)$ in the inequality (18), one has $\varphi(\rho_0) = 0$.

The conclusion (a) of the theorem 1.1 assures that there is a critical point of $\Psi + \Phi$.

Step 2: Proof of assertion (b). We apply the part (b) of theorem 1.1. From the part (a). we know that $\varphi(\rho_n) = 0$, $\forall n \in \mathbb{N}$. Then, since $\lim_{n \rightarrow \infty} \rho_n = +\infty$, one has

$$\liminf_{\rho \rightarrow \infty} \varphi(\rho) \leq \liminf_{n \rightarrow \infty} \varphi(\rho_n) = 0 < 1 = t.$$

Now, we fix h satisfying that

$$\limsup_{(\xi, \eta) \rightarrow +\infty} \frac{H(\xi, \eta) \int_{\Omega} \theta(x) dx}{D_{\tilde{p}(\cdot)}(\xi) \int_{\Omega} h(x) dx + D_{\tilde{q}(\cdot)}(\eta) \int_{\Omega} k(x) dx} > h > \max\left(\frac{1}{\underline{p}}, \frac{1}{\underline{q}}\right),$$

and we choose a sequence $(r_n, t_n)_n$ in \mathbb{R}^2 such that $\sqrt{r_n^2 + t_n^2} \geq n$ and

$$H(r_n, t_n) \int_{\Omega} \theta(x) dx > h \left(D_{\tilde{p}(\cdot)}(r_n) \int_{\Omega} h(x) dx + D_{\tilde{q}(\cdot)}(t_n) \int_{\Omega} k(x) dx \right), \quad \forall n \in \mathbb{N}.$$

If we denote by u_n and v_n the constant functions on Ω which take the r_n and t_n values respectively, we have

$$\begin{aligned} \Phi(u_n, v_n) + \Psi(u_n, v_n) &= \Phi(r_n, t_n) + \Psi(r_n, t_n) \\ &= - \int_{\Omega} F(x, r_n, t_n) dx + \int_{\Omega} \frac{1}{p_0(x)} h(x) |r_n|^{p_0(x)} dx \\ &\quad + \int_{\Omega} \frac{1}{q_0(x)} k(x) |t_n|^{q_0(x)} dx \\ &\leq \int_{\Omega} \left(\frac{1}{\underline{p}} - h \right) h(x) D_{\tilde{p}(\cdot)}(r_n) dx \\ &\quad + \int_{\Omega} \left(\frac{1}{\underline{q}} - h \right) h(x) D_{\tilde{q}(\cdot)}(t_n) dx < 0, \quad \forall n \in \mathbb{N}. \end{aligned}$$

Since $(\sqrt{r_n^2 + t_n^2})_n$ is unbounded, at least one of the two sequences $(r_n)_n$ or $(t_n)_n$ admits one divergent subsequence.

Hence $(D_{\tilde{p}(\cdot)}(r_n))_n$ and $(D_{\tilde{q}(\cdot)}(t_n))_n$ admit one divergent subsequence, thus, the functional $\Phi + \Psi$ is unbounded from below.

The conclusion (b) of the theorem 1.1 assures that there is a sequence $(x_n, y_n)_n$ of critical points of $\Phi + \Psi$ such that $\lim_{n \rightarrow +\infty} \Psi(x_n, y_n) = +\infty$.

Moreover, since Ψ is bounded on each bounded subset of X , the sequence $(x_n, y_n)_n$ must be unbounded in X .

Step 3: Proof of assertion (c). We apply the part (c) of theorem 1.1. As before, from the (a). we know that $\varphi(\rho_n) = 0$, $\forall n \in \mathbb{N}$.

Therefore after observing that $\inf_X \Psi = \Psi(u, v) = 0$, since $\lim_{n \rightarrow \infty} \rho_n = 0$, we have

$$\delta = \liminf_{\rho \rightarrow 0^+} \varphi(\rho) \leq \liminf_{n \rightarrow +\infty} \varphi(\rho_n) = 0 < 1 = t.$$

Now, we fix h satisfying

$$\limsup_{(\xi, \eta) \rightarrow (0,0)} \frac{H(\xi, \eta) \int_{\Omega} \theta(x) dx}{D_{\tilde{p}(\cdot)}(\xi) \int_{\Omega} h(x) dx + D_{\tilde{q}(\cdot)}(\eta) \int_{\Omega} k(x) dx} > h > \max\left(\frac{1}{\underline{p}}, \frac{1}{\underline{q}}\right),$$

and choose a sequence $((r_n, t_n))_n$ in $\mathbb{R}^2 \setminus \{(0, 0)\}$ such that $\sqrt{r_n^2 + t_n^2} \leq \frac{1}{n}$ and

$$H(r_n, t_n) \int_{\Omega} \theta(x) dx > h \left(|r_n|^{\underline{p}} \int_{\Omega} h(x) dx + |t_n|^{\underline{q}} \int_{\Omega} k(x) dx \right), \quad \forall n \in \mathbb{N}.$$

Once more if we denote by u_n and v_n the constant functions on Ω which equal r_n and t_n respectively.

Then, from proposition 2.1 the sequence $((u_n, v_n))_n$ strongly converges to (u, v) in X and one has

$$\begin{aligned}\Phi(u_n, v_n) + \Psi(u_n, v_n) &= \Phi(r_n, t_n) + \Psi(r_n, t_n) \\ &\leq \int_{\Omega} |r_n|^{\underline{p}} \left(\frac{1}{\underline{p}} - h \right) h(x) dx \\ &\quad + |r_n|^{\underline{q}} \int_{\Omega} \left(\frac{1}{\underline{q}} - h \right) h(x) dx < 0 \quad \forall n \in \mathbb{N}.\end{aligned}$$

Since $\Phi(u, v) + \Psi(u, v) = 0$ in virtue of the last inequality (u, v) can't be a local minimum of $\Phi + \Psi$.

Then, since (u, v) is the only global minimum of Ψ , the conclusion (c) of the theorem 1.1 assures that there is a sequence of pairwise distinct critical points of $\Phi + \Psi$ such that $\lim_{n \rightarrow \infty} \Psi(x_n, y_n) = 0$ with $x_n, y_n \rightarrow 0$, thus $(x_n, y_n)_n$ must be in norm infinitesimal. \square

The following theorem is a practicable form of theorem 4.1 part (b).

Theorem 4.2. *Let $(a_n)_n$ and $(b_n)_n$ be two sequences in \mathbb{R}^+ satisfying*

$$b_n < a_n \quad \forall n \in \mathbb{N}, \quad \lim_{n \rightarrow +\infty} b_n = +\infty, \quad \lim_{n \rightarrow +\infty} \frac{a_n}{b_n} = +\infty$$

and let

$$\begin{aligned}A_n &= \left\{ (\xi, \eta) \in \mathbb{R}^2 : F_{\vec{p}(\cdot)}(\xi) + F_{\vec{q}(\cdot)}(\eta) \leq a_n \right\} \\ B_n &= \left\{ (\xi, \eta) \in \mathbb{R}^2 : D_{\vec{p}(\cdot)}(\xi) + D_{\vec{q}(\cdot)}(\eta) \leq b_n \right\},\end{aligned}$$

be such that $\sup_{A_n \setminus \text{Int} B_n} H \leq 0$ for all $n \in \mathbb{N}$.

Finally, let us assume that

$$\limsup_{(\xi, \eta) \rightarrow +\infty} \frac{H(\xi, \eta) \int_{\Omega} \theta(x) dx}{D_{\vec{p}(\cdot)}(\xi) \int_{\Omega} h(x) dx + D_{\vec{q}(\cdot)}(\eta) \int_{\Omega} k(x) dx} > \max \left(\frac{1}{\underline{p}}, \frac{1}{\underline{q}} \right).$$

Then, Problem (P) admits an unbounded sequence of weak solutions in X .

Proof Since $b_n < a_n$ it follows that $B_n \subseteq A_n$.

Let

$$\gamma' = \min\{\alpha, \beta\} > 0 \quad \text{and} \quad \delta' = \max \left\{ \frac{\int_{\Omega} h(x) dx}{\underline{p}}, \frac{\int_{\Omega} k(x) dx}{\underline{q}} \right\} > 0.$$

Then $\frac{\delta'}{\gamma'} > 0$ and in virtue of $\lim_{n \rightarrow +\infty} \frac{a_n}{b_n} = +\infty$, then we get $\frac{\delta'}{\gamma'} < \frac{a_n}{b_n}$ for $n \in \mathbb{N}$ large enough.

Let $\rho_n = \gamma' a_n$. Then $\{\rho_n\}_n \subset \mathbb{R}^+$ is a divergent sequence and for n large enough, the following inclusions hold

$$\text{Int} B_n \subseteq B_n \subseteq B(\rho_n) \subseteq A(\rho_n) \subseteq A_n,$$

Then, since H is negative in the set $A_n \setminus \text{Int} B_n$ for all $n \in \mathbb{N}$, we have

$$\max_{\text{Int} B_n} H = \max_{A_n} H,$$

in particular, we have $\max_{\text{Int}B_n} H = \max_{A(\rho_n)} H$ for $n \in \mathbb{N}$ large enough, i.e. there exist at least one sequence $(\xi_n, \eta_n)_n \subset \text{Int}B_n$ such that for n large enough, we have

$$\max_{A(\rho_n)} H(\xi, \eta) = H(\xi_n, \eta_n)$$

Thus, the sequences $(\xi_n)_n$, $(\eta_n)_n$ and $(\rho_n)_n$ have got the properties required in theorem 4.1 part (b).

This completes the proof. \square

The following theorem is a practicable form of theorem 4.1 part (c).

Theorem 4.3. *Let $(a_n)_n$ and $(b_n)_n$ be two sequences in \mathbb{R}^+ satisfying*

$$b_n < a_n \quad \forall n \in \mathbb{N}, \quad \lim_{n \rightarrow +\infty} a_n = 0, \quad \lim_{n \rightarrow +\infty} \frac{a_n}{b_n} = +\infty$$

and let

$$A_n = \left\{ (\xi, \eta) \in \mathbb{R}^2 : F_{\vec{p}(\cdot)}(\xi) + F_{\vec{q}(\cdot)}(\eta) \leq a_n \right\}$$

$$B_n = \left\{ (\xi, \eta) \in \mathbb{R}^2 : D_{\vec{p}(\cdot)}(\xi) + D_{\vec{q}(\cdot)}(\eta) \leq b_n \right\},$$

be such that $\sup_{A_n \setminus \text{Int}B_n} H \leq 0$ for all $n \in \mathbb{N}$.

Finally, let us assume that

$$\limsup_{(\xi, \eta) \rightarrow (0,0)} \frac{H(\xi, \eta) \int_{\Omega} \theta(x) dx}{D_{\vec{p}(\cdot)}(\xi) \int_{\Omega} h(x) dx + D_{\vec{q}(\cdot)}(\eta) \int_{\Omega} k(x) dx} > \max\left(\frac{1}{p}, \frac{1}{q}\right).$$

Then, Problem (\mathcal{P}) admits a sequence of non-zero weak solutions which strongly converges to (u, v) in X .

Proof Likewise, by applying theorem 4.1 part (c), we get the theorem 4.3, whose proof will be omitted. \square

5. Conclusion and perspective

In this paper we have proved an existence result of infinitely many solutions of the elliptic problem (\mathcal{P}) in the anisotropic variable exponent Sobolev spaces by applying a variational principe of Ricceri, our future works will be devoted to the existence of infinitely many solutions elliptic and parabolic problems in the case of flux boundary conditions and the framework of anisotropic Sobolev spaces.

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