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Moroccan J. Pure and Appl. Anal.(MJPAA)

DOI 10.1515/mjpaa-2017-0006
Volume 3(1), 2017, Pages 63–69
ISSN: 2351-8227

Inequalities for the m-th derivative of the (q, k)-Gamma function

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Abstract. By using the generalized Hölder's and Minkowski's integral inequalities, some inequalities for the m-th derivative of the (q, k)-Gamma function are established. Consequently, some previous results are recovered as particular cases of the present results.

2010 Mathematics Subject Classification. 33B15, 26D15.

Key words and phrases. Gamma function, (q, k)-analogue, inequality.

1. Introduction and Preliminaries

The classical Gamma function, which is an extension of the factorial notation to noninteger values, is usually defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dx$$

satisfying the basic properties:

$$\Gamma(x+1) = x\Gamma(x), \quad x \in \mathbb{R}^+$$
$$\Gamma(n+1) = n!, \quad n \in \mathbb{Z}^+ \cup \{0\}.$$

The Jackson's q-integral from 0 to a and from 0 to ∞ are defined as [6]

$$\int_0^a f(t)\,d_qt = (1-q)a\sum_{n=0}^\infty f(aq^n)q^n$$

Received September 15, 2016 - Accepted January 12, 2016.

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$$\int_0^\infty f(t)\,d_qt = (1-q)\sum_{-\infty}^{\infty} f(q^n)q^n$$

provided that the sums converge absolutely. In a generic interval $[a,b]$, the Jackson's q-integral takes the form

$$\int_a^b f(t)\,d_qt = \int_0^b f(t)\,d_qt - \int_0^a f(t)\,d_qt.$$

The q-analogue of the Gamma function is defined for $q \in (0,1)$ and $x > 0$ by [6]

$$\Gamma_q(x) = \int_0^{\frac{1}{1-q}} t^{x-1}E_q^{-qt}\,d_qt$$

satisfying the properties

$$\Gamma_q(x+1) = [x]_q\Gamma_q(x)$$
$$\Gamma_q(1) = 1$$

where $[x]_q = \frac{1-q^x}{1-q}$ and $E_q^t = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}}\frac{t^n}{[n]_q!}$ is the q-analogue of the exponential function.

Also, the k-analogue of the Gamma function is also defined for $k > 0$ and $x \in \mathbb{C}\backslash k\mathbb{Z}^-$ as [4]

$$\Gamma_k(x) = \int_0^\infty t^{x-1}e^{-\frac{t^k}{k}}\,dt$$

satisfying the properties

$$\Gamma_k(x+k) = x\Gamma_k(x)$$
$$\Gamma_k(k) = 1.$$

Then the (q,k)-analogue of the Gamma function, $\Gamma_{q,k}(x)$ is defined for $x > 0$, $q \in (0,1)$ and $k > 0$ as [5]

$$\Gamma_{q,k}(x) = \int_0^{\left(\frac{[k]_q}{1-q^k}\right)^{\frac{1}{k}}} t^{x-1}E_{q,k}^{-\frac{q^kt^k}{[k]_q}}\,d_qt \tag{1}$$

satisfying the properties

$$\Gamma_{q,k}(x+k) = [x]_q\Gamma_{q,k}(x)$$
$$\Gamma_{q,k}(k) = 1$$

where $E_{q,k}^t = \sum_{n=0}^{\infty} q^{\frac{kn(n-1)}{2}}\frac{t^n}{[n]_{q^k}!}$ is the (q,k)-analogue of the exponential function.

The functions $\Gamma(x)$, $\Gamma_q(x)$, $\Gamma_k(x)$ and $\Gamma_{q,k}(x)$ fit into the commutative diagram [5]

$$\begin{tikzcd}
\Gamma_{q,k}(x) \arrow[r, "q\to1"] \arrow[d, "k\to1"'] & \Gamma_k(x) \arrow[d, "k\to1"] \\
\Gamma_q(x) \arrow[r, "q\to1"'] & \Gamma(x)
\end{tikzcd}$$

Then by differentiating (1) m times, we obtain

$$\Gamma_{q,k}^{(m)}(x) = \int_0^{\left(\frac{[k]_q}{1-q^k}\right)^{\frac{1}{k}}} t^{x-1}(\ln t)^m E_{q,k}^{-\frac{q^kt^k}{[k]_q}}\,d_qt, \quad m \in \mathbb{N}_0 \tag{2}$$

where $\Gamma^{(0)}_{q,k}(x) = \Gamma_{q,k}(x)$.

In this paper, the objective is to establish some inequalities for the function $\Gamma^{(m)}_{q,k}(x)$. From the established results, some known results are obtained as particular cases. We present our findings in the following section.

2. Results and Discussion

Let us begin with the following generalizations of the classical Hölder's and Minkowski's integral inequalities.

Lemma 2.1 ([3]). *Let $f_1, f_2, \ldots, f_n$ be functions such that the integrals exist. Then the inequality*

$$\int_a^b \left| \prod_{i=1}^n f_i(t) \right| dt \le \prod_{i=1}^n \left(\int_a^b |f_i(t)|^{\alpha_i} \, dt \right)^{\frac{1}{\alpha_i}} \tag{3}$$

holds for $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $\sum_{i=1}^n \frac{1}{\alpha_i} = 1$.

Lemma 2.2 ([3]). *Let $f_1, f_2, \ldots, f_n$ be functions such that the integrals exist. Then the inequality*

$$\left(\int_a^b \left| \sum_{i=1}^n f_i(t) \right|^u dt \right)^{\frac{1}{u}} \le \sum_{i=1}^n \left(\int_a^b |f_i(t)|^u \, dt \right)^{\frac{1}{u}} \tag{4}$$

holds for $u \ge 1$.

Theorem 2.1. *For $i = 1, 2, \ldots, n$, let $\alpha_i > 1$, $\sum_{i=1}^n \frac{1}{\alpha_i} = 1$ and $m_i \in \mathbb{N}_0$ such that $\sum_{i=1}^n \frac{m_i}{\alpha_i} \in \mathbb{N}_0$. Then the inequality*

$$\Gamma^{(\sum_{i=1}^n \frac{m_i}{\alpha_i})}_{q,k} \left(\sum_{i=1}^n \frac{x_i}{\alpha_i} + \beta \right) \le \prod_{i=1}^n \left(\Gamma^{(m_i)}_{q,k}(x_i + \beta) \right)^{\frac{1}{\alpha_i}} \tag{5}$$

is valid for $x_i > 0$, $\beta \ge 0$ and even m_i.

Proof. By (2) and (3), we obtain

$$\begin{aligned}
\Gamma^{(\sum_{i=1}^n \frac{m_i}{\alpha_i})}_{q,k} \left(\sum_{i=1}^n \frac{x_i}{\alpha_i} + \beta \right) &= \int_0^{\left(\frac{[k]_q}{1-q^k}\right)^{\frac{1}{k}}} t^{\sum_{i=1}^n \frac{x_i}{\alpha_i} + \beta - 1} (\ln t)^{\sum_{i=1}^n \frac{m_i}{\alpha_i}} E_{q,k}^{-\frac{q^k t^k}{[k]_q}} \, d_q t \\
&= \int_0^{\left(\frac{[k]_q}{1-q^k}\right)^{\frac{1}{k}}} t^{\sum_{i=1}^n \frac{x_i + \beta - 1}{\alpha_i}} (\ln t)^{\sum_{i=1}^n \frac{m_i}{\alpha_i}} E_{q,k}^{-\frac{q^k t^k}{[k]_q} \cdot \sum_{i=1}^n \frac{1}{\alpha_i}} \, d_q t \\
&= \int_0^{\left(\frac{[k]_q}{1-q^k}\right)^{\frac{1}{k}}} \prod_{i=1}^n \left(t^{\frac{x_i + \beta - 1}{\alpha_i}} (\ln t)^{\frac{m_i}{\alpha_i}} E_{q,k}^{-\frac{q^k t^k}{[k]_q} \cdot \frac{1}{\alpha_i}} \right) d_q t \\
&\le \prod_{i=1}^n \left[\int_0^{\left(\frac{[k]_q}{1-q^k}\right)^{\frac{1}{k}}} t^{x_i + \beta - 1} (\ln t)^{m_i} E_{q,k}^{-\frac{q^k t^k}{[k]_q}} \, d_q t \right]^{\frac{1}{\alpha_i}}
\end{aligned}$$

$$= \prod_{i=1}^{n} \left(\Gamma_{q,k}^{(m_i)}(x_i+\beta) \right)^{\frac{1}{\alpha_i}}$$

which completes the proof.

Remark 2.1. *By letting $n=2$, $\beta=0$, $m_1=m_2=m$, $\frac{1}{\alpha_1}=a$, $\frac{1}{\alpha_2}=b$, $x_1=x$ and $x_2=y$ in Theorem 2.1, we obtain Theorem 4.4 of* [2].

Remark 2.2. *By letting $n=2$, $\beta=0$, $m_1=m_2=0$, $\frac{1}{\alpha_1}=\lambda$, $\frac{1}{\alpha_2}=1-\lambda$, $x_1=x$ and $x_2=y$ in Theorem 2.1, we obtain Corollary 2.3 of* [7].

Remark 2.3. *By letting $\beta=0$, $q\to 1$ and $k\to 1$ in Theorem 2.1, we obtain a result of Theorem 2.2 of* [1].

Remark 2.4. *Let $n=2$, $\beta=0$, $\alpha_1=\alpha_2=2$, $x_1=x$ and $x_2=y$ in Theorem 2.1. Then by allowing $q\to 1$ and $k\to 1$, we obtain Theorem 2.1 of* [9].

Theorem 2.2. *For $i=1,2,\dots,n$, let $m_i\in\mathbb{N}_0$ such that m_i is even for each i. Then the inequality*

$$\left(\sum_{i=1}^{n} \Gamma_{q,k}^{(m_i)}(x_i) \right)^{\frac{1}{u}} \leq \sum_{i=1}^{n} \left(\Gamma_{q,k}^{(m_i)}(x_i) \right)^{\frac{1}{u}} \tag{6}$$

is valid for $x_i>0$ and $u\geq 1$.

Proof. We utilize the fact that $\sum_{i=1}^{n} a_i^u \leq \left(\sum_{i=1}^{n} a_i\right)^u$, for $a_i\geq 0$, $u\geq 1$ together with the generalized Minkowski's inequality (4). Then by (2) we obtain

$$\begin{aligned}
\left(\sum_{i=1}^{n} \Gamma_{q,k}^{(m_i)}(x_i) \right)^{\frac{1}{u}} &= \left(\sum_{i=1}^{n} \int_0^{\left(\frac{[k]_q}{1-q^k}\right)^{\frac{1}{k}}} t^{x_i-1} (\ln t)^{m_i} E_{q,k}^{-\frac{q^k t^k}{[k]_q}} \, d_q t \right)^{\frac{1}{u}} \\
&= \left(\int_0^{\left(\frac{[k]_q}{1-q^k}\right)^{\frac{1}{k}}} \left[\sum_{i=1}^{n} \left(t^{\frac{x_i-1}{u}} (\ln t)^{\frac{m_i}{u}} E_{q,k}^{-\frac{q^k t^k}{[k]_q}\cdot\frac{1}{u}} \right)^{u} \right] d_q t \right)^{\frac{1}{u}} \\
&\leq \left(\int_0^{\left(\frac{[k]_q}{1-q^k}\right)^{\frac{1}{k}}} \left[\sum_{i=1}^{n} \left(t^{\frac{x_i-1}{u}} (\ln t)^{\frac{m_i}{u}} E_{q,k}^{-\frac{q^k t^k}{[k]_q}\cdot\frac{1}{u}} \right) \right]^{u} d_q t \right)^{\frac{1}{u}} \\
&\leq \sum_{i=1}^{n} \left(\int_0^{\left(\frac{[k]_q}{1-q^k}\right)^{\frac{1}{k}}} \left(t^{\frac{x_i-1}{u}} (\ln t)^{\frac{m_i}{u}} E_{q,k}^{-\frac{q^k t^k}{[k]_q}\cdot\frac{1}{u}} \right)^{u} d_q t \right)^{\frac{1}{u}} \\
&= \sum_{i=1}^{n} \left(\int_0^{\left(\frac{[k]_q}{1-q^k}\right)^{\frac{1}{k}}} t^{x_i-1} (\ln t)^{m_i} E_{q,k}^{-\frac{q^k t^k}{[k]_q}} \, d_q t \right)^{\frac{1}{u}} \\
&= \sum_{i=1}^{n} \left(\Gamma_{q,k}^{(m_i)}(x_i) \right)^{\frac{1}{u}}
\end{aligned}$$

which completes the proof.

Remark 2.5. *In particular, by letting $n = 2$, $m_1 = m$, $m_2 = n$, $x_1 = x$ and $x_2 = y$ in Theorem 2.2, we obtain*

$$\left(\Gamma_{q,k}^{(m)}(x) + \Gamma_{q,k}^{(n)}(y)\right)^{\frac{1}{u}} \leq \left(\Gamma_{q,k}^{(m)}(x)\right)^{\frac{1}{u}} + \left(\Gamma_{q,k}^{(n)}(y)\right)^{\frac{1}{u}}. \tag{7}$$

In order to prove the next results, we need the following lemma which is known in the literature as the weighted AM-GM inequality.

Lemma 2.3 ([8])**.** *For $i = 1, 2, \ldots, n$, let $Q_i \geq 0$ and $\lambda_i \geq 0$ such that $\sum_{i=1}^{n} \lambda_i = 1$. Then the inequality*

$$\sum_{i=1}^{n} \lambda_i Q_i \geq \prod_{i=1}^{n} Q_i^{\lambda_i} \tag{8}$$

holds.

Theorem 2.3. *For $i = 1, 2, \ldots, n$, let $\alpha_i > 1$, $\sum_{i=1}^{n} \frac{1}{\alpha_i} = 1$ and $m_i \in \mathbb{N}$ such that $\sum_{i=1}^{n} \frac{m_i}{\alpha_i} \in \mathbb{N}$. Then the inequality*

$$\exp \Gamma_{q,k}^{(\sum_{i=1}^{n} \frac{m_i}{\alpha_i})} \left(\sum_{i=1}^{n} \frac{x_i}{\alpha_i} \right) \leq \prod_{i=1}^{n} \left(\exp \Gamma_{q,k}^{(m_i)}(x_i) \right)^{\frac{1}{\alpha_i}} \tag{9}$$

is satisfied for $x_i > 0$, where m_i and $\sum_{i=1}^{n} \frac{m_i}{\alpha_i}$ are even.

Proof. Let m_i and $\sum_{i=1}^{n} \frac{m_i}{\alpha_i}$ be even for each i. Then by (2) we obtain

$$\begin{aligned}
&\Gamma_{q,k}^{(\sum_{i=1}^{n} \frac{m_i}{\alpha_i})} \left(\sum_{i=1}^{n} \frac{x_i}{\alpha_i} \right) - \sum_{i=1}^{n} \frac{\Gamma_{q,k}^{(m_i)}(x_i)}{\alpha_i} \\
&= \int_0^{\left(\frac{[k]_q}{1-q^k}\right)^{\frac{1}{k}}} t^{\sum_{i=1}^{n} \frac{x_i}{\alpha_i} - 1} (\ln t)^{\sum_{i=1}^{n} \frac{m_i}{\alpha_i}} E_{q,k}^{-\frac{q^k t^k}{[k]_q}} \, d_q t \\
&\quad - \sum_{i=1}^{n} \frac{1}{\alpha_i} \int_0^{\left(\frac{[k]_q}{1-q^k}\right)^{\frac{1}{k}}} t^{x_i - 1} (\ln t)^{m_i} E_{q,k}^{-\frac{q^k t^k}{[k]_q}} \, d_q t \\
&= \int_0^{\left(\frac{[k]_q}{1-q^k}\right)^{\frac{1}{k}}} \prod_{i=1}^{n} t^{\frac{x_i}{\alpha_i} - 1} (\ln t)^{\frac{m_i}{\alpha_i}} E_{q,k}^{-\frac{q^k t^k}{[k]_q}} \, d_q t \\
&\quad - \int_0^{\left(\frac{[k]_q}{1-q^k}\right)^{\frac{1}{k}}} \sum_{i=1}^{n} \frac{1}{\alpha_i} t^{x_i - 1} (\ln t)^{m_i} E_{q,k}^{-\frac{q^k t^k}{[k]_q}} \, d_q t \\
&= \int_0^{\left(\frac{[k]_q}{1-q^k}\right)^{\frac{1}{k}}} \left[\prod_{i=1}^{n} t^{\frac{x_i}{\alpha_i}} (\ln t)^{\frac{m_i}{\alpha_i}} - \sum_{i=1}^{n} \frac{1}{\alpha_i} t^{x_i} (\ln t)^{m_i} \right] \frac{1}{t} E_{q,k}^{-\frac{q^k t^k}{[k]_q}} \, d_q t \\
&\leq 0
\end{aligned}$$

which results from (8). Thus,

$$\Gamma_{q,k}^{(\sum_{i=1}^{n} \frac{m_i}{\alpha_i})} \left(\sum_{i=1}^{n} \frac{x_i}{\alpha_i} \right) \leq \sum_{i=1}^{n} \frac{\Gamma_{q,k}^{(m_i)}(x_i)}{\alpha_i}. \tag{10}$$

Then by exponentiating (10), we obtain the result (9) concluding the proof.

Remark 2.6. *Upon letting $q \to 1$ and $k \to 1$ in Theorem 2.3, we obtain the result of Theorem 2.3 of* [1].

Remark 2.7. *Let $n = 2$, $\alpha_1 = \alpha_2 = 2$, $x_1 = x_2 = x$, $m_1 = m+s$ and $m_2 = m-s$ in Theorem 2.3, where m and s are even such that $m \geq s$. Then we recover the result of Theorem 3.1 of* [10].

Acknowledgements

The authors authors would like to thank the anonymous referees for their useful comments and suggestions.

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