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HERMITE-HADAMARD TYPE INEQUALITIES FOR (α, m)- CONVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

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Abstract. In this paper, we establish Hermite-Hadamard type inequalities for (α, m)-convex functions via fractional integrals.

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1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(x)+f(y)}{2} \tag{1}$$

is known in the literature as Hadamard inequality for convex function. Both inequalities hold in the reverse direction if f is concave.

In ([17]), the notion of m-convexity was introduced by G.Toader as the following:

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Definition 1.1. *The function $f : [0,b] \to \mathbb{R}$, $b > 0$, is said to be m-convex, where $m \in [0,1]$, if we have:*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

for every $x, y \in [0,b]$ and $t \in [0,1]$.

Denote by $K_m(b)$ the class of all m-convex functions on $[0,b]$ for which $f(0) \leq 0$.

The class of (α, m)-convex functions was first introduced in ([8]) and it is defined as follows.

Definition 1.2. *The function $f : [0,b] \to \mathbb{R}$, $b > 0$, is said to be (α, m)-convex, where $(\alpha, m) \in [0,1]^2$, if we have:*

$$f(tx + m(1-t)y) \leq t^{\alpha} f(x) + m(1-t^{\alpha})f(y)$$

for all $x, y \in [0,b]$ and $t \in [0,1]$.

Denote by $K_m^{\alpha}(b)$ the class of all (α, m)-convex functions on $[0,b]$ for which $f(0) \leq 0$.

Note that for $(\alpha, m) \in \{(0,0), (\alpha, 0), (1,0), (1,m), (1,1), (\alpha, 1)\}$ one obtains the following classes of functions: increasing, α-starshaped, starshaped, m-convex, convex and α-convex functions.

For recent results and generalizations concerning m-convex and (α, m)-convex functions, see ([1],[6],[9]-[11],[16]).

We give in the following some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further this paper.

Definition 1.3. *Let $f \in L[a,b]$. The Riemann-Liouville integrals $J_{a^+}^k f(x)$ and $J_{b^-}^k f(x)$ of order $k > 0$ with $k \geq 0$ are defined by*

$$J_{a^+}^k f(x) = \frac{1}{\Gamma(k)} \int_a^x (x-t)^{k-1} f(t)dt, \ x > a$$

and

$$J_{b^-}^k f(x) = \frac{1}{\Gamma(k)} \int_x^b (t-x)^{k-1} f(t)dt, \ x < b$$

respectively. Here $\Gamma(k) = \int_0^{\infty} e^{-t} u^{k-1} du$ is the Gamma function and $J_{a^+}^0 f(x) = J_{b^-}^0 f(x) = f(x)$.

The incomplete beta function is defined by

$$B(x; \alpha, \beta) = \int_0^x t^{\alpha-1}(1-t)^{\beta-1} dt.$$

For $x = 1$, the incomplete beta function coincides with the complete beta function.

For some recent results connected with fractional integral inequalities, see([2]-[5],[13]-[15]).

Sarıkaya ([12]) generalized Kırmacı's ([7]) results for fractional integral. These results are given below.

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $|f'|$ is convex on $[a,b]$ then the following inequality for fractional integrals holds:*

$$\begin{aligned} &\left| \frac{2^{k-1}\Gamma(k+1)}{(b-a)^k} \left[J_{\left(\frac{a+b}{2}\right)^+}^k f(b) + J_{\left(\frac{a+b}{2}\right)^-}^k f(a) \right] - f\left(\frac{a+b}{2}\right) \right| \qquad (2) \\ \leq\ & \frac{b-a}{4(k+1)} [|f'(a)| + |f'(b)|] \end{aligned}$$

Theorem 1.2. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $|f'|^q$ is convex on $[a, b]$ for $q > 1$, then the following inequality for fractional integrals holds:*

$$\left|\frac{2^{k-1}\Gamma(k+1)}{(b-a)^k}\left[J^k_{\left(\frac{a+b}{2}\right)^+}f(b) + J^k_{\left(\frac{a+b}{2}\right)^-}f(a)\right] - f\left(\frac{a+b}{2}\right)\right| \tag{3}$$

$$\leq \frac{b-a}{4}\left(\frac{1}{kp+1}\right)^{\frac{1}{p}}\left[\left(\frac{|f'(a)|+3|f'(b)|}{4}\right)^{\frac{1}{q}} + \left(\frac{3|f'(a)|+|f'(b)|}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{b-a}{4}\left(\frac{4}{kp+1}\right)^{\frac{1}{p}}\left[|f'(a)|+|f'(b)|\right]$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 1.3. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $|f'|^q$ is convex on $[a, b]$ for $q \geq 1$, then the following inequality for fractional integrals holds:*

$$\left|\frac{2^{k-1}\Gamma(k+1)}{(b-a)^k}\left[J^k_{\left(\frac{a+b}{2}\right)^+}f(b) + J^k_{\left(\frac{a+b}{2}\right)^-}f(a)\right] - f\left(\frac{a+b}{2}\right)\right| \tag{4}$$

$$\leq \frac{b-a}{4(k+1)}\left(\frac{1}{2(k+2)}\right)^{\frac{1}{q}}\Big[\big((k+1)|f'(a)|^q + (k+3)|f'(b)|^q\big)^{\frac{1}{q}}$$

$$+\big((k+3)|f'(a)|^q + (k+1)|f'(b)|^q\big)^{\frac{1}{q}}\Big].$$

2. Main Results

In order to prove our main theorems we need the following lemma see([12]).

Lemma 2.1. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $f' \in L[a, b]$, then the following inequality for fractional integrals holds:*

$$\frac{2^{k-1}\Gamma(k+1)}{(b-a)^k}\left[J^k_{\left(\frac{a+b}{2}\right)^+}f(b) + J^k_{\left(\frac{a+b}{2}\right)^-}f(a)\right] - f\left(\frac{a+b}{2}\right) \tag{5}$$

$$= \frac{b-a}{4}\left\{\int_0^1 t^k f'\left(\frac{t}{2}a + \frac{2-t}{2}b\right)dt - \int_0^1 t^k f'\left(\frac{2-t}{2}a + \frac{t}{2}b\right)dt\right\}$$

with $k > 0$.

We will start with the following theorem containing Hermite-Hadamard type inequality for fractional integral.

Theorem 2.1. *Let $f : I \to \mathbb{R}$ where $I \subset [0, \infty)$ be a differentiable function on I° such that $f' \in L[a, b]$ where, $a, b \in I$, $a < b$. If $|f'|$ is (α, m)-convex on $[a, b]$, for $(\alpha, m) \in [0, 1] \times (0, 1]$, then the following inequality for fractional integrals holds:*

$$\left|\frac{2^{k-1}\Gamma(k+1)}{(b-a)^k}\left[J^k_{\left(\frac{a+b}{2}\right)^+}f(b) + J^k_{\left(\frac{a+b}{2}\right)^-}f(a)\right] - f\left(\frac{a+b}{2}\right)\right| \tag{6}$$

$$\leq \frac{b-a}{4}\left\{\left[|f'(a)| - m\left|f'\left(\frac{b}{m}\right)\right|\right]\right.$$

$$\times\left[\frac{1}{2^{\alpha}(\alpha+k+1)}+2^{k+1}B\Big(\frac{1}{2};k+1,\alpha+1\Big)+\frac{2m}{k+1}\left|f'\Big(\frac{b}{m}\Big)\right|\right]\Bigg\}$$

where $B(.;.,.)$ is the incompleted beta function.

Proof. From Lemma 2.1 since $|f'|$ is (α,m)-convex, we have

$$\begin{aligned}
&\left|\frac{2^{k-1}\Gamma(k+1)}{(b-a)^k}\left[J^k_{\left(\frac{a+b}{2}\right)^+}f(b)+J^k_{\left(\frac{a+b}{2}\right)^-}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|\\
\leq\ & \frac{b-a}{4}\left\{\int_0^1 t^k\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|dt+\int_0^1 t^k\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|dt\right\}\\
\leq\ & \frac{b-a}{4}\left\{\frac{|f'(a)|}{2^{\alpha}}\int_0^1 t^{k+\alpha}dt+m\left|f'\left(\frac{b}{m}\right)\right|\int_0^1 t^k\left(1-\left(\frac{t}{2}\right)^{\alpha}\right)dt\right.\\
&\left.+|f'(a)|\int_0^1 t^k\left(1-\frac{t}{2}\right)^{\alpha}dt+m\left|f'\left(\frac{b}{m}\right)\right|\int_0^1 t^k\left(1-\left(1-\frac{t}{2}\right)^{\alpha}\right)dt\right\}\\
=\ & \frac{b-a}{4}\left\{\left[|f'(a)|-m\left|f'\left(\frac{b}{m}\right)\right|\right]\right.\\
&\left.\times\left[\frac{1}{2^{\alpha}(\alpha+k+1)}+2^{k+1}B\left(\frac{1}{2};k+1,\alpha+1\right)+\frac{2m}{k+1}\left|f'\left(\frac{b}{m}\right)\right|\right]\right\}
\end{aligned}$$

where we used the fact that

$$\int_0^1 t^k\left(1-\frac{t}{2}\right)^{\alpha}dt=2^{k+1}B\left(\frac{1}{2};k+1,\alpha+1\right)$$

which completes the proof. □

Corollary 2.1. *In Theorem 2.1, if we choose $\alpha=1$ then the inequality (6) becomes the following inequality*

$$\begin{aligned}
&\left|\frac{2^{k-1}\Gamma(k+1)}{(b-a)^k}\left[J^k_{\left(\frac{a+b}{2}\right)^+}f(b)+J^k_{\left(\frac{a+b}{2}\right)^-}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|\\
\leq\ & \frac{b-a}{4(k+1)}\left\{|f'(a)|+m\left|f'\left(\frac{b}{m}\right)\right|\right\}.
\end{aligned}$$

Remark 2.1. *In Theorem 2.1, if we choose $m=\alpha=1$ then the inequality (6) reduces to the inequality (2) of Theorem 1.1.*

Theorem 2.2. *Let $f:I\to\mathbb{R}$ where $I\subset[0,\infty)$ be a differentiable function on I° such that $f'\in L[a,b]$ where, $a,b\in I$, $a<b$. If $|f'|^q$ is (α,m)-convex on $[a,b]$, for $(\alpha,m)\in[0,1]\times(0,1]$, $q>1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left|\frac{2^{k-1}\Gamma(k+1)}{(b-a)^k}\left[J^k_{\left(\frac{a+b}{2}\right)^+}f(b)+J^k_{\left(\frac{a+b}{2}\right)^-}f(a)\right]-f\left(\frac{a+b}{2}\right)\right| \qquad (7)\\
\leq\ & \frac{b-a}{4}\left(\frac{1}{kp+1}\right)^{\frac{1}{p}}\left\{\left(\frac{|f'(a)|^q}{2^{\alpha}(\alpha+1)}+m\left|f'\left(\frac{b}{m}\right)\right|^q\left[1-\frac{1}{2^{\alpha}(\alpha+1)}\right]\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\left[\frac{2}{\alpha+1}-\frac{1}{2^{\alpha}(\alpha+1)}\right]\left[|f'(a)|^q-m\left|f'\left(\frac{b}{m}\right)\right|^q\right]+m\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2.1 using the well known Hölder inequality and $|f'|^q$ is (α,m)-convex we have

$$\begin{aligned}
&\left|\frac{2^{k-1}\Gamma(k+1)}{(b-a)^k}\left[J^k_{\left(\frac{a+b}{2}\right)^+}f(b)+J^k_{\left(\frac{a+b}{2}\right)^-}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|\\
\leq\ & \frac{b-a}{4}\left(\int_0^1 t^{kp}dt\right)^{\frac{1}{p}}\left\{\left(\int_0^1\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}+\left(\int_0^1\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right\}\\
\leq\ & \frac{b-a}{4}\left(\frac{1}{kp+1}\right)^{\frac{1}{p}}\left\{\left(\frac{|f'(a)|^q}{2^\alpha}\int_0^1 t^\alpha dt+m\left|f'\left(\frac{b}{m}\right)\right|^q\int_0^1\left(1-\left(\frac{t}{2}\right)^\alpha\right)dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(|f'(a)|^q\int_0^1\left(1-\frac{t}{2}\right)^\alpha dt+m\left|f'\left(\frac{b}{m}\right)\right|^q\int_0^1\left(1-\left(1-\frac{t}{2}\right)^\alpha\right)dt\right)^{\frac{1}{q}}\right\}\\
=\ & \frac{b-a}{4}\left(\frac{1}{kp+1}\right)^{\frac{1}{p}}\left\{\left(\frac{|f'(a)|^q}{2^\alpha(\alpha+1)}+m\left|f'\left(\frac{b}{m}\right)\right|^q\left[1-\frac{1}{2^\alpha(\alpha+1)}\right]\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\left[\frac{2}{\alpha+1}-\frac{1}{2^\alpha(\alpha+1)}\right]\left[|f'(a)|^q-m\left|f'\left(\frac{b}{m}\right)\right|^q\right]+m\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

where

$$\int_0^1\left(1-\frac{t}{2}\right)^\alpha dt=\frac{2}{\alpha+1}-\frac{1}{2^\alpha(\alpha+1)}$$

□

Corollary 2.2. *In Theorem 2.2, if we choose $\alpha=1$ then the inequality (7) becomes the following inequality,*

$$\begin{aligned}
&\left|\frac{2^{k-1}\Gamma(k+1)}{(b-a)^k}\left[J^k_{\left(\frac{a+b}{2}\right)^+}f(b)+J^k_{\left(\frac{a+b}{2}\right)^-}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|\\
\leq\ & \frac{b-a}{4}\left(\frac{1}{kp+1}\right)^{\frac{1}{p}}\left\{\left(\frac{|f'(a)|^q}{4}+\frac{3}{4}m\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}+\left(\frac{3|f'(a)|^q}{4}+\frac{1}{4}m\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Remark 2.2. *In Theorem 2.2, if we choose $m=\alpha=1$ then the inequality (7) reduces to the inequality (3) of Theorem 1.2.*

Theorem 2.3. *Let $f:I\to\mathbb{R}$ where $I\subset[0,\infty)$ be a differentiable function on I° such that $f'\in L[a,b]$ where, $a,b\in I$, $a<b$. If $|f'|^q$ is (α,m)-convex on $[a,b]$, for $(\alpha,m)\in[0,1]\times(0,1]$, $q\geq 1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left|\frac{2^{k-1}\Gamma(k+1)}{(b-a)^k}\left[J^k_{\left(\frac{a+b}{2}\right)^+}f(b)+J^k_{\left(\frac{a+b}{2}\right)^-}f(a)\right]-f\left(\frac{a+b}{2}\right)\right| \quad (8)\\
\leq\ & \frac{b-a}{4}\left(\frac{1}{k+1}\right)^{\frac{1}{p}}\left\{\left(\left[|f'(a)|^q-m\left|f'\left(\frac{b}{m}\right)\right|^q\right]\frac{1}{2^\alpha(\alpha+k+1)}+\frac{m}{k+1}\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\left(2^{k+1}B\left(\frac{1}{2};k+1,\alpha+1\right)\right)\left[|f'(a)|^q-m\left|f'\left(\frac{b}{m}\right)\right|^q\right]+\frac{m}{k+1}\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

where $B(.;.\,,.)$ is the incompleted beta function.

Proof. From Lemma 2.1 using the well known power mean inequality and $|f'|^q$ is (α, m)-convex we have

$$\begin{aligned}
&\left|\frac{2^{k-1}\Gamma(k+1)}{(b-a)^k}\left[J^k_{\left(\frac{a+b}{2}\right)^+}f(b)+J^k_{\left(\frac{a+b}{2}\right)^-}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|\\
\le\;& \frac{b-a}{4}\left(\int_0^1 t^k dt\right)^{\frac{1}{p}}\left\{\left(\int_0^1 t^k\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}+\left(\int_0^1 t^k\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right\}\\
\le\;& \frac{b-a}{4}\left(\frac{1}{k+1}\right)^{\frac{1}{p}}\left\{\left(\frac{|f'(a)|^q}{2^\alpha}\int_0^1 t^{k+\alpha}dt+m\left|f'\left(\frac{b}{m}\right)\right|^q\left[\int_0^1 t^k\left(1-\left(\frac{t}{2}\right)^\alpha\right)dt\right]\right)^{\frac{1}{q}}\right.\\
&\left.+\left(|f'(a)|^q\int_0^1 t^k\left(1-\frac{t}{2}\right)^\alpha dt+m\left|f'\left(\frac{b}{m}\right)\right|^q\int_0^1 t^k\left(1-\left(1-\frac{t}{2}\right)^\alpha\right)dt\right)^{\frac{1}{q}}\right\}\\
=\;& \frac{b-a}{4}\left(\frac{1}{k+1}\right)^{\frac{1}{p}}\left\{\left(\left[|f'(a)|^q-m\left|f'\left(\frac{b}{m}\right)\right|^q\right]\frac{1}{2^\alpha(\alpha+k+1)}+\frac{m}{k+1}\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\left(2^{k+1}B\left(\frac{1}{2};k+1,\alpha+1\right)\right)\left[|f'(a)|^q-m\left|f'\left(\frac{b}{m}\right)\right|^q\right]+\frac{m}{k+1}\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

where we used the fact that

$$\int_0^1 t^k\left(1-\frac{t}{2}\right)^\alpha dt=2^{k+1}B\left(\frac{1}{2};k+1,\alpha+1\right)$$

which completes the proof. □

Corollary 2.3. *In Theorem 2.3, if we choose $\alpha=1$ then the inequality* (8) *becomes the following inequality*

$$\begin{aligned}
&\left|\frac{2^{k-1}\Gamma(k+1)}{(b-a)^k}\left[J^k_{\left(\frac{a+b}{2}\right)^+}f(b)+J^k_{\left(\frac{a+b}{2}\right)^-}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|\\
\le\;& \frac{b-a}{4}\left(\frac{1}{k+1}\right)^{\frac{1}{p}}\left\{\left(\frac{|f'(a)|^q}{2(k+2)}+\frac{k+3}{2(k+1)(k+2)}m\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\frac{k+3}{2(k+1)(k+2)}|f'(a)|^q+\frac{1}{2(k+2)}m\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Remark 2.3. *In Theorem 2.3, if we choose $m=\alpha=1$ then the inequality* (8) *reduces to the inequality* (4) *of Theorem 1.3.*

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