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m-Convexity and Functional Equations

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Abstract. In this research we aim to explore some properties of m-convex functions from the point of view of functional equations or better, functional inequalities. So far studies of m-convexity have been devoted mainly to establish properties, inequalities and examples on the topic, but not to look at the problem from the perspective of functional inequalities.

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1. Introduction

The concept of m-convex function was introduced in [15] and since then many properties, especially inequalities, have been obtained for them [6, 8, 11, 12], and many more. We present it here since is one of the key definitions, together with functional equations, used along the whole paper.

Definition 1. A function $f : [0, b] \to \mathbf{R}$ $(b > 0)$ is said to be m-convex in the interval $[0, b]$, $0 \leq m \leq 1$, if for any x, $y \in [0, b]$ and $t \in [0, 1]$ we have

$$f(tx + m(1 - t)y) \leq tf(x) + m(1 - t)f(y). \tag{1.1}$$

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Remark 2. *In the foregoing definition it may happen that $f : [0, +\infty) \to \mathbf{R}$ and everything runs in the same fashion.*

The set

$$\{f : [0, b] \to \mathbb{R} \mid f \text{ is } m\text{-convex in } [0, b] \text{ and } f(0) \le 0\}$$

is usually denoted by $K_m(b)$ and in case $f : [0, +\infty) \to \mathbb{R}$, the corresponding set is named $K_m(+\infty)$.

We may say that the beginning of the theory of functional equations is connected with the work of an excellent specialist in this field, the Hungarian mathematician J. Aczél. In his numerous papers he treats whole classes of functional equations, gives general methods for solving them and criteria on the existence and uniqueness of solutions. He also indicates new applications of this important topic [1, 2, 3, 4, 5].

Definition 3. ([9]) A functional equation is an equality, say, $T_1 = T_2$ between two terms T_1 and T_2 which contains at least one unknown function and a finite number of independent variables. This equality is to be satisfied identically with respect to all occurring variables in a certain set (of any sort).

The solution of a functional equation may depend on the set in which the equation is postulated. One should also precisely state in what function class the solution is sought. The number and behavior of solutions depend on this class. It is one of the important differences between differential and functional equations [9, 10].

In this paper we establish some properties of m-convex functions from de point of view of both, funcional equations and functional inequalities.

2. Main Results

Here we set and prove our main results, basically we shall deal with the concept of m-convex function $(0 < m < 1)$ and the expression $tx + m(1-t)y$, where $x, y \in [0, b]$ or $x, y \in (0, +\infty)$ and, as usual, $t \in [0, 1]$, even in some cases t will be chosen arbitrary but in $(0, 1)$.

Proposition 4. *Let $f : [0, +\infty) \to \mathbb{R}$ given as $f(x) = x^p$, with $p \in \mathbb{R}$ and $m \in [0, 1]$ fixed. The function f is m-convex if and only if $p \ge 1$.*

Proof. Let $x, y \in [0, +\infty)$, the function f is in $K_m(+\infty)$ if and only if

$$f(tx + m(1-t))y) \le tf(x) + m(1-t)f(y)$$

or equivalently

$$tx^p + m(1-t)y^p - (tx + m(1-t)y)^p \ge 0. \tag{2.1}$$

We may assume $y > 0$, in such way that (2.1) can be rewritten as

$$tz^p + m(1-t) - (tz + m(1-t))^p \ge 0, \text{ with } z = \frac{x}{y}.$$

Now define the function

$$F(z) = tz^p + m(1-t) - (tz + m(1-t))^p,$$

it is easy to see that

$$F'(z) = pt[z^{p-1} - (tz + m(1-t))^{p-1}],$$

hence the only critical point is $z = m$. Further,

$$F''(z) = p(p-1)t[z^{p-2} - t(tz + m(1-t))^{p-2}]$$

and consequently, $F''(m) = p(p-1)tm^{p-2}(1-t) \geq 0$ for any value of $t \in [0,1]$ if and only if $p \geq 1$. Therefore, for $p \geq 1$, F has an absolute minimum at $z = m$; indeed $F(m) = (1-t)m(1-m^{p-1}) \geq 0$, so F is m-convex under this condition. $\square$

An immediate consequence of the convexity of $f(x) = x^p$, $p \geq 1$ is

Proposition 5. *For $p \geq 1$, the following inequality holds*

$$(tx + (1-t)my)^p \leq tx^p + (1-t)(my)^p$$

for every $x, y \in [0, +\infty)$ and $t \in [0,1]$.

Theorem 6. *Let $m \in (0,1)$ and $F : [0,b] \to \mathbb{R}$ be a function given by $F(x) = tx + mb(1-t)$, $x \in [0,b]$, and $t \in (0,1)$ arbitrary but fixed. Then, the sequence $\{F^n(x)\}_{n\geq 1}$ defined recursively by $F^1(x) = F(x)$, $F^{n+1}(x) = F(F^n(x))$ has limit as $n \to +\infty$, actually*

$$\lim_{n\to+\infty} F^n(x) = mb \tag{2.2}$$

Proof. We shall show by induction on n the following formula,

$$F^n(x) = t^n x + mb(1-t)\sum_{k=0}^{n-1} t^k. \tag{2.3}$$

For $n = 1$ and $n = 2$, (2.3) follows at once. Let us assume it true for n and show it for $n+1$. Indeed,

$$F^{n+1}(x) = F(F^n(x)) = tF^n(x) + m(1-t)b = t[t^n x + mb(1-t)\sum_{k=0}^{n-1} t^k] + m(1-t)b,$$

(the last equality because of the induction hypothesis) hence,

$$F^{n+1}(x) = t^{n+1}x + mb(1-t)\left[\sum_{k=0}^{n-1} t^{k+1} + 1\right] = t^{n+1}x + mb(1-t)\sum_{k=0}^{n} t^k.$$

So (2.3) is true, in other words,

$$F^{n+1}(x) = t^{n+1}x + mb[1 - t^{n+1}]$$

consequently, and taking into account that $0 < t < 1$, conclusion follows. $\square$

Remark 7. *If $t = 1$ then $F(x) = x$ for any $x \in [0,b]$ and, of course, $F^n(x) = x$, while $t = 0$ makes the sequence a constant, in fact, $F^n(x) = mb$, $x \in [0,b]$.*

Theorem 8. *Let m and M be two real numbers with $0 < m < 1$ and $1 < M$, $K, L : [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ two functions given, respectively, as*

$$K(x,y) = tx + m(1-t)y, \text{ and, } L(x,y) = tx + M(1-t)y,$$

where $t \in (0,1)$ is arbitrary but fixed.

Define now two new functions $T, Q : [0,+\infty) \times [0,+\infty) \to [0,+\infty) \times [0,+\infty)$ given by

$$T(x,y) = (K(x,y), K(y,x)), \text{ and, } Q(x,y) = (K(x,y), L(x,y))$$

respectively. Then the sets

$$\Delta_T = \{(0,0)\}$$

and

$$\Delta_Q = \begin{cases} \{(0,0)\} & \text{if } M \in (1, 1+m] \text{ and } t \in (0, t_1) \cup (t_1, 1) \\ \{(cy, y) : y \geq 0\} & \text{if } M \in (1, 1+m] \text{ and } t = t_1 \\ \{(0,0)\} & \text{if } M \in (1+m, +\infty) \text{ and } t \in (0, t_0) \\ \{(0,0)\} & \text{if } M \in (1+m, +\infty) \text{ and } t = t_0 \\ \{(0,0)\} & \text{if } M \in (1+m, +\infty) \text{ and } t \in (t_0, t_1) \cup (t_1, 1) \\ \{(cy, y) : y \geq 0\} & \text{if } M \in (1+m, +\infty) \text{ and } t = t_1 \end{cases}$$

are the set of fixed points of T and Q respectively, where $t_0 = \frac{M-1-m}{M-m}$, $t_1 = \frac{M-1}{M-m}$, and $c = 1 - (M-m)(1-t)$.

Proof. The point (x, y) is a fixed point of T if and only if $T(x, y) = (x, y)$ which is equivalent to the following pair of equations

$$tx + m(1-t)y = x, \text{ and, } ty + m(1-t)x = y,$$

now by subtracting the second equation from the first

$$t(x-y) + m(1-t)(y-x) = x - y.$$

If $x \neq y$ we may divide the whole expression by $x - y$ and the foregoing equation becomes $t - m(1-t) = 1$, hence $m = -1$ which is impossible. Therefore the only solution is $x = y$. Then, we have the inclusion $\Delta_T \subseteq D := \{(x, x) : x \geq 0\}$ but the only point of D that is fixed by T is $(0, 0)$ which proves

$$\Delta_T = \{(0,0)\}.$$

The set of fixed points of Q is determined by considering the equations

$$tx + m(1-t)y = x, \text{ and, } tx + M(1-t)y = y.$$

Now we proceed as in the foregoing case by now subtracting the first equation from the second one and arrive to

$$(M-m)(1-t)y = y - x,$$

and from here, $x = (1 - (M-m)(1-t))y$ which indicates that $\Delta_Q \subseteq E := \{(cy, y) : y \geq 0\}$. From this point on, we will consider different conditions under which a point of set E belongs to Δ_Q. Observe that $(0, 0) \in E$ always belongs to Δ_Q. Otherwise, $(cy, y) \in \Delta_Q$ iff simultaneously the equations (after replacing x by cy and dividing by $y > 0$ the original equations)

$$tc + m(1-t) = c, \text{ and, } tc + M(1-t) = 1$$

are satisfied. Observe they are equivalent to each other. Let us proceed by considering the six cases indicated above and the equation $tc + M(1-t) = 1$.

1st case: $M \in (1, 1+m]$ and $t \in (0, t_1) \cup (t_1, 1)$. In this case, $c > 0$ but the given equation can not be solved for t. Then, $\Delta_Q = \{(0,0)\}$.

2nd case: $M \in (1, 1+m]$ and $t = t_1$. Now, $c > 0$ and the equation has solution for $t = t_1$. Therefore, $\Delta_Q = E$

3rd case: $M \in (1+m, +\infty)$ and $t \in (0, t_0)$. $\Delta_Q = \{(0,0)\}$, because $c < 0$ and the equation is unsolvable.

4th case: $M \in (1+m, +\infty)$ and $t = t_0$. Again, $\Delta_Q = \{(0,0)\}$, because $c = 0$ and the equation is unsolvable.

5th case: $M \in (1+m, +\infty)$ and $t \in (t_0, t_1) \cup (t_1, 1)$. One more time, $\Delta_Q = \{(0,0)\}$, because the equation is unsolvable even though $c > 0$.

6th case: $M \in (1+m, +\infty)$ and $t = t_1$. As in the second case, $\Delta_Q = E$. □

Observe that we can condense the part related to function Q as shows next corollary.

Corollary 9. *For the function Q defined in Theorem 8 follows:*

$$\Delta_Q = \begin{cases} \{(my, y) : y \geq 0\} & \text{if } t = t_1 \\ \{(0,0)\} & \text{if } t \in (0,1) - \{t_1\} \end{cases}$$

A couple of things may be said about this result; in the case of function T, its set of fixed points, $\Delta_T = \{(0,0)\}$, differs considerably from the case when $m = 1$ (classical convexity) which consists of half of the diagonal in the plane, namely, the origin and all those points with both components equal and positive. Similarly, the set Δ_Q is made of a half line in the first quadrant whose equation is $y = \frac{1}{m}x$ (corresponding to $t = t_1$) and just the origin for the remaining cases.

Remark 10. *If we define $R : [0, +\infty) \times [0, +\infty) \to [0, +\infty) \times [0, +\infty)$ by $R(x, y) = (L(x, y), L(y, x))$ is easy to verify that the only fixed point of this application is the origin*

Remark 11. (1) *If $t = 0$ then, $K(x, y) = my$ and $T(x, y) = (my, mx)$. So, $\Delta_T = \{(0,0)\}$.*

(2) *If $t = 1$ then, $K(x, y) = x$ and $T(x, y) = (x, y)$. So, $\Delta_T = D$.*

Remark 12. (1) *If $t = 0$ then, $K(x, y) = my$, $L(x, y) = My$, and $Q(x, y) = (my, My)$. So, $\Delta_Q = \{(0,0)\}$.*

(2) *If $t = 1$ then, $K(x, y) = x$, $L(x, y) = x$, and $Q(x, y) = (x, x)$. So, $\Delta_Q = D$.*

Next step is to characterize or determine which functions $f : [0, +\infty) \to \mathbb{R}$ satisfy, for given real numbers $0 < m < 1 < M$ and $t \in [0, 1]$, the following two inequalities,

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y), \; x, y \geq 0 \tag{2.4}$$

and

$$f(tx + M(1-t)y) \leq tf(x) + M(1-t)f(y), \; x, y \geq 0. \tag{2.5}$$

Actually the following result shows up,

Theorem 13. *If $f : [0, +\infty) \to \mathbb{R}$ satisfies (2.4) and (2.5) simultaneously then $f(x) = f(1)x$, for any $x \geq 0$. In other words, the only funcional solutions of set of inequalities (2.4) and (2.5) are the linear functions.*

Proof. First of all, we notice that if f satisfies (2.4) and (2.5), then $f(0) = 0$; now we set $u = tx + m(1-t)y$, $v \geq 0$, $s \in [0, 1]$ and t as in the hypothesis, hence from (2.5),

$$f(su + M(1-s)v) \leq sf(u) + M(1-s)f(v),$$

or better

$$\begin{aligned} f(stx + sm(1-t)y + M(1-s))v) &\leq sf(tx + m(1-t)y) + M(1-s)f(v) \\ &\leq stf(x) + sm(1-t)f(y) + M(1-s)f(v) \\ &= stf(x) + [sm(1-t) + M(1-s)]f(y), \end{aligned}$$

the last equality holds once we set $v = y$. Therefore,

$$f(stx + sm(1-t)y + M(1-s))y) \leq stf(x) + [sm(1-t) + M(1-s)]f(y),$$

from this inequality, if $x = 0$

$$f((sm(1-t)y + M(1-s)y)) \leq [sm(1-t) + M(1-s)]f(y) \tag{2.6}$$

and for $y = 0$,

$$f(stx) \leq stf(x), \text{ for any } x \geq 0 \text{ and } s, t \in [0, 1].$$

Of course if, $s = 1$ $(t = 1)$ then $f(tx) \leq tf(x)$ $(f(sx) \leq sf(x)$ respectively). Choose now $s, t \in [0, 1]$ in such a way that $b = ms(1-t) + M(1-s) > 1$, $a = t$ and $\frac{\log b}{\log a}$ is irrational. Use now (2.6) and the fact that $f(tx) \leq tf(x)$ to show that

$$f(a^n b^m x) = a^n b^m f(x), \; n, m \in \mathbb{N}, \; x \geq 0.$$

Because of the hypothesis assumed on a and b the set

$$A = \{a^n b^m : n, m \in \mathbb{N}\}$$

is dense in $(0, +\infty)$ [13, 14]; thus for any $r > 0$ there exists a sequence $\{n_k, m_k\}_{k \in \mathbb{N}}$ such that $r = \lim_{k \to +\infty} a^{n_k} b^{m_k}$. But then

$$\lim_{k \to +\infty} f(a^{n_k} b^{m_k} x) \leq \lim_{k \to +\infty} a^{n_k} b^{m_k} f(x),$$

consequently

$$f(rx) \leq rf(x), \; \text{for any } x \geq 0 \text{ and } r > 0 \text{ arbitrary.}$$

By considering, in this last inequality, $\frac{x}{r}$ instead x and $\frac{1}{r}$ instead r the following two inequalities show up,

$$\frac{1}{r} f(x) \leq f\left(\frac{x}{r}\right), \text{ and } f\left(\frac{x}{r}\right) \leq f\left(\frac{x}{r}\right),$$

hence, $\frac{1}{r} f(x) = f(\frac{x}{r})$, or $f(rx) = rf(x)$. The proof concludes by choosing $x = 1$. □

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