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Bounds for the weighted Dragomir–Fedotov functional

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Abstract. In literature the Dragomir-Fedotov functional is well known as

$$\mathcal{D}(f;u) := \int_a^b f(x)\,du(x) - \frac{u(b)-u(a)}{b-a}\int_a^b f(t)\,dt.$$

In this work a generalization of $\mathcal{D}(f;u)$ is established. Namely, we define the weighted Dragomir–Fedotov functional such as:

$$\mathcal{OD}(f,g;u) := \frac{1}{u(b)-u(a)} \cdot \int_a^b f(x)\,du(x) - \frac{1}{\int_a^b g(t)\,dt} \cdot \int_a^b f(t)\,g(t)\,dt,$$

and hence several bounds are proved.

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1. Introduction

In order to approximate the Stieltjes integral $\int_a^b f(x)\,du(x)$ by the Riemann integral $\int_a^b f(t)\,dt$, Dragomir and Fedotov [13], have established the following functional:

$$\mathcal{D}(f;u) := \int_a^b f(x)\,du(x) - \frac{u(b)-u(a)}{b-a}\int_a^b f(t)\,dt, \tag{1}$$

provided that the Stieltjes integral $\int_a^b f(x)\,du(x)$ and the Riemann integral $\int_a^b f(t)\,dt$ exist.

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In the same paper [13], the authors have proved the following inequality:

Theorem 1.1. *Let $f, u : [a, b] \to \mathbb{R}$ be such that u is of bounded variation on $[a, b]$ and f is Lipschitzian with the constant $K > 0$. Then we have*

$$|\mathcal{D}(f; u)| \le \frac{1}{2} K (b - a) \bigvee_a^b (u). \tag{2}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

In his interesting work [15], Dragomir has obtained the following inequality:

Theorem 1.2. *Let $f, u : [a, b] \to \mathbb{R}$ be such that u is Lipschitzian on $[a, b]$, i.e.,*

$$|u(y) - u(x)| \le L |x - y|, \forall x, y \in [a, b], \quad (L > 0)$$

and f is Riemann integrable on $[a, b]$.

If $m, M \in \mathbb{R}$, are such that $m \le f(x) \le M$, for any $x \in [a, b]$, then the inequality

$$|\mathcal{D}(f; u)| \le \frac{1}{2} L (M - m)(b - a). \tag{3}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

For other recent inequalities of Gruss type, see [2]–[6], [9]–[16] and [18] and the references therein.

1.1. A weighted Dragomir functional.

In order to compare the Stieltjes integral mean with the weighted Riemann integral mean, we define the functional $\mathcal{OD}(f, g; u)$, as follows:

$$\mathcal{OD}(f, g; u) := \frac{1}{u(b) - u(a)} \cdot \int_a^b f(x)\, du(x) - \frac{1}{\int_a^b g(t)\, dt} \cdot \int_a^b f(t)\, g(t)\, dt, \tag{4}$$

provided that the both integrals exist and $g(t) \neq 0$, for all $t \in [a, b]$.

In particular, as special cases; we are interested in two functionals:

1: The Dragomir-Fedotov functional:

$$\begin{aligned} \mathcal{D}(f; u) &:= \int_a^b f(x)\, du(x) - \frac{u(b) - u(a)}{b - a} \int_a^b f(t)\, dt \\ &= [u(b) - u(a)] \cdot \mathcal{OD}(f, 1; u). \end{aligned} \tag{5}$$

2: The weighted integral functional:

$$\begin{aligned} \mathcal{E}(f, g; w) &:= \frac{\int_a^b f(t)\, w(t)\, dt}{\int_a^b w(t)\, dt} - \frac{\int_a^b f(t)\, g(t)\, dt}{\int_a^b g(t)\, dt} \\ &= \frac{1}{\int_a^b w(t)\, dt \int_a^b g(t)\, dt} \cdot \mathcal{OD}\left(f, g; \int_a^x w(s)\, ds\right), \end{aligned} \tag{6}$$

where, $u(x)=\int_a^x w(s)\,ds$, $w:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$, and $g(t), w(t)\neq 0$, for all $t\in[a,b]$. For more works about this type of integral functional the reader may refer to [1], [7, 8] and [17].

In fact, the functional $\mathcal{OD}(f,g;u)$ is a natural generalization of Dragomir functional $\mathcal{D}(f;u)$; so that in this paper, several new bounds with various type of integrators for the functional $\mathcal{OD}(f,g;u)$ are proved. More specifically, the obtained results deal with integrands of r-H–Hölder type, and integrators of bounded variation, Lipschitzian and monotonic types. Through the assumptions for the functions involved in the functionals (5) and (6) several bounds may be obtained by a direct substitution and we shall omit the details.

2. The case of bounded variation integrators

The following result holds:

Theorem 2.1. *Let $f,u,g:[a,b]\to\mathbb{R}$ be mappings such that f is of r-H–Hölder type on $[a,b]$, where $r\in(0,1]$ and $H>0$ are given, and u is of bounded variation on $[a,b]$. Then we have the inequality:*

$$|\mathcal{OD}(f,g;u)| \tag{7}$$
$$\leq \frac{H}{[u(b)-u(a)]\cdot\int_a^b g(t)\,dt}\begin{cases}\frac{(b-a)^{r+1}}{r+1}\cdot\|g\|_\infty\cdot\bigvee_a^b(u), & \text{if } g\in L_\infty[a,b],\\[2ex] \frac{(b-a)^{(qr+1)/q}}{(qr+1)^{1/q}}\cdot\|g\|_p\cdot\bigvee_a^b(u), & \text{if } g\in L_p[a,b],\\ & p>1, \frac{1}{p}+\frac{1}{q}=1,\\[2ex] (b-a)^r\|g\|_1\cdot\bigvee_a^b(u), & \text{if } g\in L_1[a,b].\end{cases},$$

where, $\bigvee_a^b(u)$ is the total variation of u over $[a,b]$.

Proof. It is well-known that for a continuous function $p:[a,b]\to\mathbb{R}$ and a function $\nu:[a,b]\to\mathbb{R}$ of bounded variation, one has the inequality

$$\left|\int_a^b p(t)\,d\nu(t)\right|\leq\sup_{t\in[a,b]}|p(t)|\bigvee_a^b(\nu).$$

Therefore, as u is of bounded variation on $[a,b]$, we have

$$\begin{aligned}&[u(b)-u(a)]\cdot\left(\int_a^b g(t)\,dt\right)\cdot|\mathcal{OD}(f,g;u)|\\ &=\left|\int_a^b\left[f(x)\int_a^b g(t)\,dt-\int_a^b f(t)g(t)\,dt\right]du(x)\right|\\ &\leq\sup_{x\in[a,b]}\left|\int_a^b[f(x)-f(t)]\,g(t)\,dt\right|\cdot\bigvee_a^b(u).\end{aligned}\tag{8}$$

As f is of r-H-Hölder type on $[a,b]$ and $g \in L_\infty[a,b]$, then we have

$$\begin{aligned}\left|\int_a^b [f(x) - f(t)]\, g(t)\, dt\right| &\le \int_a^b |f(x) - f(t)|\, |g(t)|\, dt \\ &\le H \int_a^b |x-t|^r\, |g(t)|\, dt \\ &\le H \sup_{t\in[a,b]} |g(t)| \cdot \int_a^b |x-t|^r\, dt \\ &= \frac{H}{r+1}\left[(x-a)^{r+1} + (b-x)^{r+1}\right] \cdot \|g\|_\infty . \end{aligned} \tag{9}$$

It follows that

$$\sup_{x\in[a,b]} \left|\int_a^b [f(x) - f(t)]\, g(t)\right| dx \le \frac{H}{r+1} (b-a)^{r+1} \cdot \|g\|_\infty . \tag{10}$$

Combining (8) and (10), we get the first inequality in (7).

To prove the second inequality in (7). As f is of r-H-Hölder type on $[a,b]$, then we have

$$\begin{aligned}\left|\int_a^b [f(x) - f(t)]\, g(t)\, dt\right| &\le \int_a^b |f(x) - f(t)|\, |g(t)|\, dt \\ &\le H \int_a^b |x-t|^r\, |g(t)|\, dt.\end{aligned}$$

Now, as $g \in L_p[a,b]$ therefore, by applying the well-known Hölder integral inequality, we get

$$\begin{aligned}\left|\int_a^b [f(x) - f(t)]\, g(t)\, dt\right| &\le H \int_a^b |x-t|^r\, |g(t)|\, dt \\ &\le H \left(\int_a^b |x-t|^{rq}\, dt\right)^{1/q} \left(\int_a^b |g(t)|^p\, dt\right)^{1/p} \\ &= \frac{H}{(qr+1)^{1/q}} \left[(x-a)^{qr+1} + (b-x)^{qr+1}\right]^{1/q} \cdot \|g\|_p .\end{aligned}$$

It follows that

$$\begin{aligned}\sup_{x\in[a,b]} \left|\int_a^b [f(x) - f(t)]\, g(t)\, dt\right| &\le \frac{H}{(qr+1)^{1/q}} \cdot \|g\|_p \cdot \sup_{x\in[a,b]} \left[(x-a)^{qr+1} + (b-x)^{qr+1}\right]^{1/q} \\ &\le H \frac{(b-a)^{(qr+1)/q}}{(qr+1)^{1/q}} \cdot \|g\|_p .\end{aligned} \tag{11}$$

Combining (8) and (11), we get the second inequality in (7).

Finally, to prove the third inequality in (7). By assumptions we have:

$$\left|\int_a^b [f(x) - f(t)]\, g(t)\, dt\right| \le \int_a^b |f(x) - f(t)|\, |g(t)|\, dt$$

$$
\begin{aligned}
&\leq H \int_a^b |x-t|^r \, |g(t)| \, dt \\
&\leq H \sup_{t\in[a,b]} \{|x-t|^r\} \int_a^b |g(t)| \, dt \\
&= H \|g\|_1 \max_{t\in[a,b]} \{(x-a)^r, (b-x)^r\} \\
&= H \|g\|_1 \left[\max_{t\in[a,b]} \{(x-a), (b-x)\}\right]^r \\
&= H \|g\|_1 \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]^r \\
&\leq H \|g\|_1 (b-a)^r . \qquad (12)
\end{aligned}
$$

Combining (8) and (12), we get the third inequality in (7) and thus the theorem is proved. □

Corollary 2.1. *Let u as in Theorem 2.1 and $f : [a,b] \to \mathbb{R}$ be an L–Lipschitzian mapping on $[a,b]$. Then we have the inequality*

$$
|\mathcal{OD}(f,g;u)| \qquad (13)
$$

$$
\leq \frac{L}{[u(b)-u(a)] \cdot \int_a^b g(t)\,dt}
\begin{cases}
\frac{(b-a)^2}{2} \cdot \|g\|_\infty \cdot \bigvee_a^b(u), & \text{if} \quad g \in L_\infty[a,b], \\[2ex]
\frac{(b-a)^{(q+1)/q}}{(q+1)^{1/q}} \cdot \|g\|_p \cdot \bigvee_a^b(u), & \text{if} \quad g \in L_p[a,b], \\
 & p>1, \frac{1}{p}+\frac{1}{q}=1, \\[2ex]
(b-a)\|g\|_1 \cdot \bigvee_a^b(u), & \text{if} \quad g \in L_1[a,b].
\end{cases} .
$$

Corollary 2.2. *Assume f as in Theorem 2.1. Let $u \in C^{(1)}[a,b]$. Then we have the inequality*

$$
|\mathcal{OD}(f,g;u)| \qquad (14)
$$

$$
\leq \frac{H}{[u(b)-u(a)] \cdot \int_a^b g(t)\,dt}
\begin{cases}
\frac{(b-a)^{r+1}}{r+1} \cdot \|g\|_\infty \cdot \|u'\|_{1,[a,b]}, & \text{if} \quad g \in L_\infty[a,b], \\[2ex]
\frac{(b-a)^{(qr+1)/q}}{(qr+1)^{1/q}} \cdot \|g\|_p \cdot \|u'\|_{1,[a,b]}, & \text{if} \quad g \in L_p[a,b], \\
 & p>1, \frac{1}{p}+\frac{1}{q}=1, \\[2ex]
(b-a)^r \|g\|_1 \cdot \|u'\|_{1,[a,b]}, & \text{if} \quad g \in L_1[a,b].
\end{cases} ,
$$

where $\|\cdot\|_1$ is the L_1 norm, namely $\|u'\|_{1,[a,b]} := \int_a^b |u'(t)|\,dt$.

Corollary 2.3. *Assume f as in Theorem 2.1. Let $u : [a,b] \to \mathbb{R}$ be a K-Lipschitzian mapping with the constant $K > 0$. Then we have the inequality*

$$
|\mathcal{OD}(f,g;u)| \qquad (15)
$$

$$\leq \frac{HK\left(b-a\right)}{\left[u\left(b\right)-u\left(a\right)\right]\cdot\int_a^b g\left(t\right)dt}\begin{cases}\frac{(b-a)^{r+1}}{r+1}\cdot\left\|g\right\|_{\infty}, & \text{if} \quad g\in L_{\infty}[a,b],\\[2ex] \frac{(b-a)^{(qr+1)/q}}{(qr+1)^{1/q}}\cdot\left\|g\right\|_{p}, & \text{if} \quad g\in L_{p}[a,b], \\ & p>1, \frac{1}{p}+\frac{1}{q}=1,\\[2ex] \left(b-a\right)^{r}\left\|g\right\|_{1}, & \text{if} \quad g\in L_{1}[a,b].\end{cases}.$$

Corollary 2.4. *Assume f as in Theorem 2.1. Let $u:[a,b]\to\mathbb{R}$ be a monotonic increasing mapping. Then we have the inequality*

$$\left|\mathcal{OD}\left(f,g;u\right)\right| \tag{16}$$

$$\leq \frac{H}{\int_a^b g\left(t\right)dt}\begin{cases}\frac{(b-a)^{r+1}}{r+1}\cdot\left\|g\right\|_{\infty}, & \text{if} \quad g\in L_{\infty}[a,b],\\[2ex] \frac{(b-a)^{(qr+1)/q}}{(qr+1)^{1/q}}\cdot\left\|g\right\|_{p}, & \text{if} \quad g\in L_{p}[a,b], \\ & p>1, \frac{1}{p}+\frac{1}{q}=1,\\[2ex] \left(b-a\right)^{r}\left\|g\right\|_{1}, & \text{if} \quad g\in L_{1}[a,b].\end{cases}.$$

Remark 2.1. *For the last three inequalities, one may deduce several inequalities for L-Lipschitzian integrands by setting $r=1$ and replace H by L. We left the details to the interested reader.*

Remark 2.2. *Under the assumptions of Theorem 2.1, one may deduce several inequalities for the Dragomir-Fedotov functional (5) and the weighted integral functional (6).*

3. The case of Lipschitzian integrators

Theorem 3.1. *Let $f:[a,b]\to\mathbb{R}$ be an r-H–Hölder type mapping on $[a,b]$, and $u:[a,b]\to\mathbb{R}$ be an L-Lipschitzian mapping on $[a,b]$, where r and $H,L>0$ are given. Then we have the inequality*

$$\left|\mathcal{OD}\left(f,g;u\right)\right| \tag{17}$$

$$\leq \frac{LH}{\left[u\left(b\right)-u\left(a\right)\right]\cdot\int_a^b g\left(t\right)dt}\begin{cases}\frac{2(b-a)^{r+2}}{(r+1)(r+2)}\cdot\left\|g\right\|_{\infty}, & , \quad \text{if} \quad g\in L_{\infty}[a,b],\\[2ex] \frac{2q}{(qr+1)^{1/q}}\cdot\frac{(b-a)^{(q(r+1)+1)/q}}{(q(r+1)+1)}\cdot\left\|g\right\|_{p}, & \text{if} \quad g\in L_{p}[a,b], \\ & p>1, \frac{1}{p}+\frac{1}{q}=1,\\[2ex] \frac{\left(2^{r+1}-1\right)}{2^{r}(r+1)}\left(b-a\right)^{r+1}\left\|g\right\|_{1}, & \text{if} \quad g\in L_{1}[a,b].\end{cases}.$$

Proof. It is well-known that for a Riemann integrable function $p : [a,b] \to \mathbb{R}$ and L–Lipschitzian function $\nu : [a,b] \to \mathbb{R}$, one has the inequality

$$\left|\int_a^b p(t)\,d\nu(t)\right| \le L\int_a^b |p(t)|\,dt.$$

Therefore, as u is L-Lipschitzian on $[a,b]$, we have

$$\begin{aligned}
&[u(b)-u(a)]\cdot\left(\int_a^b g(t)\,dt\right)\cdot|\mathcal{OD}(f,g;u)| \\
&= \left|\int_a^b \left[f(x)\int_a^b g(t)\,dt - \int_a^b f(t)\,g(t)\,dt\right]du(x)\right| \\
&\le L\int_a^b \left|\int_a^b [f(x)-f(t)]\,g(t)\,dt\right|dx. \qquad (18)
\end{aligned}$$

As f is of r-H-Hölder type on $[a,b]$ and $g \in L_\infty[a,b]$, by (9) we have

$$\left|\int_a^b [f(x)-f(t)]\,g(t)\,dt\right| \le \frac{H}{r+1}\left[(x-a)^{r+1}+(b-x)^{r+1}\right]\cdot\|g\|_\infty. \qquad (19)$$

It follows that

$$\begin{aligned}
\int_a^b \left|\int_a^b [f(x)-f(t)]\,g(t)\,dt\right|dx &\le \frac{H}{r+1}\cdot\|g\|_\infty\cdot\int_a^b \left[(x-a)^{r+1}+(b-x)^{r+1}\right]dx \\
&\le \frac{2H}{(r+1)(r+2)}(b-a)^{r+2}\cdot\|g\|_\infty. \qquad (20)
\end{aligned}$$

Combining (18) and (20), we get the first inequality in (17).

To prove the second inequality in (17). As f is of r-H-Hölder type on $[a,b]$, then we have

$$\begin{aligned}
\left|\int_a^b [f(x)-f(t)]\,g(t)\,dt\right| &\le \int_a^b |f(x)-f(t)|\,|g(t)|\,dt \\
&\le H\int_a^b |x-t|^r\,|g(t)|\,dt.
\end{aligned}$$

Now, as $g \in L_p[a,b]$ therefore, by applying the well-known Hölder integral inequality, we get

$$\begin{aligned}
\left|\int_a^b [f(x)-f(t)]\,g(t)\,dt\right| &\le H\int_a^b |x-t|^r\,|g(t)|\,dt \\
&\le H\left(\int_a^b |x-t|^{rq}\,dt\right)^{1/q}\left(\int_a^b |g(t)|^p\,dt\right)^{1/p} \\
&= \frac{H}{(qr+1)^{1/q}}\left[(x-a)^{qr+1}+(b-x)^{qr+1}\right]^{1/q}\cdot\|g\|_p.
\end{aligned}$$

$$= \frac{H}{(qr+1)^{1/q}}\left[\left((x-a)^{\left(r+\frac{1}{q}\right)}\right)^{q}+\left((b-x)^{\left(r+\frac{1}{q}\right)}\right)^{q}\right]^{1/q}\cdot\|g\|_{p}. \tag{21}$$

Using the fact that $(A^s+B^s)^{1/s} \le (A+B)$, for all $A, B \ge 0$ and $s \ge 1$, it follows that

$$\begin{aligned}
&\int_a^b \left|\int_a^b [f(x)-f(t)]\,g(t)\,dt\right| dx \\
&\le \frac{H}{(qr+1)^{1/q}}\cdot\|g\|_p \int_a^b \left[\left((x-a)^{\left(r+\frac{1}{q}\right)}\right)^{q}+\left((b-x)^{\left(r+\frac{1}{q}\right)}\right)^{q}\right]^{1/q} dx \\
&\le \frac{H}{(qr+1)^{1/q}}\cdot\|g\|_p\cdot\int_a^b \left[(x-a)^{\left(r+\frac{1}{q}\right)}+(b-x)^{\left(r+\frac{1}{q}\right)}\right] dx \\
&\le H\frac{2q}{(qr+1)^{1/q}}\cdot\frac{(b-a)^{(q(r+1)+1)/q}}{(q(r+1)+1)}\cdot\|g\|_p.
\end{aligned} \tag{22}$$

Combining (18) and (22), we get the second inequality in (17).

Finally, to prove the third inequality in (17). By assumptions we have:

$$\left|\int_a^b [f(x)-f(t)]\,g(t)\,dt\right| \le H\|g\|_1\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]^r \tag{23}$$

It follows that

$$\begin{aligned}
\int_a^b \left|\int_a^b [f(x)-f(t)]\,g(t)\,dt\right| dx &\le H\|g\|_1\cdot\int_a^b\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]^r dx \\
&= H\|g\|_1\frac{(2^{r+1}-1)}{2^r(r+1)}(b-a)^{r+1}.
\end{aligned} \tag{24}$$

Combining (18) and (24), we get the third inequality in (17) and thus the theorem is proved.

□

Corollary 3.1. *Let u as in Theorem 3.1 and $f:[a,b]\to\mathbb{R}$ be an K–Lipschitzian mapping on $[a,b]$. Then we have the inequality*

$$\begin{aligned}
&|\mathcal{OD}(f,g;u)| \\
&\le \frac{LK}{[u(b)-u(a)]\cdot\int_a^b g(t)\,dt}\begin{cases}
\frac{(b-a)^3}{3}\cdot\|g\|_\infty, & \text{if } g\in L_\infty[a,b],\\
\frac{2q}{(q+1)^{1/q}}\cdot\frac{(b-a)^{(2q+1)/q}}{(2q+1)}\cdot\|g\|_p, & \text{if } g\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1,\\
\frac{3}{4}(b-a)^2\|g\|_1, & \text{if } g\in L_1[a,b].
\end{cases}
\end{aligned} \tag{25}$$

Remark 3.1. *Under the assumptions of Theorem 3.1, one may deduce several inequalities for the functionals (5) and (6).*

4. The case of monotonic integrators

Theorem 4.1. *Let $f : [a,b] \to \mathbb{R}$ be an r-H–Hölder type mapping on $[a,b]$, and $u : [a,b] \to \mathbb{R}$ be a monotonic mapping on $[a,b]$, where r and $H > 0$ are given. Then we have the inequality*

$$|\mathcal{OD}(f,g;u)| \tag{26}$$
$$\leq \frac{H}{[u(b)-u(a)]\cdot\int_a^b g(t)\,dt}\begin{cases} 2\frac{(b-a)^{r+1}}{r+1}\cdot\|g\|_\infty\cdot[u(b)-u(a)], & \text{if } \ g\in L_\infty[a,b], \\ \frac{2(b-a)^{r+\frac{1}{q}}}{(qr+1)^{1/q}}\cdot\|g\|_p\cdot[u(b)-u(a)], & \text{if } \ g\in L_p[a,b]; \\ & p>1, \frac{1}{p}+\frac{1}{q}=1, \\ (b-a)^r\|g\|_1\cdot[u(b)-u(a)], & \text{if } \ g\in L_1[a,b]. \end{cases}.$$

Proof. It is well-known that for a monotonic non-decreasing function $\nu : [a,b] \to \mathbb{R}$ and continuous function $p : [a,b] \to \mathbb{R}$, one has the inequality

$$\left|\int_a^b p(t)\,d\nu(t)\right| \leq \int_a^b |p(t)|\,d\nu(t).$$

Therefore, as u is monotonic non-decreasing on $[a,b]$, we have

$$\begin{aligned} &[u(b)-u(a)]\cdot\left(\int_a^b g(t)\,dt\right)\cdot|\mathcal{OD}(f,g;u)| \\ &= \frac{1}{(b-a)^2}\left|\int_a^b\left[f(x)\int_a^b g(t)\,dt - \int_a^b f(t)\,g(t)\,dt\right]du(x)\right| \\ &\leq \int_a^b\left|f(x)\int_a^b g(t)\,dt - \int_a^b f(t)\,g(t)\,dt\right|du(x) \\ &= \int_a^b\left|\int_a^b [f(x)-f(t)]\,g(t)\,dt\right|du(x). \end{aligned} \tag{27}$$

As f is of r-H-Hölder type on $[a,b]$ and $g \in L_\infty[a,b]$, by (9) we have

$$\left|\int_a^b [f(x)-f(t)]\,g(t)\,dt\right| \leq \frac{H}{r+1}\left[(x-a)^{r+1}+(b-x)^{r+1}\right]\cdot\|g\|_\infty. \tag{28}$$

It follows that

$$\begin{aligned} &\int_a^b\left|\int_a^b [f(x)-f(t)]\,g(t)\,dt\right|du(x) \\ &\qquad\leq \frac{H}{r+1}\cdot\|g\|_\infty\cdot\int_a^b\left[(x-a)^{r+1}+(b-x)^{r+1}\right]du(x). \end{aligned} \tag{29}$$

Now, using Riemann–Stieltjes integral we have

$$\int_a^b (x-a)^{r+1}\,du(x) = (b-a)^{r+1}\,u(b) - (r+1)\int_a^b (x-a)^r\,u(x)\,dx, \tag{30}$$

and

$$\int_a^b (b-x)^{r+1}\,du(x) = -(b-a)^{r+1}\,u(a) + (r+1)\int_a^b (b-x)^r\,u(x)\,dx. \tag{31}$$

Adding (30) and (31), we get

$$\int_a^b \left[(x-a)^{r+1} + (b-x)^{r+1}\right] du(x).$$
$$= (b-a)^{r+1}\left[u(b)-u(a)\right] + (r+1)\int_a^b \left[(b-x)^r - (x-a)^r\right] u(x)\,dx. \tag{32}$$

Now, by the monotonicity property of u we have

$$\int_a^b (x-a)^r\,u(x)\,dx \geq u(a)\int_a^b (x-a)^r\,dx = \frac{(b-a)^{r+1}}{r+1}\cdot u(a), \tag{33}$$

and

$$\int_a^b (b-x)^r\,u(x)\,dx \leq u(b)\int_a^b (b-x)^r\,dx = \frac{(b-a)^{r+1}}{r+1}\cdot u(b). \tag{34}$$

Substituting (33) and (34) in (32), we get

$$\int_a^b \left[(x-a)^{r+1} + (b-x)^{r+1}\right] du(x) \leq 2(b-a)^{r+1}\left[u(b)-u(a)\right]. \tag{35}$$

Substituting (35) in (29), we get

$$\int_a^b \left|\int_a^b [f(x)-f(t)]\,g(t)\,dt\right| du(x) \leq 2H\frac{(b-a)^{r+1}}{r+1}\cdot \|g\|_\infty \cdot \left[u(b)-u(a)\right],$$

and therefore, by (27) we get the first inequality in (26).

To prove the second inequality in (26). As f is of r-H-Hölder type on $[a,b]$ and $g \in L_p[a,b]$ therefore, by (21), we have

$$\left|\int_a^b [f(x)-f(t)]\,g(t)\,dt\right|$$
$$\leq \frac{H}{(qr+1)^{1/q}}\left[\left((x-a)^{\left(r+\frac{1}{q}\right)}\right)^q + \left((b-x)^{\left(r+\frac{1}{q}\right)}\right)^q\right]^{1/q}\cdot \|g\|_p. \tag{36}$$

It follows by (22), that

$$\int_a^b \left|\int_a^b [f(x)-f(t)]\,g(t)\,dt\right| du(x)$$
$$\leq \frac{H}{(qr+1)^{1/q}}\cdot \|g\|_p \int_a^b \left[\left((x-a)^{\left(r+\frac{1}{q}\right)}\right)^q + \left((b-x)^{\left(r+\frac{1}{q}\right)}\right)^q\right]^{1/q} du(x)$$
$$\leq \frac{H}{(qr+1)^{1/q}}\cdot \|g\|_p \cdot \int_a^b \left[(x-a)^{\left(r+\frac{1}{q}\right)} + (b-x)^{\left(r+\frac{1}{q}\right)}\right] du(x). \tag{37}$$

Now, using Riemann–Stieltjes integral we have

$$\int_a^b (x-a)^{r+\frac{1}{q}}\,du(x) = (b-a)^{r+\frac{1}{q}}\,u(b) - \left(r+\frac{1}{q}\right)\int_a^b (x-a)^{r+\frac{1}{q}-1}\,u(x)\,dx, \quad (38)$$

and

$$\int_a^b (b-x)^{r+\frac{1}{q}}\,du(x) = -(b-a)^{r+\frac{1}{q}}\,u(a) + \left(r+\frac{1}{q}\right)\int_a^b (b-x)^{r+\frac{1}{q}-1}\,u(x)\,dx. \quad (39)$$

Adding (38) and (39), we get

$$\int_a^b \left[(x-a)^{r+1} + (b-x)^{r+1}\right] du(x). = (b-a)^{r+\frac{1}{q}}\left[u(b)-u(a)\right] + \left(r+\frac{1}{q}\right)\int_a^b \left[(b-x)^{r+\frac{1}{q}-1} - (x-a)^{r+\frac{1}{q}-1}\right] u(x)\,dx. \quad (40)$$

Now, by the monotonicity property of u we have

$$\int_a^b (x-a)^{r+\frac{1}{q}-1}\,u(x)\,dx \geq u(a)\int_a^b (x-a)^{r+\frac{1}{q}-1}\,dx = \frac{q}{qr+1}(b-a)^{r+\frac{1}{q}}\cdot u(a), \quad (41)$$

and

$$\int_a^b (b-x)^{r+\frac{1}{q}-1}\,u(x)\,dx \leq u(b)\int_a^b (b-x)^{r+\frac{1}{q}-1}\,dx = \frac{q}{qr+1}(b-a)^{r+\frac{1}{q}}\cdot u(b). \quad (42)$$

Substituting (41) and (42) in (40), we get

$$\int_a^b \left[(x-a)^{r+1} + (b-x)^{r+1}\right] du(x) \leq 2(b-a)^{r+\frac{1}{q}}\left[u(b)-u(a)\right]. \quad (43)$$

Substituting (43) in (37), we get

$$\int_a^b \left|\int_a^b \left[f(x)-f(t)\right] g(t)\,dt\right| du(x) \leq \frac{2H}{(qr+1)^{1/q}}(b-a)^{r+\frac{1}{q}}\cdot \|g\|_p \cdot \left[u(b)-u(a)\right],$$

and therefore, by (27) we get the second inequality in (26).

Finally, to prove the third inequality in (26). As f is of r-H-Hölder type on $[a,b]$ and $g \in L_1[a,b]$ therefore, by (23), we have

$$\left| \int_a^b \left[f\left(x\right) - f\left(t\right) \right] g\left(t\right) dt \right| \le H \left\| g \right\|_1 \left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right]^r, \tag{44}$$

which gives by (27), that

$$\int_a^b \left| \int_a^b \left[f\left(x\right) - f\left(t\right) \right] g\left(t\right) dt \right| du\left(x\right) \le H \left\| g \right\|_1 \cdot \int_a^b \left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right]^r du\left(x\right). \tag{45}$$

Now, using Riemann–Stieltjes integral we have

$$\begin{aligned}
&\int_a^b \left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right]^r du\left(x\right) \\
&= \int_a^{\frac{a+b}{2}} \left(b-x\right)^r du\left(x\right) + \int_{\frac{a+b}{2}}^b \left(x-a\right)^r du\left(x\right) \\
&= \left(\frac{b-a}{2} \right)^r u\left(\frac{a+b}{2} \right) - \left(b-a\right)^r u\left(a\right) + r \int_a^{\frac{a+b}{2}} \left(b-x\right)^{r-1} u\left(x\right) dx \\
&\qquad + \left(b-a\right)^r u\left(b\right) - \left(\frac{b-a}{2} \right)^r u\left(\frac{a+b}{2} \right) - r \int_{\frac{a+b}{2}}^b \left(x-a\right)^{r-1} u\left(x\right) dx \\
&= \left(b-a\right)^r \left[u\left(b\right) - u\left(a\right) \right] + r \left[\int_a^{\frac{a+b}{2}} \left(b-x\right)^{r-1} u\left(x\right) dx - \int_{\frac{a+b}{2}}^b \left(x-a\right)^{r-1} u\left(x\right) dx \right].
\end{aligned} \tag{46}$$

Now, by the monotonicity property of u we have

$$\begin{aligned}
\int_a^{\frac{a+b}{2}} \left(b-x\right)^{r-1} u\left(x\right) dx &\le u\left(\frac{a+b}{2} \right) \int_a^{\frac{a+b}{2}} \left(b-x\right)^{r-1} dx \\
&= \frac{\left(2^r - 1\right)}{r 2^r} \left(b-a\right)^r \cdot u\left(\frac{a+b}{2} \right),
\end{aligned} \tag{47}$$

and

$$\begin{aligned}
\int_{\frac{a+b}{2}}^b \left(x-a\right)^{r-1} u\left(x\right) dx &\ge u\left(\frac{a+b}{2} \right) \int_{\frac{a+b}{2}}^b \left(x-a\right)^{r-1} dx \\
&= \frac{\left(2^r - 1\right)}{r 2^r} \left(b-a\right)^r \cdot u\left(\frac{a+b}{2} \right),
\end{aligned} \tag{48}$$

Substituting (47) and (48) in (46), we get

$$\int_a^b \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]^r du\,(x) \le (b-a)^r \left[u\,(b) - u\,(a)\right]. \tag{49}$$

Substituting (49) in (45), we get

$$\int_a^b \left|\int_a^b \left[f\,(x) - f\,(t)\right] g\,(t)\,dt\right| du\,(x) \le H\,\|g\|_1 \cdot (b-a)^r \left[u\,(b) - u\,(a)\right].$$

and therefore, by (27) we get the third inequality in (26), and thus the theorem is proved. □

Remark 4.1. *Under the assumptions of Theorem 4.1, one may deduce several inequalities for the functionals (5) and (6).*

References

[1] M.W. Alomari, Difference between two Stieltjes integral means, *Kragujevac Journal of Mathematics*, 38(1) (2014), 35–49.

[2] M.W. Alomari, New Grüss type inequalities for double integrals, *Appl. Math. Comp.*, 228 (2014) 102–107.

[3] M.W. Alomari and S.S. Dragomir, New Grüss type inequalities for Riemann–Stieltjes integral with monotonic integrators and applications, *Ann. Funct. Anal.*, 5 (2014), no. 1, 77–93.

[4] M.W. Alomari and S.S. Dragomir, Mercer-Trapezoid rule for Riemann–Stieltjes integral with applications, *Journal of Advances in Mathematics*, 2 (2) (2013), 67–85.

[5] N.S. Barnett, S.S. Dragomir and I. Gomma, A companion for the Ostrowski and the generalised trapezoid inequalities, *Mathematical and Computer Modelling*, 50 (2009), 179–187.

[6] N.S. Barnett, W.-S. Cheung, S.S. Dragomir, A. Sofo, Ostrowski and trapezoid type inequalities for the Stieltjes integral with Lipschitzian integrands or integrators, *Comp. Math. Appl.* , 57 (2009), 195–201.

[7] N.S. Barnett, P. Cerone, S.S. Dragomir and A.M. Fink, Comparing two integral means for absolutely continuous mappings whose derivatives are in $L_\infty[a,b]$ and applications, *Comp. and Math. Appl.*, 44 (1/2) (2002), 241–251.

[8] P. Cerone and S.S. Dragomir, Differences between means with bounds from a Riemann–Stieltjes integral, *Comp. Math. Appl.*, 46 (2003) 445–453.

[9] P. Cerone, W.S. Cheung, S.S. Dragomir, On Ostrowski type inequalities for Stieltjes integrals with absolutely continuous integrands and integrators of bounded variation, *Comp. Math. Appl.*, 54 (2007), 183–191.

[10] P. Cerone, S.S. Dragomir, New bounds for the three–point rule involving the Riemann–Stieltjes integrals, in: C. Gulati, et al. (Eds.), Advances in Statistics Combinatorics and Related Areas, World Science Publishing, 2002, pp. 53–62.

[11] P. Cerone, S.S. Dragomir, Approximating the Riemann–Stieltjes integral via some moments of the integrand, *Mathematical and Computer Modelling*, 49 (2009), 242–248.

[12] S.S. Dragomir and Th.M. Rassias (Ed.), Ostrowski Type Inequalities and Applications in Numerical Integration, Kluwer Academic Publishers, Dordrecht, 2002.

[13] S.S. Dragomir, I. Fedotov, An inequality of Grüss type for Riemann–Stieltjes integral and applications for special means, *Tamkang J. Math.*, 29 (4) (1998) 287–292

[14] S.S. Dragomir, I. Fedotov, A Grüss type inequality for mappings of bounded variation and applications to numerical analysis, *Nonlinear Funct. Anal. Appl.*, 6 (3) (2001) 425–433.

[15] S.S. Dragomir, Inequalities of Grüss type for the Stieltjes integral and applications, *Kragujevac J. Math.*, 26 (2004) 89–112.

[16] S.S. Dragomir, C. Buşe, M.V. Boldea, L. Braescu, A generalisation of the trapezoid rule for the Riemann–Stieltjes integral and applications, *Nonlinear Anal. Forum* 6 (2) (2001) 33–351.

[17] A.I. Kechriniotis and N.D. Assimakis, On the inequality of the difference of two integral means and applications for pdfs, *J. Ineq. pure Appl. math.*, 8 (1) (2007), Article 10, 6 pp.

[18] Z. Liu, Refinement of an inequality of Grüss type for Riemann–Stieltjes integral, *Soochow J. Math.*, 30 (4) (2004) 483–489.