DOI 10.7603/s40956-016-0004-2

Moroccan J. Pure and Appl. Anal.(MJPAA) RESEARCH ARTICLE
Volume 2(1), 2016, Pages 34–46
ISSN: 2351-8227

Generalized Hermite-Hadamard-Fejer type inequalities for GA-convex functions via Fractional integral

İmdat İşcan[a] and Sercan Turhan[b]

Abstract. In this paper, we give a new identity for differentiable and GA-convex functions. As a result of this identity, we obtain some new fractional integral inequalities for differentiable GA-convex functions.

2010 Mathematics Subject Classification. Primary 26D15; Secondary 26A51, Third 26D10, Fourth 26A15.

Key words and phrases. GA-convex, Hermite-Hadamard-Fejer type inequalities, Fractional Integral.

1. Introduction

The classical or the usual convexity is defined as follows:
A function $f : \emptyset \neq I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$, is said to be convex on I if the inequality

$$f\left(tx + (1-t)\,y\right) \leq t f(x) + (1-t)\, f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

A number of papers have been written on inequalities using the classical convexity and one of the most fascinating inequalities in mathematical analysis is stated as

Received February 6, 2016 - Accepted May 02, 2016.

©The Author(s) 2016. This article is published with open access by Sidi Mohamed Ben Abdallah University

[a] *Department of Mathematics, Faculty of Arts and Sciences, Giresun University, 28100, Giresun, TURKEY.*
e-mail: imdati@yahoo.com, imdat.iscan@giresun.edu.tr

[b] *Dereli Vocational High School, Giresun University, 28100, Giresun, TURKEY.*
e-mail: sercanturhan28@gmail.com, sercan.turhan@giresun.edu.tr

Published online: 31 May 2016

follows:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1}$$

where $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex mapping and $a, b \in I$ with $a < b$. Both the inequalities in (1) hold in reversed direction if f is concave. The inequalities stated in (1) are known as Hermite-Hadamard inequalities.

For more results on (1) which provide new proof, significant extensions, generalizations, refinements, counterparts, new Hermite-Hadamard-type inequalities and numerous applications, we refer the interested reader to [1]-[5],[7, 11, 12, 14] and the references therein.

The usual notion of convex functions have been generalized in diverse manners. One of them is called GA-convex functions and is stated in the definition below.

Definition 1.1. [11, 12]*A function $f : I \subseteq \mathbb{R}_+ = (0, \infty) \longrightarrow \mathbb{R}$ is said to be GA-convex function on I if*

$$f\left(x^{\lambda}y^{1-\lambda}\right) \leq \lambda f(x) + (1-\lambda) f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$, where $x^{\lambda}y^{1-\lambda}$ and $\lambda f(x) + (1-\lambda) f(y)$ are respectively the weighted geometric mean of two positive numbers x and y and the weighted arithmetic mean of $f(x)$ and $f(y)$.

The definition of GA-convexity is further generalized as GA-s-convexity in the second sense as follows.

Definition 1.2. [14] *A function $f : I \subseteq \mathbb{R}_+ = (0, \infty) \longrightarrow \mathbb{R}$ is said to be GA-s-convex function on I if*

$$f\left(x^{\lambda}y^{1-\lambda}\right) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$ and for some $s \in (0, 1]$.

For the properties of GA-convex functions and GA-s-convex function, we refer the reader to [6, 9, 10, 14, 15, 16] and the references therein.

Most recently, a number of findings have been seen on Hermite-Hadamard type integral inequalities for GA-convex and for GA-s-convex functions.

Zhang et all. in [15] established the following Hermite-Hadamard type integral inequalities for GA-convex function.

Theorem 1.1. [15] *Let $f : I \subseteq \mathbb{R}_+ = (0, \infty) \rightarrow \mathbb{R}$ be differentiable on I°, and $a, b \in I$ with $a < b$ and $f' \in L[a, b]$. If $|f'|^q$ is GA-convex on $[a, b]$ for $q \geq 1$, then*

$$\left| bf(b) - af(a) - \int_a^b f(x)dx \right| \leq \frac{\left[(b-a) A(a, b)\right]^{1-1/q}}{2^{1/q}} \tag{2}$$

$$\times \left\{ \left[L(a^2, b^2) - a^2\right] |f'(a)|^q + \left[b^2 - L(a^2, b^2)\right] |f'(b)|^q \right\}^{1/q}.$$

Theorem 1.2. [15] *Let $f : I \subseteq \mathbb{R}_+ = (0,\infty) \rightarrow \mathbb{R}$ be differentiable on I°, and $a, b \in I$ with $a < b$ and $f' \in L[a,b]$. If $|f'|^q$ is GA-convex on $[a,b]$ for $q > 1$, then*

$$\left| bf(b) - af(a) - \int_a^b f(x)dx \right| \leq (\ln b - \ln a) \tag{3}$$

$$\times \left[L(a^{2q/(q-1)}, b^{2q/(q-1)}) - a^{2q/(q-1)} \right]^{1-1/q} \left[A\left(|f'(a)|^q, |f'(b)|^q \right) \right]^{1/q}.$$

Theorem 1.3. [15] *Let $f : I \subseteq \mathbb{R}_+ = (0,\infty) \rightarrow \mathbb{R}$ be differentiable on I°, and $a, b \in I$ with $a < b$ and $f' \in L[a,b]$. If $|f'|^q$ is GA-convex on $[a,b]$ for $q > 1$ and $2q > p > 0$, then*

$$\left| bf(b) - af(a) - \int_a^b f(x)dx \right| \leq \frac{(\ln b - \ln a)^{1-1/q}}{p^{1/q}} \tag{4}$$

$$\times \left[L(a^{(2q-p)/(q-1)}, b^{(2q-p)/(q-1)}) \right]^{1-1/q}$$

$$\times \left\{ [L(a^p, b^p) - a^p] |f'(a)|^q + [b^p - L(a^p, b^p)] |f'(b)|^q \right\}^{1/q}.$$

Theorem 1.4. [6] *Suppose that $f : I \subseteq \mathbb{R}_+ = (0,\infty) \longrightarrow \mathbb{R}$ is GA-s-convex function in the second sense, where $s \in [0,1)$ and let $a, b \in [0,\infty)$, $a < b$. If $f \in L[a,b]$, then the following inequalities hold*

$$2^{s-1} f\left(\sqrt{ab}\right) \leq \frac{1}{lnb - lna} \int_a^b \frac{f(x)}{x} dx \leq \frac{f(a) + f(b)}{s+1}, \tag{5}$$

the constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1).

If f is GA-convex function in Theorem 4, then we get the following inequalities

$$f\left(\sqrt{ab}\right) \leq \frac{1}{lnb - lna} \int_a^b \frac{f(x)}{x} dx \leq \frac{f(a) + f(b)}{2}. \tag{6}$$

For more results on GA-convex function and GA-s-convex function see e.g [6, 9, 14].

Definition 1.3. [9] *A function $f : I \subseteq \mathbb{R}_+ = (0,\infty) \longrightarrow \mathbb{R}$ is said to be geometrically symmetric with respect to $\sqrt{ab}$ if the inequality*

$$g\left(\frac{ab}{x}\right) = g(x)$$

holds for all $x \in [a,b]$.

Definition 1.4. [8] *Let $f \in L[a,b]$. The right-hand side and left-hand side Hadamard fractional integrals $J_{a+}^\alpha f$ and $J_{b-}^\alpha f$ of order $\alpha > 0$ with $b > a \geq 0$ are defined by*

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t} \right)^{\alpha-1} f(t) \frac{dt}{t}, \; x > a$$

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{t}{x}\right)^{\alpha-1} f(t) \frac{dt}{t},\ x < b$$

respectively where $\Gamma(\alpha)$ *is the Gamma function defined by* $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$ *and* $J_{a^+}^0 f(x) = J_{b^-}^0 f(x) = f(x)$.

Lemma 1.1. [13] *For* $0 < \theta \leq 1$ *and* $0 \leq a < b$ *we have*

$$\left|a^\theta - b^\theta\right| \leq (b-a)^\theta .$$

In [4], D. Y. Hwang established a new identity for convex functions. In this study, we will prove a similar identity and will obtain Hermite-Hadamard-Fejér inequality for GA-convex functions via fractional integrals based on this new identity.

2. Main Results

Throughout in this section, we will use the notations $L(t) = a^t G^{1-t}$, $U(t) = b^t G^{1-t}$ and $G = G(a,b) = \sqrt{ab}$.

Lemma 2.1. *Let* $f : I \subseteq \mathbb{R}_+ = (0,\infty) \longrightarrow \mathbb{R}$ *be a differentiable function on* I^o, $a, b \in I^o$ *with* $a < b$. *If* $h : [a,b] \longrightarrow [0,\infty)$ *is a differentiable function and* $f' \in L([a,b])$, *the following inequality holds:*

$$[h(b) - 2h(a)] \frac{f(a)}{2} + h(b) \frac{f(b)}{2} - \int_a^b f(x) h'(x) dx \tag{7}$$

$$= \frac{\ln b - \ln a}{4} \left\{ \int_0^1 \left[2h\left(a^t G^{1-t}\right) - h(b)\right] f'\left(a^t G^{1-t}\right) a^t G^{1-t} dt \right.$$

$$\left. + \int_0^1 \left[2h\left(b^t G^{1-t}\right) - h(b)\right] f'\left(b^t G^{1-t}\right) b^t G^{1-t} dt \right\}.$$

Proof. We calculate the integrals on the right side of (7), as follows

$$\begin{aligned} I_1 &= \int_0^1 \left[2h\left(a^t G^{1-t}\right) - h(b)\right] d\left(f\left(a^t G^{1-t}\right)\right) \\ &= \left[2h\left(a^t G^{1-t}\right) - h(b)\right] f\left(a^t G^{1-t}\right)\Big|_0^1 \\ &- 2\ln\left(\frac{a}{G}\right) \int_0^1 f\left(a^t G^{1-t}\right) h'\left(a^t G^{1-t}\right) a^t G^{1-t} dt \end{aligned}$$

and

$$\begin{aligned} I_2 &= \int_0^1 \left[2h\left(b^t G^{1-t}\right) - h(b)\right] d\left(f\left(b^t G^{1-t}\right)\right) \\ &= \left[2h\left(b^t G^{1-t}\right) - h(b)\right] f\left(b^t G^{1-t}\right)\Big|_0^1 \\ &- 2\ln\left(\frac{b}{G}\right)\int_0^1 f\left(b^t G^{1-t}\right) h'\left(b^t G^{1-t}\right) b^t G^{1-t} dt. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{I_1+I_2}{2} &= \left[h(b)-2h(a)\right]\frac{f(a)}{2} + h(b)\frac{f(b)}{2} \\ &- \frac{\ln b - \ln a}{2}\left\{\int_0^1 f\left(a^t G^{1-t}\right) h'\left(a^t G^{1-t}\right) a^t G^{1-t} dt \right. \\ &\left. + \int_0^1 f\left(b^t G^{1-t}\right) h'\left(b^t G^{1-t}\right) b^t G^{1-t} dt\right\}. \end{aligned} \tag{8}$$

This completes the proof of the lemma. □

Theorem 2.1. *Let $f : I \subseteq \mathbb{R}_+ = (0,\infty) \longrightarrow \mathbb{R}$ be differentiable function on I^o and $a, b \in I^o$ with $a < b$. If $h : [a,b] \longrightarrow [0,\infty)$ is a differentiable function and $|f'|$ is GA-convex on $[a,b]$, the following inequality holds*

$$\left|\left[h(b)-2h(a)\right]\frac{f(a)}{2} + h(b)\frac{f(b)}{2} - \int_a^b f(x)h'(x)dx\right| \tag{9}$$

$$\leq \frac{\ln b - \ln a}{4}\left[\zeta_1(a,b)\left|f'(a)\right| + \zeta_2(a,b)\left|f'(G)\right| + \zeta_3(a,b)\left|f'(b)\right|\right],$$

where

$$\zeta_1(a,b) = \int_0^1 t a^t G^{1-t}\left|2h\left(a^t G^{1-t}\right) - h(b)\right| dt, \tag{10}$$

$$\zeta_2(a,b) = \int_0^1 (1-t) a^t G^{1-t}\left|2h\left(a^t G^{1-t}\right) - h(b)\right| dt \tag{11}$$

$$+ \int_0^1 (1-t) b^t G^{1-t} \left| 2h\left(b^t G^{1-t}\right) - h(b) \right| dt$$

$$\zeta_3(a,b) = \int_0^1 t b^t G^{1-t} \left| 2h\left(b^t G^{1-t}\right) - h(b) \right| dt.$$

and

Proof. We get the following inequality by taking the absolute value on both sides of the equality in (7):

$$\left| [h(b) - 2h(a)] \frac{f(a)}{2} + h(b)\frac{f(b)}{2} - \int_a^b f(x)h'(x)dx \right| \tag{12}$$

$$\leq \frac{\ln b - \ln a}{4} \left\{ \int_0^1 \left| 2h\left(a^t G^{1-t}\right) - h(b) \right| \left| f'\left(a^t G^{1-t}\right) a^t G^{1-t} \right| dt \right.$$

$$\left. + \int_0^1 \left| 2h\left(b^t G^{1-t}\right) - h(b) \right| \left| f'\left(b^t G^{1-t}\right) b^t G^{1-t} \right| dt \right\}.$$

Since $|f'|$ is GA-convex on $[a,b]$ in (12), we have for all $t \in [a,b]$ that

$$\left| [h(b) - 2h(a)] \frac{f(a)}{2} + h(b)\frac{f(b)}{2} - \int_a^b f(x)h'(x)dx \right| \tag{13}$$

$$\leq \frac{\ln b - \ln a}{4} \left\{ \int_0^1 \left| 2h\left(a^t G^{1-t}\right) - h(b) \right| \left[t\left|f'(a)\right| + (1-t)\left|f'(G)\right| \right] a^t G^{1-t} dt \right.$$

$$\left. + \int_0^1 \left| 2h\left(b^t G^{1-t}\right) - h(b) \right| \left[t\left|f'(b)\right| + (1-t)\left|f'(G)\right| \right] b^t G^{1-t} dt \right\}.$$

This completes the proof of the theorem. □

Corollary 2.1. *Suppose that $g : [a,b] \longrightarrow [0,\infty)$ is a continuous positive mapping and geometrically symmetric with respect to $\sqrt{ab}$ (i.e. $g\left(\frac{ab}{x}\right) = g(x)$ holds for all $x \in [a,b]$ with $a < b$). Choosing $h(x) = \int_a^x \left[\left(\ln \frac{b}{t}\right)^{\alpha-1} + \left(\ln \frac{t}{a}\right)^{\alpha-1} \right] \frac{g(t)}{t} dt$ for all $x \in [a,b]$ and $\alpha > 0$ in Theorem 5, we obtain*

$$\left| \left(\frac{f(a)+f(b)}{2} \right) \left[J_{a+}^{\alpha} g(b) + J_{b-}^{\alpha} g(a) \right] - \left[J_{a+}^{\alpha} (fg)(b) + J_{b-}^{\alpha} (fg)(a) \right] \right| \tag{14}$$

$$\leq \frac{(\ln b-\ln a)^{\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+1)}\|g\|_{\infty}\left[C_1(\alpha)\left|f'(a)\right|+C_2(\alpha)\left|f'(G)\right|+C_3(\alpha)\left|f'(b)\right|\right],$$

where

$$\begin{aligned} C_1(\alpha) &= \int_0^1 \left[(1+t)^{\alpha}-(1-t)^{\alpha}\right] t a^t G^{1-t} dt, \\ C_2(\alpha) &= \int_0^1 (1-t)\left[(1+t)^{\alpha}-(1-t)^{\alpha}\right]\left[a^t G^{1-t}+b^t G^{1-t}\right] dt \end{aligned}$$

and

$$C_3(\alpha)=\int_0^1 \left[(1+t)^{\alpha}-(1-t)^{\alpha}\right] t b^t G^{1-t} dt.$$

Specially, if we use Lemma 1 in (14), for $0<\alpha\leq 1$, *we have*

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a^+}^{\alpha}g(b)+J_{b^-}^{\alpha}g(a)\right]-\left[J_{a^+}^{\alpha}(fg)(b)+J_{b^-}^{\alpha}(fg)(a)\right]\right| \tag{15}$$

$$\leq \frac{(\ln b-\ln a)^{\alpha+1}}{2\Gamma(\alpha+1)}\|g\|_{\infty}\left[D_1(\alpha)\left|f'(a)\right|+D_2(\alpha)\left|f'(G)\right|+D_3(\alpha)\left|f'(b)\right|\right]$$

where

$$\begin{aligned} D_1(\alpha) &= \int_0^1 t^{\alpha+1} a^t G^{1-t}, \\ D_2(\alpha) &= \int_0^1 \left[(1-t)t^{\alpha}a^t G^{1-t}+(1-t)t^{\alpha}b^t G^{1-t}\right] dt \end{aligned}$$

and

$$D_3(\alpha)=\int_0^1 t^{\alpha+1} b^t G^{1-t}.$$

Proof. If we take $h(x)=\int_a^x\left[\left(\ln\frac{b}{t}\right)^{\alpha-1}+\left(\ln\frac{t}{a}\right)^{\alpha-1}\right]\frac{g(t)}{t}dt$ for all $x\in[a,b]$ in the inequality (9), we have

$$\left|\Gamma(\alpha)\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a^+}^{\alpha}g(b)+J_{b^-}^{\alpha}g(a)\right]-\Gamma(\alpha)\left[J_{a^+}^{\alpha}(fg)(b)+J_{b^-}^{\alpha}(fg)(a)\right]\right| \tag{16}$$

$$\leq \frac{\ln b-\ln a}{4}\left\{\begin{array}{l} \int\limits_0^1 \left|\begin{array}{c} 2\int\limits_a^{a^tG^{1-t}}\left[\left(\ln\frac{b}{x}\right)^{\alpha-1}+\left(\ln\frac{x}{a}\right)^{\alpha-1}\right]\frac{g(x)}{x}dx \\ -\int\limits_a^b\left[\left(\ln\frac{b}{x}\right)^{\alpha-1}+\left(\ln\frac{x}{a}\right)^{\alpha-1}\right]\frac{g(x)}{x}dx \end{array}\right| \\ \times\left[t\left|f'(a)\right|+(1-t)\left|f'(G)\right|\right]a^tG^{1-t}dt \\ +\int\limits_0^1 \left|\begin{array}{c} 2\int\limits_a^{b^tG^{1-t}}\left[\left(\ln\frac{b}{x}\right)^{\alpha-1}+\left(\ln\frac{x}{a}\right)^{\alpha-1}\right]\frac{g(x)}{x}dx \\ -\int\limits_a^b\left[\left(\ln\frac{b}{x}\right)^{\alpha-1}+\left(\ln\frac{x}{a}\right)^{\alpha-1}\right]\frac{g(x)}{x}dx \end{array}\right| \\ \times\left[t\left|f'(b)\right|+(1-t)\left|f'(G)\right|\right]b^tG^{1-t}dt \end{array}\right\}.$$

Since $g(x)$ is geometrically symmetric with respect to $x=\sqrt{ab}$, we have

$$\begin{aligned} &\left|2\int\limits_a^{a^tG^{1-t}}\left[\left(\ln\frac{b}{x}\right)^{\alpha-1}+\left(\ln\frac{x}{a}\right)^{\alpha-1}\right]\frac{g(x)}{x}dx\right. \\ &-\left.\int\limits_a^b\left[\left(\ln\frac{b}{x}\right)^{\alpha-1}+\left(\ln\frac{x}{a}\right)^{\alpha-1}\right]\frac{g(x)}{x}dx\right| \\ &=\left|\int\limits_{a^tG^{1-t}}^{b^tG^{1-t}}\left[\left(\ln\frac{b}{x}\right)^{\alpha-1}+\left(\ln\frac{x}{a}\right)^{\alpha-1}\right]\frac{g(x)}{x}dx\right| \end{aligned} \tag{17}$$

and

$$\begin{aligned} &\left|2\int\limits_a^{b^tG^{1-t}}\left[\left(\ln\frac{b}{x}\right)^{\alpha-1}+\left(\ln\frac{x}{a}\right)^{\alpha-1}\right]\frac{g(x)}{x}dx\right. \\ &-\left.\int\limits_a^b\left[\left(\ln\frac{b}{x}\right)^{\alpha-1}+\left(\ln\frac{x}{a}\right)^{\alpha-1}\right]\frac{g(x)}{x}dx\right| \\ &=\left|\int\limits_{a^tG^{1-t}}^{b^tG^{1-t}}\left[\left(\ln\frac{b}{x}\right)^{\alpha-1}+\left(\ln\frac{x}{a}\right)^{\alpha-1}\right]\frac{g(x)}{x}dx\right|. \end{aligned} \tag{18}$$

for all $t\in[0,1]$. By using (17)-(18) in (16), we have

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]-\left[J_{a+}^{\alpha}\left(fg\right)(b)+J_{b-}^{\alpha}\left(fg\right)(a)\right]\right| \tag{19}$$

$$\leq \frac{\ln b-\ln a}{4\Gamma(\alpha)}\left\{\int_0^1 \left|\int_{a^tG^{1-t}}^{b^tG^{1-t}}\left[\left(\ln\tfrac{b}{x}\right)^{\alpha-1}+\left(\ln\tfrac{x}{a}\right)^{\alpha-1}\right]\tfrac{g(x)}{x}dx\right| \times\left[t\left|f'(a)\right|+(1-t)\left|f'(G)\right|\right]a^tG^{1-t}dt\right.$$
$$\left.+\int_0^1 \left|\int_{a^tG^{1-t}}^{b^tG^{1-t}}\left[\left(\ln\tfrac{b}{x}\right)^{\alpha-1}+\left(\ln\tfrac{x}{a}\right)^{\alpha-1}\right]\tfrac{g(x)}{x}dx\right| \times\left[t\left|f'(b)\right|+(1-t)\left|f'(G)\right|\right]b^tG^{1-t}dt\right\}$$
$$\leq \frac{\ln b-\ln a}{4\Gamma(\alpha)}\left\|g\right\|_\infty\left\{\int_0^1 \left[\int_{a^tG^{1-t}}^{b^tG^{1-t}}\left[\left(\ln\tfrac{b}{x}\right)^{\alpha-1}+\left(\ln\tfrac{x}{a}\right)^{\alpha-1}\right]\tfrac{1}{x}dx\right] \times\left[t\left|f'(a)\right|+(1-t)\left|f'(G)\right|\right]a^tG^{1-t}dt\right.$$
$$\left.+\int_0^1 \left[\int_{a^tG^{1-t}}^{b^tG^{1-t}}\left[\left(\ln\tfrac{b}{x}\right)^{\alpha-1}+\left(\ln\tfrac{x}{a}\right)^{\alpha-1}\right]\tfrac{1}{x}dx\right] \times\left[t\left|f'(b)\right|+(1-t)\left|f'(G)\right|\right]b^tG^{1-t}dt\right\}.$$

In the last inequality, we calculate integrals simply as follows:

$$\begin{aligned}
&\int_{a^tG^{1-t}}^{b^tG^{1-t}}\left[\left(\ln\frac{b}{x}\right)^{\alpha-1}+\left(\ln\frac{x}{a}\right)^{\alpha-1}\right]\frac{1}{x}dx \qquad (20)\\
=&\int_{a^tG^{1-t}}^{b^tG^{1-t}}\left(\ln\frac{b}{x}\right)^{\alpha-1}\frac{1}{x}dx+\int_{a^tG^{1-t}}^{b^tG^{1-t}}\left(\ln\frac{x}{a}\right)^{\alpha-1}\frac{1}{x}dx\\
=&\frac{2.\left(\ln b-\ln a\right)^{\alpha}}{2^{\alpha}.\alpha}\left[(1+t)^{\alpha}-(1-t)^{\alpha}\right].
\end{aligned}$$

By Lemma 1, for $0<\alpha\leq 1$, we have

$$\begin{aligned}
&\int_{a^tG^{1-t}}^{b^tG^{1-t}}\left[\left(\ln\frac{b}{x}\right)^{\alpha-1}+\left(\ln\frac{x}{a}\right)^{\alpha-1}\right]\frac{1}{x}dx\\
=&\int_{a^tG^{1-t}}^{b^tG^{1-t}}\left(\ln\frac{b}{x}\right)^{\alpha-1}\frac{1}{x}dx+\int_{a^tG^{1-t}}^{b^tG^{1-t}}\left(\ln\frac{x}{a}\right)^{\alpha-1}\frac{1}{x}dx\\
&\leq\frac{2.\left(\ln b-\ln a\right)^{\alpha}}{\alpha}t^{\alpha}.
\end{aligned}$$

A combination of (19) and (20), we have (14) and (15). Thus the proof is completed. □

Corollary 2.2. *(1)If we take $\alpha=1$, we obtain the following Hermite-Hadamard-Fejer type inequality for GA-convex functions related to (15):*

$$\left|\left[\frac{f(a)+f(b)}{2}\right]\int_a^b \frac{g(x)}{x}dx-\int_a^b f(x)\frac{g(x)}{x}dx\right| \tag{21}$$
$$\leq \frac{(\ln b-\ln a)^2}{4}\left\|g\right\|_\infty \left[D_1(1)\left|f'(a)\right|+D_2(1)\left|f'(G)\right|+D_3(1)\left|f'(b)\right|\right],$$

where for $a,b>0$, we have

$$\begin{aligned} D_1(1) &= \int_0^1 t^2 a^t G^{1-t}dt=\frac{2}{\ln b-\ln a}\left\{-a-\frac{4a}{\ln b-\ln a}-\frac{8a-8G}{(\ln b-\ln a)^2}\right\}, \\ D_2(1) &= \int_0^1 t(1-t)a^tG^{1-t}dt+\int_0^1 t(1-t)b^tG^{1-t}dt \\ &= \frac{2}{\ln b-\ln a}\left\{\frac{2(a+b+2G)}{\ln b-\ln a}+\frac{8(a-b)}{(\ln b-\ln a)^2}\right\}, \\ D_3(1) &= \int_0^1 t^2 a^t G^{1-t}dt=\frac{2}{\ln b-\ln a}\left\{b-\frac{4b}{\ln b-\ln a}+\frac{8b-8G}{(\ln b-\ln a)^2}\right\}. \end{aligned}$$

(2)If we take $g(x)=1$ in (14), we obtain the following inequality

$$\left|\left(\frac{f(a)+f(b)}{2}\right)-\frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^\alpha}\left[J^\alpha_{a^+}f(b)+J^\alpha_{b^-}f(a)\right]\right| \tag{22}$$
$$\leq \frac{(\ln b-\ln a)}{2^{\alpha+2}}\left[C_1(\alpha)\left|f'(a)\right|+C_2(\alpha)\left|f'(G)\right|+C_3(\alpha)\left|f'(b)\right|\right], \qquad \alpha>0.$$

(3)If we take $g(x)=1$ and $\alpha=1$ in (15), we obtain the following inequality

$$\left|\left(\frac{f(a)+f(b)}{2}\right)-\frac{1}{(\ln b-\ln a)}\int_a^b \frac{f(x)}{x}dx\right| \tag{23}$$
$$\leq \frac{(\ln b-\ln a)}{4}\left[D_1(1)\left|f'(a)\right|+D_2(1)\left|f'(G)\right|+D_3(1)\left|f'(b)\right|\right].$$

Theorem 2.2. *Let $f: I\subseteq \mathbb{R}_+=(0,\infty)\longrightarrow \mathbb{R}$ be a differentiable function on I^o, $a,b\in I^o$ with $a<b$. If $h:[a,b]\longrightarrow[0,\infty)$ is a differentiable function and $|f'|^q$ is GA-convex on $[a,b]$ for $q\geq 1$, the following inequality holds*

$$\left|[h(b)-2h(a)]\frac{f(a)}{2}+h(b)\frac{f(b)}{2}-\int_a^b f(x)h'(x)dx\right| \tag{24}$$

$$\leq \frac{\ln b-\ln a}{4}\left\{\begin{array}{c}\left(\int\limits_0^1\left|2h(a^tG^{1-t})-h(b)\right|dt\right)^{1-\frac{1}{q}} \\ \times\left(\int\limits_0^1\begin{array}{c}\left(\left|2h(a^tG^{1-t})-h(b)\right|dt\right) \\ \left(ta^{qt}G^{q(1-t)}\left|f'(a)\right|^q+(1-t)a^{qt}G^{q(1-t)}\left|f'(G)\right|^q\right)\end{array}\right)^{\frac{1}{q}} \\ +\left(\int\limits_0^1\left|2h(b^tG^{1-t})-h(b)\right|dt\right)^{1-\frac{1}{q}} \\ \times\left(\int\limits_0^1\begin{array}{c}\left(\left|2h(b^tG^{1-t})-h(b)\right|dt\right)\times \\ \left(tb^{qt}G^{q(1-t)}\left|f'(b)\right|^q+(1-t)b^{qt}G^{q(1-t)}\left|f'(G)\right|^q\right)\end{array}\right)^{\frac{1}{q}}\end{array}\right\}.$$

Proof. Continuing from (13) in proof of Theorem 5, the power mean inequality and using the fact that $|f'|^q$ is GA-convex on $[a,b]$, we get the required result. This completes the proof of the theorem. □

Corollary 2.3. *Let $g:[a,b]\longrightarrow[0,\infty)$ be a positive continuous mapping and geometrically symmetric with respect to $\sqrt{ab}$ (i.e. $g\left(\frac{ab}{x}\right)=g(x)$ holds for all $x\in[a,b]$ with $a<b$). If $h(x)=\int\limits_a^x\left[\left(ln\frac{b}{t}\right)^{\alpha-1}+\left(ln\frac{t}{a}\right)^{\alpha-1}\right]\frac{g(t)}{t}dt$ for all $x\in[a,b]$, $\alpha>0$ in Theorem 6, we obtain*

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a^+}^{\alpha}g(b)+J_{b^-}^{\alpha}g(a)\right]-\left[J_{a^+}^{\alpha}\left(fg\right)(b)+J_{b^-}^{\alpha}\left(fg\right)(a)\right]\right| \tag{25}$$
$$\leq \frac{(\ln b-\ln a)^{\alpha+1}\left\|g\right\|_{\infty}}{2^{\alpha+1}\Gamma(\alpha+1)}\left(\frac{2^{\alpha+2}-2^2}{\alpha+1}\right)^{1-\frac{1}{q}}\times$$
$$\left[C_1(\alpha,q)\left|f'(a)\right|^q+C_2(\alpha,q)\left|f'(G)\right|^q+C_3(\alpha,q)\left|f'(b)\right|^q\right]^{\frac{1}{q}}$$

where for $q\geq1$, $\alpha>0$,

$$C_1(\alpha,q) = \int\limits_0^1\left[(1+t)^{\alpha}-(1-t)^{\alpha}\right]ta^{qt}G^{q(1-t)}dt,$$
$$C_2(\alpha,q) = \int\limits_0^1\left[(1+t)^{\alpha}-(1-t)^{\alpha}\right](1-t)\left(a^{qt}G^{q(1-t)}+b^{qt}G^{q(1-t)}\right)dt$$

and

$$C_3(\alpha,q)=\int\limits_0^1\left[(1+t)^{\alpha}-(1-t)^{\alpha}\right]tb^{qt}G^{q(1-t)}dt.$$

Proof. When we use the equality (20) in (24), we obtain

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a^+}^{\alpha}g(b)+J_{b^-}^{\alpha}g(a)\right]-\left[J_{a^+}^{\alpha}\left(fg\right)(b)+J_{b^-}^{\alpha}\left(fg\right)(a)\right]\right| \tag{26}$$

$$\leq \frac{(\ln b-\ln a)^{\alpha+1}\|g\|_{\infty}}{2^{\alpha+1}\Gamma(\alpha+1)}$$

$$\left\{\begin{array}{c} \times\left(\int_0^1 [(1+t)^{\alpha}-(1-t)^{\alpha}]\,dt\right)^{1-\frac{1}{q}} \\ \times\left(\int_0^1 \begin{array}{c}[(1+t)^{\alpha}-(1-t)^{\alpha}]\times \\ \left(ta^{qt}G^{q(1-t)}|f'(a)|^q+(1-t)a^{qt}G^{q(1-t)}|f'(G)|^q\right)dt\end{array}\right)^{\frac{1}{q}}\end{array}\right.$$

$$+\left.\begin{array}{c}\left(\int_0^1 [(1+t)^{\alpha}-(1-t)^{\alpha}]\,dt\right)^{1-\frac{1}{q}} \\ \times\left(\int_0^1 \begin{array}{c}[(1+t)^{\alpha}-(1-t)^{\alpha}]\times \\ \left(tb^{qt}G^{q(1-t)}|f'(b)|^q+(1-t)b^{qt}G^{q(1-t)}|f'(G)|^q\right)dt\end{array}\right)^{\frac{1}{q}}\end{array}\right\}$$

$$\leq \frac{(\ln b-\ln a)^{\alpha+1}\|g\|_{\infty}}{2^{\alpha+1}\Gamma(\alpha+1)}\left(\frac{2^{\alpha+1}-2}{\alpha+1}\right)^{1-\frac{1}{q}}$$

$$\left[\begin{array}{c}\times\left(\int_0^1 \begin{array}{c}[(1+t)^{\alpha}-(1-t)^{\alpha}]\times \\ \left[ta^{qt}G^{q(1-t)}|f'(a)|^q+(1-t)a^{qt}G^{q(1-t)}|f'(G)|^q\right]\end{array} dt\right)^{\frac{1}{q}} \\ +\left(\int_0^1 \begin{array}{c}[(1+t)^{\alpha}-(1-t)^{\alpha}]\times \\ \left[tb^{qt}G^{q(1-t)}|f'(b)|^q+(1-t)b^{qt}G^{q(1-t)}|f'(G)|^q\right]\end{array} dt\right)^{\frac{1}{q}}\end{array}\right].$$

By using the inequality $a^r+b^r\leq 2^{1-r}(a+b)^r$ for $a,b>0, r\leq 1$, we have

$$\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a^+}^{\alpha}g(b)+J_{b^-}^{\alpha}g(a)\right]-\left[J_{a^+}^{\alpha}(fg)(b)+J_{b^-}^{\alpha}(fg)(a)\right] \tag{27}$$

$$\leq \frac{(\ln b-\ln a)^{\alpha+1}\|g\|_{\infty}}{2^{\alpha+1}\Gamma(\alpha+1)}\left(\frac{2^{\alpha+2}-2^2}{\alpha+1}\right)^{1-\frac{1}{q}}$$

$$\times\left[\int_0^1\left(\begin{array}{c}[(1+t)^{\alpha}-(1-t)^{\alpha}]\,ta^{qt}G^{q(1-t)}|f'(a)|^q+ \\ [(1+t)^{\alpha}-(1-t)^{\alpha}](1-t)\begin{pmatrix}a^{qt}G^{q(1-t)} \\ +b^{qt}G^{q(1-t)}\end{pmatrix}|f'(G)|^q \\ +[(1+t)^{\alpha}-(1-t)^{\alpha}]\,tb^{qt}G^{q(1-t)}|f'(b)|^q\end{array}\right)dt\right]^{\frac{1}{q}}.$$

□

Corollary 2.4. *When $\alpha=1$ and $g(x)=\frac{1}{\ln b-\ln a}$ are taken in Corollary 3, we obtain*

$$\left|\left(\frac{f(a)+f(b)}{2}\right)-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}dx\right| \tag{28}$$

$$\leq\frac{(\ln b-\ln a)}{2^{2+\frac{1}{q}}}\left[C_1(1,q)|f'(a)|^q+C_2(1,q)|f'(G)|^q+C_3(1,q)|f'(b)|^q\right]^{\frac{1}{q}}.$$

References

[1] S.S. Dragomir, Hermite-Hadamard's type inequalities for convex funtions of selfadjoint operators in Hilbert spaces, Linear Algebra Appl. 436 (2012), no. 5, 1503-1515.
[2] S.S. Dragomir and C.E.M. Pearce, Selected topics on Hermite-Hadamard type inequalities and applications, RGMIA Monographs, 2000. Available online at http://rgmia.vu.edu.au/monographs/hermite hadamard.html.
[3] J. Hua, B.-Y. Xi, and F. Qi, Hermite-Hadamard type inequalities for geometrically s-convex functions, Commun.Korean Math.Soc.29 (2014), No.1, pp.51-63.
[4] D-Y. Hwang, Some inequalities for differentiable convex mapping with application to weighted trapezoidal formula and higher moments of random variables, Applied Mathematics and Computation, 217 (2011), 9598-9605.
[5] İ. İşcan, New estimates on generalization of some integral inequalities for s-convex functions and their applications, International Journal of Pure and Applied Mathematics, 86, No.4 (2013).
[6] İ. İşcan, Hermite-Hadamard type inequalities for GA-s-convex functions, Le Matematiche, LXIX (2014)-Fasc. II, pp. 129-146.
[7] A. P. Ji, T. Y. Zhang, F. Qi, Integral Inequalities of Hermite Hadamard Type (α, m)-GA convex Functions, Journal of Function Space and Applications, 2013 (2013), Article ID 823856, 8 pages.
[8] A. A. Kilbas H. M. Srivastava, J. J. Trujillo, Theory and applications of fractional differential equations.Elsevier, Amsterdam (2006)
[9] M.A. Latif, S. S. Dragomir, E. Momoniat, Some Fejér type integral inequalities related with geometrically-arithmetically-convex functions with applications, (Submitted).
[10] M.A. Latif, New Hermite Hadamard type integral inequalities for GA-convex functions with applications. Volume 34, Issue 4 (Nov 2014).
[11] C. P. Niculescu, Convexity according to the geometric mean, Math. Inequal. Appl. 3 (2) (2000), 155–167. Available online at http://dx.doi.org/10.7153/mia-03-19.
[12] C. P. Niculescu, Convexity according to means, Math. Inequal. Appl. 6 (4) (2003), 571–579. Available online at http://dx.doi.org/10.7153/mia-06-53.
[13] A. P. Prudnikov, Y. A. Brychkov, O. J. Marichev, Integral and series, Elementary Functions, Vol. 1,Nauka, Moscow, 1981.
[14] Y. Shuang, H.-P. Yin, and F. Qi, Hermite-Hadamard type integral inequalities for geometric-arithmetically s-convex functions, Analysis (Munich) 33 (2) (2013), 197-208. Available online at http://dx.doi.org/10.1524/anly.2013.1192.
[15] T.-Y. Zhang, A.-P. Ji and F. Qi, Some inequalities of Hermite-Hadamard type for GA-convex functions with applications to means, Le Matematiche, Vol. LXVIII (2013) – Fasc. I, pp. 229–239. doi: 10.4418/2013.68.1.17
[16] X.-M. Zhang, Y.-M. Chu, and X.-H. Zhang, The Hermite-Hadamard Type Inequality of GA-Convex Functions and Its Application, Journal of Inequalities and Applications, Volume 2010, Article ID 507560, 11 pages. doi:10.1155/2010/507560.