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Property (gab) through localized SVEP

Pietro Aiena[a] and Salvatore Triolo[a]

Abstract. In this article we study the property (gab) for a bounded linear operator $T \in L(X)$ on a Banach space X which is a stronger variant of Browder's theorem. We shall give several characterizations of property (gab). These characterizations are obtained by using typical tools from local spectral theory. We also show that property (gab) holds for large classes of operators and prove the stability of property (gab) under some commuting perturbations.

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1. Introduction

Property (gab), for bounded linear operators $T \in L(X)$ defined on a Banach spaces X, has been introduced by Berkani and Zariouh in [24]. Other variants of Browder's theorem are property (b), property (ab), and property (gb), that have been introduced in [22] and [24]. All these properties may be thought as stronger versions than the classical Browder's theorem or of a-Browder's theorem. The properties (b), (ab) and (gb), and Browder type theorems have been also studied, by using methods of local spectral theory in [9], [10], [7], [5], [8], and [11]. In this paper we continue, in the same vein, the study of property (gab) by means of methods of local spectral theory, and, in particular, we show that $T \in L(X)$ satisfies property (gab) precisely when the dual T^* has the SVEP at the points of a certain subset $\Sigma_a^g(T)$ of the spectrum $\sigma(T)$, or equivalently $\Sigma_a^g(T)$ consists of isolated points of $\sigma(T)$. The relationships between the various versions of Browder's type theorem are then obtained by using easy inclusions of suitable subsets of the spectrum.

Further, property (gab) for T may be characterized by means of some properties

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[a]DEIM, Università di Palermo Viale delle Scienze, I-90128 Palermo (Italy).
e-mail: pietro.aiena@unipa.itz, salvatore.triolo@unipa.it

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of the analytic core $K(\lambda I - T)$, or of the hyper-range $(\lambda I - T)^\infty(X)$, as λ ranges in $\Sigma_a^g(T)$. It is also shown that the property (gab) for T^* holds if and only if the quasi-nilpotent part $H_0(\lambda I - T)$ is finite-dimensional as λ ranges in $\Sigma_a^g(T^*)$. In the last part we consider the permanence of property (gab) under some commuting perturbations and the characterization of this property assuming some polaroid assumptions for T.

2. Definitions and preliminary results

Let $T \in L(X)$ be a bounded linear operator defined on an infinite-dimensional complex Banach space X, and denote by $\alpha(T)$ and $\beta(T)$, the dimension of the kernel $\ker T$ and the codimension of the range $R(T) := T(X)$, respectively. Let

$$\Phi_+(X) := \{T \in L(X) : \alpha(T) < \infty \text{ and } T(X) \text{ is closed}\}$$

denote the class of all *upper semi-Fredholm* operators, and let

$$\Phi_-(X) := \{T \in L(X) : \beta(T) < \infty\}$$

denote the class of all *lower semi-Fredholm* operators. If $T \in \Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X)$, the *index* of T is defined by $\text{ind}\,(T) := \alpha(T) - \beta(T)$. If $\Phi(X) := \Phi_+(X) \cap \Phi_-(X)$ denotes the set of all *Fredholm* operators, the set of *Weyl operators* is defined by

$$W(X) := \{T \in \Phi(X) : \text{ind}\, T = 0\},$$

the class of *upper semi-Weyl operators* is defined by

$$W_+(X) := \{T \in \Phi_+(X) : \text{ind}\, T \le 0\},$$

and class of *lower semi-Weyl operators* is defined by

$$W_-(X) := \{T \in \Phi_-(X) : \text{ind}\, T \ge 0\}.$$

Clearly, $W(X) = W_+(X) \cap W_-(X)$. The classes of operators above defined generate the following spectra: the *Weyl spectrum*, defined by

$$\sigma_{\mathrm{w}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W(X)\};$$

and the *upper semi-Weyl spectrum*, defined by

$$\sigma_{\mathrm{uw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W_+(X)\}.$$

Let $p(T) := p$ be the *ascent* of an operator T; i.e. the smallest non-negative integer p such that $\ker T^p = \ker T^{p+1}$. If such integer does not exist we put $p(T) = \infty$. Analogously, let $q(T) := q$ be the *descent* of T; i.e the smallest non-negative integer q such that $T^q(X) = T^{q+1}(X)$, and if such integer does not exist we put $q(T) = \infty$. It is well known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$, see [1, Theorem 3.3]. Moreover, if $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ then λ is a pole of the resolvent, see [28, Proposition 50.2], and in particular an isolated point of $\sigma(T)$.

The class of all *Browder operators* is defined

$$B(X) := \{T \in \Phi(X) : p(T), q(T) < \infty\};$$

while the class of all *upper semi-Browder operators* is defined

$$B_+(X) := \{T \in \Phi_+(X) : p(T) < \infty\}.$$

The *Browder spectrum* is denoted by $\sigma_{\rm b}(T)$, while the *upper semi- Browder spectrum* is denoted by $\sigma_{\rm ub}(T)$. Obviously, $B(X) \subseteq W(X)$ and $B_+(X) \subseteq W_+(X)$, see [1, Theorem 3.4], so $\sigma_{\rm w}(T) \subseteq \sigma_{\rm b}(T)$ and $\sigma_{\rm uw}(T) \subseteq \sigma_{\rm ub}(T)$.

In the sequel we denote by $\sigma_{\rm a}(T)$ the *approximate point spectrum*, defined by

$$\sigma_{\rm a}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below}\},$$

where an operator is said to be *bounded below* if it is injective and has closed range. The classical *surjective spectrum* of T is denoted by $\sigma_{\rm s}(T)$.

An operator $T \in L(X)$ is said to satisfy *Browder's theorem* if $\sigma_{\rm w}(T) = \sigma_{\rm b}(T)$, or equivalently $\Delta(T) = p_{00}(T)$, where

$$\Delta(T) := \sigma(T) \setminus \sigma_{\rm w}(T) \quad \text{and} \quad p_{00}(T) = \sigma(T) \setminus \sigma_{\rm b}(T).$$

The operator $T \in L(X)$ is said to satisfy *a- Brower's theorem* if $\sigma_{\rm uw}(T) = \sigma_{\rm ub}(T)$, or equivalently $\Delta_a(T) = p^a_{00}(T)$, where

$$\Delta_a(T) := \sigma_{\rm a}(T) \setminus \sigma_{\rm uw}(T) \quad \text{and} \quad p^a_{00}(T) := \sigma_{\rm a}(T) \setminus \sigma_{\rm ub}(T).$$

It is known that a-Browder's theorem entails Browder's theorem.

Semi-Fredholm operators have been generalized by Berkani ([17], [20] and [19]) in the following way: for every $T \in L(X)$ and a nonnegative integer n let us denote by $T_{[n]}$ the restriction of T to $T^n(X)$, viewed as a map from the space $T^n(X)$ into itself (we set $T_{[0]} = T$). $T \in L(X)$ is said to be *semi B-Fredholm*, (resp. *B-Fredholm, upper semi B-Fredholm*, *lower semi B-Fredholm,*) if for some integer $n \geq 0$ the range $T^n(X)$ is closed and $T_{[n]}$ is a semi-Fredholm operator (resp. Fredholm, upper semi-Fredholm, lower semi-Fredholm). In this case $T_{[m]}$ is a semi-Fredholm operator for all $m \geq n$ ([20]) with the same index of $T_{[n]}$. This enables one to define the index of a semi B-Fredholm as ind $T = $ ind $T_{[n]}$.

A bounded operator $T \in L(X)$ is said to be *B-Weyl* (respectively, *upper semi B-Weyl*, *lower semi B-Weyl*) if for some integer $n \geq 0$ the range $T^n(X)$ is closed and $T_{[n]}$ is Weyl (respectively, upper semi-Weyl. lower semi-Weyl). The *B-Weyl spectrum* is defined by

$$\sigma_{\rm bw}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not B-Weyl}\},$$

and the *upper semi B-Weyl spectrum* of T is defined by

$$\sigma_{\rm ubw}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi B-Weyl}\}.$$

Analogously, the *lower semi B-Weyl spectrum* of T is defined by

$$\sigma_{\rm lbw}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not lower semi B-Weyl}\}.$$

The concept of Drazin invertibility has been introduced in a more abstract setting than operator theory. In the case of the Banach algebra $L(X)$, $T \in L(X)$ is said to be *Drazin invertible* (with a finite index) if $p(T) = q(T) < \infty$, and this is equivalent to saying that $T = T_0 \oplus T_1$, where T_0 is invertible and T_1 is nilpotent, see [29, Corollary 2.2] and [32, Prop. A]. Every B-Fredholm operator T admits the representation $T = T_0 \oplus T_1$, where T_0 is Fredholm and T_1 is nilpotent [19], so every Drazin invertible operator is B-Fredholm. Drazin invertibility for bounded operators suggests the following definition:

The concept of pole of the resolvent suggests the following definition:

Definition 2.1. *An operator $T \in L(X)$, is said to be left Drazin invertible if $p := p(T) < \infty$ and $T^{p+1}(X)$ is closed. $T \in L(X)$, is said to be right Drazin invertible if $q := q(T) < \infty$ and $T^q(X)$ is closed. If $\lambda I - T$ is left Drazin invertible and $\lambda \in \sigma_{\mathrm{a}}(T)$ then λ is said to be a* left pole. *A left pole λ is said to have finite rank if $\alpha(\lambda I - T) < \infty$. If $\lambda I - T$ is right Drazin invertible and $\lambda \in \sigma_{\mathrm{s}}(T)$ then λ is said to be a* right pole. *A right pole λ is said to have finite rank if $\beta(\lambda I - T) < \infty$.*

The *Drazin spectrum* is then defined as

$$\sigma_{\mathrm{d}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Drazin invertible}\},$$

the *left Drazin spectrum* is defined as

$$\sigma_{\mathrm{ld}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not left Drazin invertible}\},$$

while the *right Drazin spectrum* is defined as

$$\sigma_{\mathrm{rd}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not right Drazin invertible}\},$$

It should be noted that the concepts of Drazin invertibility may be expressed in terms of B-Fredholm theory. Indeed, $T \in L(X)$ is Drazin invertible (respectively, left Drazin invertible, right Drazin invertible) if and only if T is B-Browder (respectively, upper semi B-Browder, lower semi B-Browder), see [18] or [6].

In the following we need the following elementary lemma:

Lemma 2.2. *Let $T \in L(X)$.*

(i) *If T is upper semi B-Fredholm and $\alpha(T) < \infty$ then T is upper semi-Fredholm.*

(ii) *If T is injective and upper semi B-Fredholm then T is bounded below.*

Proof. (i) If T is upper semi B-Fredholm then there exists $n \in \mathbb{N}$ such that $T^n(X)$ is closed. By assumption $\alpha(T) < \infty$, and this implies that $\alpha(T^n) < \infty$, so T^n is upper semi-Fredholm and by the classical Fredholm theory we deduce that T is upper semi-Fredholm.

(ii) Being $\alpha(T) = 0$ and T upper semi-Fredholm from part (i), then $T(X)$ is closed, so T is bounded below. ■

Define

$$\Gamma(T) := \{n \in \mathbb{N} : m \geq n, m \in \mathbb{N} \Rightarrow T^n(X) \cap \ker T \subseteq T^m(X) \cap \ker T\}.$$

The *degree of stable iteration* is defined as $\mathrm{dis}(T) := \inf \Gamma(T)$ if $\Gamma(T) \neq \emptyset$, while $\mathrm{dis}(T) = \infty$ if $\Gamma(T) = \emptyset$.

Definition 2.3. *$T \in L(X)$ is said to be* quasi-Fredholm of degree d, *in symbol $T \in QF(d)$, if there exists $d \in \mathbb{N}$ such that:*

(a) *$dis(T) = d$,*

(b) $T^n(X)$ *is a closed subspace of* X *for each* $n \geq d$,
(c) $T(X) + \ker T^d$ *is a closed subspace of* X.

It is known that $T \in QF(d)$ if and only if $T^* \in QF(d)$, see [34]. It is also known that every semi B-Fredholm operator, and in particular every left or right Drazin invertible operator, is quasi-Fredholm [20].

An operator $T \in L(X)$ is said to have *the single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open disc U of λ_0, the only analytic function $f : U \to X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in U$ is the function $f \equiv 0$. An operator $T \in L(X)$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$. Evidently, an operator $T \in L(X)$ has SVEP at every point of the resolvent $\rho(T) := \mathbf{C} \setminus \sigma(T)$, and both T and T^* have SVEP at the isolated points of the spectrum. From definition of SVEP we easily obtain:

$$\sigma_{\mathrm{a}}(T) \text{ does not cluster at } \lambda \Rightarrow T \text{ has SVEP at } \lambda,$$

and, by duality,

$$\sigma_{\mathrm{s}}(T) \text{ does not cluster at } \lambda \Rightarrow T^* \text{ has SVEP at } \lambda.$$

In the sequel we shall freely use the following characterizations of SVEP for quasi-Fredholm operators, in particular semi B-Fredholm operators . These characterizations have been proved in [2].

Theorem 2.4. *Let* $T \in L(X)$ *and suppose that* $\lambda_0 I - T$ *is quasi-Fredholm. Then the following statements are equivalent:*

(i) T *has SVEP at* λ_0;
(ii) $p(\lambda_0 I - T) < \infty$;
(iii) $\sigma_{\mathrm{a}}(T)$ *does not cluster at* λ_0;
(iv) $H_0(\lambda_0 I - T)$ *is closed;*
(v) *there exists* $\nu \in \mathbb{N}$ *such that* $H_0(\lambda_0 I - T) = \ker(\lambda_0 I - T)^\nu$.

Dually, the following statements are equivalent:

(v) T^* *has SVEP at* λ_0;
(vii) $q(\lambda_0 I - T) < \infty$;
(viii) $\sigma_{\mathrm{s}}(T)$ *does not cluster at* λ_0;
(ix) *there exists* $\nu \in \mathbb{N}$ *such that* $K(\lambda_0 I - T) = (\lambda_0 I - T)^\nu(X)$.

In the sequel we shall need the following result:

Theorem 2.5. *For an operator* $T \in L(X)$ *the following statements hold:*

(i) *If* $\lambda I - T$ *is upper semi B-Weyl and* T^* *has SVEP at* λ *then,* $\lambda I - T$ *is Drazin invertible.*

(ii) *If* $\lambda I - T$ *is lower semi B-Weyl and* T *has SVEP at* λ, *then* $\lambda I - T$ *is Drazin invertible.*

(iii) *If* $\lambda I - T$ *is B-Weyl and* T *and* T^* *has either SVEP at* λ, *then* $\lambda I - T$ *is Drazin invertible.*

Proof. (i) There is no harm to suppose that $\lambda = 0$. Suppose that T is upper semi B-Weyl and T^* has SVEP at 0. Then, by Theorem 2.4, $q := q(T) < \infty$. Then there exists $n \in \mathbb{N}$ such that $T^n(X)$ is closed and $T_{[n]}$ is upper semi-Weyl, i.e. $T_{[n]}$ is upper semi-Fredholm with $\operatorname{ind} T_{[n]} \leq 0$. Moreover, $T_{[m]}$ is upper semi-Weyl for all $m \geq n$. Consider the operator $T_{[m]} : T^m(X) \to T^m(X)$. It is evident that

$$R(T_{[m]}) = T^{m+1}(X) = T^m(X), \quad \text{for all } m \geq q,$$

thus $T_{[m]}$ is onto, i.e $q(T_{[m]}) = 0$. Now, choose $m \geq \max\{n, q\}$ then $T_{[m]}$ is both onto and upper semi- Weyl. Then $\operatorname{ind} T_{[m]} = \alpha(T_{[m]}) \geq 0$, from which we obtain $\operatorname{ind} T_{[m]} = 0$ and hence $\alpha(T_{[m]}) = \beta(T_{[m]}) = 0$, i.e., $T_{[m]}$ is invertible. Consequently, $T^k_{[m]}$ is invertible for all $k \in \mathbb{N}$. Therefore, $\ker T^k \cap T^m(X) = \ker T^k_{[m]} = \{0\}$ for all $k \in \mathbb{N}$. This implies, by [1, Lemma 3.2, part (i)], that $p(T) < \infty$. Therefore, T is Drazin invertible.

(ii) Also here we can assume that $\lambda = 0$. Assume that T is lower semi B-Weyl and T has SVEP at 0. Then, by Theorem 2.4, $p = p(T) < \infty$. Then there exists $n \in \mathbb{N}$ such that $T^n(X)$ is closed and $T_{[n]}$ is lower semi-Fredholm and $\operatorname{ind} T_{[n]} \geq 0$. Moreover, $T_{[m]}$ is lower semi-Weyl for all $m \geq n$. By [1, Lemma 3.2, part (i)] the condition $p := p(T) < \infty$ entails that $\ker T_{[m]} = \ker T \cap T^m(X) = \{0\}$ for all natural $m \geq p$. Choosing $m \geq \max\{n, p\}$ then $T_{[m]}$ is both injective and lower semi-Weyl, hence $\operatorname{ind} T_{[m]} = -\beta(T_{[m]}) \leq 0$, so $\operatorname{ind} T_{[m]} = 0$, and hence $T_{[m]}$ is invertible, since $\alpha(T_{[m]}) = \beta(T_{[m]}) = 0$. Consequently,

$$T^{m+1}(X) = R(T_{[m]}) = T^m(X),$$

and hence $q(T) < \infty$. Hence T is Drazin invertible.

(iii) It is evident from part (i) and (ii), since $\lambda I - T$ is both upper and lower semi B-Weyl. ∎

Theorem 2.6. *Let $T \in L(X)$. The following statements hold:*

(i) *If T^* has SVEP then*

$$\sigma_{\mathrm{d}}(T) = \sigma_{\mathrm{ubw}}(T) = \sigma_{\mathrm{bw}}(T). \tag{1}$$

(ii) *If T has SVEP then*

$$\sigma_{\mathrm{d}}(T) = \sigma_{\mathrm{lbw}}(T) = \sigma_{\mathrm{bw}}(T). \tag{2}$$

Proof. (i) Suppose that T^* has SVEP. Then, by [16, Corollary 2.4], $\sigma_{\mathrm{d}}(T) = \sigma_{\mathrm{bw}}(T)$. Obviously, $\sigma_{\mathrm{ubw}}(T) \subseteq \sigma_{\mathrm{bw}}(T)$ for every $T \in L(X)$. Suppose that $\lambda \notin \sigma_{\mathrm{ubw}}(T)$. Then $\lambda I - T$ is upper semi B-Weyl and the SVEP for T^* implies, by part (i) of Theorem 2.5, that $\lambda \notin \sigma_{\mathrm{d}}(T)$. Therefore, $\sigma_{\mathrm{ubw}}(T) = \sigma_{\mathrm{d}}(T)$.

(ii) Assume that T has SVEP. Again by [16, Corollary 2.4], $\sigma_{\mathrm{d}}(T) = \sigma_{\mathrm{bw}}(T)$. Obviously, $\sigma_{\mathrm{lbw}}(T) \subseteq \sigma_{\mathrm{bw}}(T)$ for every $T \in L(X)$. Suppose that $\lambda \notin \sigma_{\mathrm{lbw}}(T)$. Then $\lambda I - T$ is lower semi B-Weyl and the SVEP for T implies, by part (ii) of Theorem 2.5, that $\lambda \notin \sigma_{\mathrm{d}}(T)$. Therefore, $\sigma_{\mathrm{lbw}}(T) = \sigma_{\mathrm{d}}(T)$. ∎

3. property (gab)

Denote by $\Pi(T)$ and $\Pi_a(T)$ the set of all poles and the set of left poles of T, respectively. Clearly, $\Pi_a(T) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ld}}(T)$. Obviously, $\Pi(T) \subseteq \operatorname{iso}\sigma(T)$, and analogously we have

$$\Pi_a(T) \subseteq \operatorname{iso}\sigma_{\mathrm{a}}(T) \quad \text{for all } T \in L(X). \tag{3}$$

In fact, if $\lambda_0 \in \Pi_a(T)$ then $\lambda I - T$ is left Drazin invertible and hence $p(\lambda_0 I - T) < \infty$. Since $\lambda I - T$ has topological uniform descent (see [27], for definition and details), it then follows, from [27, Corollary 4.8], that $\lambda I - T$ is bounded below in a punctured disc centered at λ_0. Obviously,

$$p_{00}^a(T) \subseteq \Pi_a(T) \quad \text{and} \quad p_{00}(T) \subseteq \Pi(T)$$

for every $T \in L(X)$.

Define

$$\Delta^g(T) := \sigma(T) \setminus \sigma_{\mathrm{bw}}(T).$$

and

$$\Delta_a^g(T) := \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ubw}}(T).$$

It should be noted that the set $\Delta_a^g(T)$ may be empty. This is, for instance, the case of a unilateral shift right R on $\ell^2(\mathbb{N})$. Since, R has SVEP then, by [11, Corolary 2.12], $\sigma_{\mathrm{a}}(R) = \sigma_{\mathrm{ubw}}(R)$, so $\Delta_a^g(T) = \emptyset$.

An operator $T \in L(X)$ is said to verify *generalized Browder's theorem* if $\sigma_{\mathrm{bw}}(T) = \sigma_{\mathrm{d}}(T)$, or equivalently $\Delta^g(T) = \Pi(T)$. Generalized Browder's theorem and Browder's theorem are equivalent, see [15] or Theorem 3.2 of [6].

Lemma 3.1. *If $T \in L(X)$ then*

$$\Delta_a^g(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is upper semi B-Weyl and } 0 < \alpha(\lambda I - T)\}. \tag{4}$$

Furthermore,

$$\Delta(T) \subseteq \Delta_a^g(T), \quad \Delta(T) \subseteq \Delta^g(T) \subseteq \sigma_{\mathrm{a}}(T),$$

and $\Pi_a(T) \subseteq \Delta_a^g(T)$.

Proof. The inclusion $\supseteq$ in (4) is obvious. To show the opposite inclusion, suppose that $\lambda \in \Delta_a^g(T)$. There is no harm if we assume $\lambda = 0$. Then T is upper semi B-Weyl and $0 \in \sigma_{\mathrm{a}}(T)$. Both conditions entail that $\alpha(T) > 0$, otherwise if $\alpha(T) = 0$, by Lemma 2.2, we would have $0 \notin \sigma_{\mathrm{a}}(T)$, a contradiction. Therefore the equality (4) holds.

The inclusion $\Delta(T) \subseteq \Delta_a^g(T)$ is evident: if $\lambda \in \sigma(T) \setminus \sigma_{\mathrm{w}}(T)$ then $\lambda I - T$ is Weyl and hence upper semi B-Weyl. Moreover, $\alpha(\lambda I - T) > 0$, otherwise $\alpha(\lambda I - T) = \beta(\lambda I - T) = 0$, in contradiction with the assumption $\lambda \in \sigma(T)$.

The inclusion $\Delta(T) \subseteq \Delta^g(T)$ is clear, since $\sigma_{\mathrm{bw}}(T) \subseteq \sigma_{\mathrm{w}}(T)$. To show the inclusion $\Delta^g(T) \subseteq \sigma_{\mathrm{a}}(T)$, observe that if $\lambda \in \Delta^g(T)$ then $\lambda I - T$ is B-Weyl. This implies that $\alpha(\lambda I - T) > 0$. Indeed, if were $\alpha(\lambda I - T) = 0$, then $p(\lambda I - T) = 0$ and by part (iii) of Theorem 2.5, $\lambda I - T$ would be Drazin invertible. Therefore, $p(\lambda I - T) = q(\lambda I - T) = 0$, so $\lambda \notin \sigma(T)$, which is impossible.

To show the inclusion $\Pi_a(T) \subseteq \Delta_a^g(T)$, let assume that $\lambda \in \Pi_a(T) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ld}}(T)$.

Then $\lambda \in \sigma_{\rm a}(T)$, and $\lambda I-T$ is left Drazin invertible, in particular, upper semi B-Weyl. Therefore $\Pi_a(T) \subseteq \Delta_a^g(T)$. ∎

The following properties, introduced in [22] and [24], may be though as stronger variants than Browder type theorems.

Definition 3.2. *Let $T \in L(X)$.*

(i) *T is said to satisfy* property (b) *if $\Delta_a(T) = p_{00}(T)$.*

(ii) *T is said to satisfy* property (gb) *if $\Delta_a^g(T) = \Pi(T)$.*

(iii) *T is said to satisfy* property (ab) *if $\Delta(T) = p_{00}^a(T)$.*

(iv) *T is said to satisfy* property (gab) *if $\Delta^g(T) = \Pi_a(T)$.*

Define

$$\Sigma_a(T) := \Delta(T) \cup p_{00}^a(T).$$

Property (ab) may be characterized by the SVEP at the points of $\Sigma_a(T)$.

Theorem 3.3. *If $T \in L(X)$ then the following statements are equivalent:*

(i) *T satisfies property (ab);*

(ii) *$\Sigma_a(T) \subseteq iso\, \sigma(T)$,*

(iii) *T^* has SVEP at every point $\lambda \in \Sigma_a(T)$*

Proof. The equivalence (i) ⇔ (ii) has been proved in [9, Theorem 3.2]

(ii) ⇔ (iii) T^* has SVEP at every point $\lambda \in \Sigma_a(T)$ since $\lambda \in \operatorname{iso}\sigma(T) = \operatorname{iso}\sigma(T^*)$. Conversely, suppose that T^* has SVEP at every point $\lambda \in \Sigma_a(T)$. Then T^* has SVEP at the points of $\Delta(T)$, as well as at the points of $p_{00}^a(T)$. It is easily seen that that $\Delta(T) \subseteq p_{00}^a(T)$. Indeed, if $\lambda \in \Delta(T)$ then $\lambda I - T$ is Weyl, and the SVEP for T^* at λ entails by Theorem 2.4 that $q(\lambda I - T) < \infty$. By [1, Theorem 3.4], then $\lambda I - T$ is Browder, in particular upper semi-Browder, so $\lambda \in p_{00}^a(T)$. The SVEP for T^* at every $\lambda \in p_{00}^a(T)$ then entails, by [9, Theorem 3.2], that T satisfies property (ab). ∎

Set

$$\Sigma_a^g(T) := \Delta^g(T) \cup \Pi_a(T).$$

Lemma 3.4. *If $T \in L(X)$ then $\Sigma_a^g(T) \subseteq \Delta_a^g(T)$.*

Proof. By Lemma 3.1 we have $\Pi_a(T) \subseteq \Delta_a^g(T)$. It remains only to prove that $\Delta^g(T) \subseteq \Delta_a^g(T)$. If $\lambda \in \Delta^g(T)$ then $\lambda \in \sigma_{\rm a}(T)$, always by Lemma 3.1. On the other hand, we have $\lambda \notin \sigma_{\rm bw}(T)$ and hence $\lambda \notin \sigma_{\rm ubw}(T)$, since $\sigma_{\rm ubw}(T) \subseteq \sigma_{\rm bw}(T)$. ∎

Also the set $\Sigma_a^g(T)$ may be empty. Indeed, In the case of the unilateral right shift $R \in L(\ell_2(\mathbb{N}))$ it has been observed that $\Delta_a^g(R) = \emptyset$, so by Lemma 3.4 $\Sigma_a^g(T) = \emptyset$

Property (gab) may be characterized by means of localized SVEP as follows.

Theorem 3.5. *For a bounded operator $T \in L(X)$ the following statements are equivalent:*

(i) *T satisfies property (gab);*

(ii) *T^* has SVEP at every $\lambda \in \Pi_a(T)$ and $\Delta^g(T) \subseteq \Pi_a(T)$;*

(iii) *T^* has SVEP at every $\lambda \in \Sigma_a^g(T)$;*

(iv) *Browder's theorem holds for T and $\Pi(T) = \Pi_a(T)$;*

(v) *Browder's theorem holds for T and $\Pi_a(T) \subseteq iso\,\sigma(T)$;*

(vi) *Browder's theorem holds for T and $\Pi_a(T) \subseteq \partial\sigma(T)$, $\partial\sigma(T)$ the boundary of $\sigma(T)$;*

(vii) $\Sigma_a^g(T) \subseteq \mathrm{iso}\,\sigma(T)$*;*

(viii) $\Sigma_a^g(T) \subseteq \mathrm{iso}\,\sigma_{\mathrm{s}}(T)$*;*

(ix) $\Sigma_a^g(T) \subseteq \Pi(T)$.

Proof. To show the equivalence (i) $\Leftrightarrow$ (ii), suppose first that T has property (gab), i.e. $\Delta^g(T) = \Pi_a(T)$. If $\lambda \in \Pi_a(T)$ then $\lambda I - T$ is B-Weyl, in particular lower semi B-Weyl. Since $p(\lambda I - T) < \infty$ then T has SVEP at λ, so, by Theorem 2.5, $\lambda I - T$ is Drazin invertible, in particular $q(\lambda I - T) < \infty$ and hence T^* has SVEP at λ. Obviously, $\Delta^g(T) \subseteq \Pi_a(T)$, by assumption.

Conversely, suppose that T^* has SVEP at every $\lambda \in \Pi_a(T)$ and $\Delta^g(T) \subseteq \Pi_a(T)$. If $\lambda \in \Pi_a(T)$ then λ is a left pole, so $p(\lambda I - T) < \infty$, and $\lambda I - T$ is left Drazin invertible, or equivalently, upper semi B-Browder. Since T^* has SVEP at λ then $q(\lambda I - T) < \infty$, by Theorem 2.4. Therefore, $\lambda I - T$ is Drazin invertible and hence $\Pi_a(T) \subseteq \Pi(T)$. The opposite inclusion holds for every operator, so $\Pi_a(T) = \Pi(T)$. If $\lambda \in \Pi_a(T)$ then $\lambda I - T$ is Drazin invertible operator, and hence B-Weyl. Now, $\lambda \in \sigma(T)$ so we have $\lambda \in \sigma(T) \setminus \sigma_{\mathrm{bw}}(T) = \Delta^g(T)$. Thus, $\Pi_a(T) \subseteq \Delta^g(T)$, and since the opposite inclusion holds by assumption we then conclude that $\Pi_a(T) = \Delta^g(T)$.

(ii) $\Rightarrow$ (iii) Since $\Delta^g(T) \subseteq \Pi_a(T)$ then $\Sigma_a^g(T) = \Delta^g(T) \cup \Pi_a(T) = \Pi_a(T)$, hence T^* has SVEP at every $\lambda \in \Sigma_a^g(T)$.

(iii) $\Rightarrow$ (iv) Suppose that T^* has SVEP at every $\lambda \in \Sigma_a^g(T)$. Let $\lambda \in \Pi_a(T)$. Then λ is a left pole and hence $\lambda I - T$ is left Drazin invertible, so $p(\lambda I - T) < \infty$. Since $\Pi_a(T) \subseteq \Sigma_a^g(T)$ the SVEP of T^* at λ implies $q(\lambda I - T) < \infty$, by Theorem 2.4, thus $\lambda \in \Pi(T)$ and consequently $\Pi_a(T) \subseteq \Pi(T)$. The opposite inclusion holds for every $T \in L(X)$, hence $\Pi(T) = \Pi_a(T)$. It remains to prove Browder's theorem for T. Let $\lambda \notin \sigma_{\mathrm{w}}(T)$. Clearly, we can suppose that $\lambda \in \sigma(T)$. Then $\lambda \notin \sigma_{\mathrm{bw}}(T)$, since $\sigma_{\mathrm{bw}}(T) \subseteq \sigma_{\mathrm{w}}(T)$, hence $\lambda \in \Delta^g(T)$. Since $\lambda I - T$ is B-Weyl, the SVEP of T^* at λ, always by Theorem 2.4, implies that $\lambda I - T$ is Drazin invertible. But $\alpha(\lambda I - T) < \infty$, so by [1, Theorem 3.4] $\lambda I - T$ is Browder, i.e $\lambda \notin \sigma_{\mathrm{b}}(T)$, from which we conclude that $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T)$, i. e., the operator T satisfies Browder's theorem.

(iv) $\Rightarrow$ (v). If $\lambda \in \Pi_a(T)$ then $\lambda I - T$ is left Drazin invertible, hence upper semi B-Weyl. Since T^* has SVEP at λ then $\lambda I - T$ is Drazin invertible, by Theorem 2.5, so $\lambda \in \Pi(T)$. Therefore, $\Pi_a(T) \subseteq \Pi(T) \subseteq \mathrm{iso}\,\sigma(T)$.

(v) $\Rightarrow$ (vi) Obvious, since $\mathrm{iso}\,\sigma(T) \subseteq \partial\sigma(T)$.

(vi) $\Rightarrow$ (ii) T^* has SVEP at every $\lambda \in \partial\sigma(T) = \partial\sigma(T^*)$, in particular T^* has SVEP at every point $\lambda \in \Pi_a(T)$. Browder's theorem for T is equivalent to generalized Browder's theorem, so $\Delta^g(T) = \Pi(T) \subseteq \Pi_a(T)$.

Therefore, all the statements (i)-(vi) are equivalent.

(v) $\Rightarrow$ (vii) Assume that T satisfies Browder's theorem, or equivalently, generalized Browder's theorem and $\Pi_a(T) \subseteq \operatorname{iso}\sigma(T)$. Then, see [8, Theorem 3.5], $\Delta^g(T) \subseteq \operatorname{iso}\sigma(T)$, and consequently, $\Sigma_a^g(T) \subseteq \operatorname{iso}\sigma(T)$.

(vii) $\Rightarrow$ (viii) It is well known that every isolated point of the spectrum belongs to the surjectivity spectrum.

(viii) $\Rightarrow$ (ix) If $\Sigma_a^g(T) \subseteq \operatorname{iso}\sigma\mathrm{s}(T)$ then T^* has SVEP at every $\lambda \in \Sigma_a^g(T)$. Since $\lambda I - T$ is either B-Weyl or left Drazin invertible then, by Theorem 2.5, $\lambda I - T$ is Drazin invertible. Hence $\Sigma_a^g(T) \subseteq \Pi(T)$.

(ix) $\Rightarrow$ (iii) T^* has SVEP at every $\lambda \in \Sigma_a^g(T)$, since $q(\lambda I - T) < \infty$. ∎

The following implications was shown in [22] and [24]. We prove of all these implications simply by observing some inclusion of subsets of the spectrum.

Corollary 3.6. *Property $(gb) \Rightarrow (gab) \Rightarrow (ab) \Rightarrow$ Browder's theorem for T.*

Proof. The condition $\Delta_a^g(T) \subseteq \operatorname{iso}\sigma(T)$ characterizes property (gb), by [10, Theorem 3.4]. Hence, by Lemma 3.4 and Theorem 3.5, property (gb) entails property (gab). Property (ab) is equivalent to the inclusion $\Sigma_a(T) \subseteq \operatorname{iso}\sigma(T)$, by Theorem 3.3, and obviously, by Lemma 3.1 $\Sigma_a(T) = \Delta(T) \cup p_{00}^a(T) \subseteq \Delta^g(T) \cup \Pi_a(T) = \Sigma_a^g(T)$. Hence, by Theorem 3.5 property (gab) entails property (ab). The last implication follows once noted that $\Delta(T) \subseteq \Sigma_a(T)$ and hence property (ab) entails $\Delta(T) \subseteq \operatorname{iso}\sigma(T)$, and this inclusion is equivalent to Browder's theorem for T, see [5, Theorem 2.9]. ∎

An operator $T \in L(X)$ is said to be a Riesz operator if $\lambda I - T \in \Phi(X)$ for every $\lambda \in \mathbb{C} \setminus \{0\}$. Examples of Riesz operators are compact and quasi-nilpotent operators

Corollary 3.7. *Suppose that $T, K \in L(X)$ commutes and K is a Riesz operator. If T^* has SVEP then $T + K$ satisfies property (gab).*

Proof. The dual of a Riesz operator is also a Riesz operator, see [1, Corollary 3.114]. The SVEP for T^* is transferred to $T^* + K^* = (T + K)^*$, see [12]. By Theorem 3.5 then $T + K$ has property (gab).

∎

Every operator $T \in L(X)$ has the SVEP at the isolated points of the spectrum, and, by Theorem 3.5, property (gab) is equivalent to the inclusion $\Sigma_a^g(T) \subseteq \operatorname{iso}\sigma(T)$. Therefore, if T has (gab) then T has SVEP at every point of $\Sigma_a^g(T)$. The converse is false. In the next we give an example of an operator which has SVEP but the property (gab) fails for T.

Example 3.8. Let $R \in L(\ell_2(\mathbb{N}))$ denote the unilateral right shift. It is known that $\sigma(R) = D(0,1)$, where $D(0,1)$ denotes the closed unit disc of $\mathbb{C}$, while $\sigma_{\mathrm{a}}(R) = \Gamma$, where Γ denotes the unit circle. Define $T := 0 \oplus R$. Clearly, T has SVEP, since T is the direct sum of operators having SVEP, and $\sigma(T) = D(0,1)$. Let $\lambda \notin \sigma_{\mathrm{bw}}(T)$, and suppose that $\lambda \in \sigma(T)$. Then, by Theorem 2.5, $\lambda I - T$ is Drazin invertible, and hence λ is pole of the resolvent of T, in particular an isolated point of $\sigma(T)$, which is impossible. Therefore $\sigma_{\mathrm{bw}}(T) = \sigma(T) = D(0,1)$. On the other hand

$$\sigma_{\mathrm{a}}(T) = \sigma_{\mathrm{a}}(R) \cup \{0\} = \Gamma \cup \{0\}.$$

We show that $\Pi_a(T) = \{0\}$, i.e. 0 is a left pole. Evidently, $p := p(T) = p(R)+p(0) = 1$. We have $T(X) = \{0\} \oplus R(X)$, so $T(X) = T^2(X)$ is closed, since $R(X)$ is closed. Hence 0 is a left pole, and

$$\Pi_a(T) = \{0\} \neq \Delta^g(T) = \sigma(T) \setminus \sigma_{\rm bw}(T) = \emptyset$$

Consequently, T does not satisfy property (gab).

Theorem 3.9. *If $T \in L(X)$ the following statements are equivalent:*

(i) *T has property (gb);*

(ii) *T has property (gab) and $\sigma_{\rm bw}(T) \cap \Delta_a^g(T) = \emptyset$;*

(iii) *T satisfies a-Browder's theorem and $\sigma_{\rm bw}(T) \cap \Delta_a^g(T) = \emptyset$;*

(iv) *T satisfies Browder's theorem and $\sigma_{\rm bw}(T) \cap \Delta_a^g(T) = \emptyset$.*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) has been proved in [24, Theorem 2.11]. The implications (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are clear, since property (gb) implies a-Browder's theorem, see [24, Theorem 2.4]. To show the implication (iv) $\Rightarrow$ (i), suppose that $\lambda \in \Delta_a^g(T)$. Then $\lambda \notin \sigma_{\rm bw}(T)$ by assumption, and since Browder's theorem is equivalent to generalized Browder's theorem then $\lambda \notin \sigma_{\rm d}(T)$, so $\lambda I - T$ is Drazin invertible, and hence $\lambda \in \operatorname{iso} \sigma(T)$. The inclusion $\Delta_a^g(T) \subseteq \operatorname{iso} \sigma(T)$ is equivalent, by [10, Theorem 3.4], to the property (gb). ■

The *quasi-nilpotent part* of T, is defined as follows:

$$H_0(T) := \{x \in X : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

It is easily seen that $ker\, T^n \subseteq H_0(T)$ for every $n \in \mathbb{N}$, so $\mathcal{N}^\infty(T) \subseteq H_0(T)$, where $\mathcal{N}^\infty(T) := \bigcup_{n=1}^\infty ker\, T^n$ denotes the *hyper-kernel* of T.

An other important subspace in local spectral theory is given by the *analytic core* $K(T)$. To define this subspace, recall that the *local resolvent set* of T at the point $x \in X$, is the set $\rho_T(x)$ defined as the union of all open subsets U of $\mathbf{C}$ for which there exists an analytic function $f : U \to X$ with the property that $(\lambda I - T)f(\lambda) = x$ for all $\lambda \in U$. The *local spectrum* $\sigma_T(x)$ of T at x is the set defined by $\sigma_T(x) := \mathbf{C} \setminus \rho_T(x)$. For every subset F of $\mathbf{C}$, then the *analytic spectral subspace* of T associated with F is defined as the set $X_T(F) := \{x \in X : \sigma_T(x) \subseteq F\}$. The *analytic core* $K(T)$ of $T \in L(X)$ is defined as $K(T) := X_T(\mathbb{C} \setminus \{0\})$. Note that $K(T) \subseteq T^\infty(X) \subseteq T^n(X)$ for all $n \in \mathbb{N}$, where $T^\infty(X) := \bigcap_{n=0}^\infty T^n(X)$ denotes the *hyper-range* of T, and $T(K(T)) = K(T)$, see [1, Theorem 1.21].

The two subspaces $H_0(T)$ and $K(T)$ are, in general, not closed, and, by [1, Theorem 2.31]

$$H_0(\lambda I - T) \text{ closed} \Rightarrow T \text{ has SVEP at } \lambda. \tag{5}$$

Furthermore, if $\lambda \in \operatorname{iso} \sigma(T)$ then $X = H_0(\lambda I - T) \oplus K(\lambda I - T)$, see [1, Theorem 3.74]. Consequently, if T has property (gab), by Theorem 3.5 we have $\Sigma_a^g(T) \subseteq \operatorname{iso} \sigma(T)$, and hence

$$X = H_0(\lambda I - T) \oplus K(\lambda I - T) \quad \text{for all } \lambda \in \Sigma_a^g(T). \tag{6}$$

The following results show that property (gab) may be characterized by some conditions that are formally weaker than the one expressed by the decomposition (6).

Theorem 3.10. *For an operators $T \in L(X)$ the following statements are equivalent:*

(i) *T satisfies property (gab);*

(ii) *$X = H_0(\lambda I - T) + K(\lambda I - T)$ for all $\lambda \in \Sigma_a^g(T)$;*

(iii) *there exists a natural $\nu := \nu(\lambda)$ such that $K(\lambda I - T) = (\lambda I - T)^\nu(X)$ for all $\lambda \in \Sigma_a^g(T)(T)$;*

(iv) *$X = \mathcal{N}^\infty(\lambda I - T) + (\lambda I - T)^\infty(X)$ for all $\lambda \in \Sigma_a^g(T)$;*

(v) *there exists a natural $\nu := \nu(\lambda)$ such that $(\lambda I - T)^\infty(X) = (\lambda I - T)^\nu(X)$ for all $\lambda \in \Sigma_a^g(T)$.*

Proof. (i) $\Rightarrow$ (ii) Clear, as observed in (6).

(ii) $\Rightarrow$ (i) By [26, Theorem 5], the condition $X = H_0(\lambda I - T) + K(\lambda I - T)$ is equivalent to the inclusion $\lambda \in \operatorname{iso} \sigma_{\mathrm{s}}(T)$. Hence $\Sigma_a^g(T) \subseteq \operatorname{iso} \sigma_{\mathrm{s}}(T)$ and from Theorem 3.5 it immediately follows that T satisfies property (gab).

(i) $\Leftrightarrow$ (iii) If T satisfies property (gab) then, by Theorem 3.5, T^* has SVEP at every $\lambda \in \Sigma_a^g(T)$. Since for every $\lambda \in \Sigma_a^g(T)$, $\lambda I - T$ is quasi-Fredholm then, by Theorem 2.4, $q := q(\lambda I - T) < \infty$ for all $\lambda \in \Sigma_a^g(T)$, so $(\lambda I - T)^\infty(X) = (\lambda I - T)^q(X)$. Since for every $\lambda \in \Sigma_a^g(T)$ the operator $\lambda I - T$ is upper semi B-Fredholm, then there exists $\nu \in \mathbb{N}$ such that $(\lambda I - T)^n(X)$ is closed for all $n \geq \nu$, hence $(\lambda I - T)^\infty(X)$ is closed. As observed above, for every $\lambda \in \Sigma_a^g(T)$, $\lambda I - T$ is quasi-Fredholm and hence has topological uniform descent, see [18]. Furthermore, by [27, Theorem 3.4], the restriction $(\lambda I - T)|(\lambda I - T)^\infty(X)$ is onto, so $(\lambda I - T)((\lambda I - T)^\infty(X)) = (\lambda I - T)^\infty(X)$. By [1, Theorem 1.22] it then follows that $(\lambda I - T)^\infty(X) \subseteq K(\lambda I - T)$, and, since the reverse inclusion holds for every operator, we then conclude that

$$(\lambda I - T)^\infty(X) = K(\lambda I - T) = (\lambda I - T)^q(X),$$

for all $\lambda \in \Sigma_a^g(T)$.

Conversely, let $\lambda \in \Sigma_a^g(T)$ be arbitrary given and suppose that there exists a natural $\nu := \nu(\lambda)$ such that $K(\lambda I - T) = (\lambda I - T)^\nu(X)$. Then we have

$$(\lambda I - T)^\nu(X) = K(\lambda I - T) = (\lambda I - T)(K(\lambda I - T)) = (\lambda I - T)^{\nu+1}(X),$$

thus $q(\lambda I - T) \leq \nu$, so T^* has SVEP at λ, hence T satisfies (gab) by Theorem 3.5.

(ii) $\Leftrightarrow$ (iv) Every semi B-Fredholm operator has topological uniform ascent ([20]), so, by [31, Corollary 2.8],

$$H_0(\lambda I - T) + K(\lambda I - T) = \mathcal{N}^\infty(\lambda I - T) + (\lambda I - T)^\infty(X),$$

for every $\lambda \in \Sigma_a^g(T)$.

(i) $\Leftrightarrow$ (v) Suppose that T satisfies property (gab). By Theorem 3.5 then T^* has SVEP at every $\lambda \in \Sigma_a^g(T)$, hence, by Theorem 2.4, $q := q(\lambda I - T) < \infty$ for all $\lambda \in \Sigma_a^g(T)$. Therefore, $(\lambda I - T)^\infty(X) = (\lambda I - T)^q(X)$ for all $\lambda \in \Sigma_a^g(T)$.

Conversely, suppose that (v) holds and $\lambda \in \Sigma_a^g(T)$. Then

$$(\lambda I - T)^\nu(X) = (\lambda I - T)^\infty(X) \subseteq (\lambda I - T)^{\nu+1}(X),$$

and since $(\lambda I - T)^{n+1}(X) \subseteq (\lambda I - T)^n(X)$ holds for all $n \in \mathbb{N}$, we then obtain that $(\lambda I - T)^\nu(X) = (\lambda I - T)^{\nu+1}(X)$. Hence $q(\lambda I - T) \leq \nu$, so, by Theorem 2.4, T^* has SVEP at every $\lambda \in \Sigma_a^g(T)$. Consequently, by Theorem 3.5, T satisfies property (gab) .
■

Theorem 3.11. *If $T \in L(X)$ then T^* has property (gab) if and only if $H_0(\lambda I - T)$ is closed for all $\lambda \in \Sigma_a^g(T^*)$.*

Proof. Suppose that T^* has property (gab). By Theorem 3.5 then $\Sigma_a^g(T^*) \subseteq \operatorname{iso}\sigma(T^*) = \operatorname{iso}\sigma(T)$, so both T and T^* have SVEP at the points of $\Sigma_a^g(T^*)$. Let $\lambda \in \Sigma_a^g(T^*) = \Delta^g(T^*) \cup \Pi_a(T^*)$. If $\lambda \in \Delta^g(T^*)$ then $\lambda I - T^*$ is B-Weyl, hence quasi-Fredholm. Then also $\lambda I - T$ is quasi-Fredholm, by [34, Theorem2.1], and since T has SVEP at λ, Theorem 2.4 implies that $H_0(\lambda I - T)$ is closed. If $\lambda \in \Pi_a(T^*)$ then $\lambda I - T^*$ is left Drazin invertible and hence by [4, Theorem 2.1], $\lambda I - T$ is right Drazin invertible, in particular quasi-Fredholm, so the SVEP of T at λ entails, always by Theorem 2.4, that $H_0(\lambda I - T)$ is closed.

Conversely, suppose that $H_0(\lambda I - T)$ is closed for all $\lambda \in \Sigma_a^g(T^*)$. If $\lambda \in \Delta^g(T^*)$ then $\lambda I - T^*$ is B-Weyl, and hence, as above, $\lambda I - T$ is quasi- Fredholm and the condition $H_0(\lambda I - T)$ closed implies, always by Theorem 2.4, that T has SVEP at λ. By Theorem 2.5 we then conclude that $\lambda I - T$ is Drazin invertible, and hence $\lambda \in \operatorname{iso}\sigma(T) = \operatorname{iso}\sigma(T^*)$. If $\lambda \in \Pi_a(T^*)$ then $\lambda I - T^*$ is left Drazin invertible, so $\lambda I - T$ is right Drazin invertible. Since $H_0(\lambda I - T)$ is closed, then T has SVEP at λ, and hence $\lambda I - T$ is Drazin invertible, always by Theorem 2.5. Consequently, $\lambda \in \operatorname{iso}\sigma(T) = \operatorname{iso}\sigma(T^*)$. Therefore, $\Sigma_a^g(T^*) \subseteq \operatorname{iso}\sigma(T^*)$ and hence T^* has property (gab) by Theorem 3.5.

■

An operator T is said *finite polaroid* if every $\lambda \in \operatorname{iso}\sigma(T)$ is a pole of finite rank, or equivalently $\lambda I - T$ is Browder.

Theorem 3.12. *Let $T \in L(X)$ be finite polaroid. Then T satisfies property (gab) if and only if $K(\lambda I - T)$ has finite codimension for all $\lambda \in \Sigma_a^g(T)$.*

Proof. By Theorem 3.5 property (gab) entails the inclusion $\Sigma_a^g(T) \subseteq \operatorname{iso}\sigma(T)$, so, if $\lambda \in \Sigma_a^g(T)$ then $\lambda I - T$ is Browder. Observe that $\beta(\lambda I - T) < \infty$ implies $\beta(\lambda I - T)^n < \infty$ for every $n \in \mathbb{N}$. Since λ is a pole, then $K(\lambda I - T) = (\lambda I - T)^p(X)$ has finite codimension, where p is the order of the pole, see [1].

Conversely, suppose that $K(\lambda I - T)$ has finite codimension for all $\lambda \in \Sigma_a^g(T)$. If $\lambda \in \Sigma_a^g(T)$ then either $\lambda \in \Delta^g(T)$ or $\lambda \in \Pi_a(T)$. If $\lambda \in \Delta^g(T)$, from the inclusion $K(\lambda I - T) \subseteq (\lambda I - T)(X)$ we see that also $(\lambda I - T)(X)$ has finite codimension, hence $\beta(\lambda I - T) < \infty$. Since $\lambda I - T$ is B-Weyl then $\alpha(\lambda I - T) = \beta(\lambda I - T) < \infty$, so $\lambda I - T$ is Weyl. The condition $\operatorname{codim} K(\lambda I - T) < \infty$ entails, that T^* has SVEP at λ, or equivalently $q(\lambda I - T) < \infty$, see [1, Theorem 3.18]. By [1, Theorem 3.4] it then follows that λ is a pole, hence $\Delta^g(T) \subseteq \operatorname{iso}\,\sigma(T)$. Consider the other case that $\lambda \in \Pi_a(T)$. Then $p(\lambda I - T) < \infty$ and, as above, the inclusion $K(\lambda I - T) \subseteq (\lambda I - T)(X)$ implies that $\beta(\lambda I - T) < \infty$. Therefore, $\lambda I - T$ is lower semi-Fredholm and hence the condition

$K(\lambda I - T)$ has finite codimension implies, again by [1, Theorem 3.18], implies that $q(\lambda I - T) < \infty$, from which we conclude that also $\Pi_a(T) \subseteq \operatorname{iso}\sigma(T)$. Consequently, $\Sigma_a^g(T) \subseteq$ iso $\sigma(T)$, and Theorem 3.5 then implies that T has property (gab).

■

An operator $T \in L(X)$ is said to be *a-polaroid* if every isolated point of $\sigma_{\rm a}(T)$ is a pole, while $T \in L(X)$ is said to be *polaroid* is every isolated point of $\sigma(T)$ is a pole. Since $\operatorname{iso}\sigma(T) \subseteq \operatorname{iso}\sigma_{\rm a}(T)$ (it is known that every isolated point of $\sigma(T)$ belongs to $\sigma_{\rm a}(T)$), then every a-polaroid operator is polaroid, while the converse, in general, is not true. Note that $T \in L(X)$ is polaroid if and only if T^* is polaroid. Evidently, if T is a-polaroid then $\Pi_a(T) = \Pi(T)$, since every $\lambda \in \Pi_a(T)$ is an isolated point of $\sigma_{\rm a}(T)$.

Theorem 3.13. *Let $T \in L(X)$ be a-polaroid.*

(i) *Property* (gb)*, property* (b) *and a-Browder's theorem are equivalent for T.*

(ii) *Property* (gab)*, property* (ab) *and Browder's theorem are equivalent for T.*

Proof. The equivalences in (i) has been observed in [10, Theorem 3.10]. By Corollary 3.6, to show the equivalences in part (ii), we need only to show that Browder's theorem implies property (gab). If T satisfies Browder's theorem, or equivalently generalized Browder's theorem, then $\Delta^g(T) = \Pi(T)$. Since T is a-polaroid then $\Pi_a(T) = \Pi(T)$, so $\Delta^g(T) = \Pi_a(T)$, hence T has property (gab). ■

The results (i) and (ii) of Theorem 3.13 cannot be extended to polaroid operators. In [10, Example 3.7] is given an example of polaroid operator that satisfies a-Browder's theorem but not property (gb). If T is defined as in Example 3.8, then T is polaroid and satisfies Browder's theorem, since T has SVEP, while property (gab) does not hold for T.

In the sequel we denote by $\operatorname{acc} F$ the set of all cluster points of a subset $F \subseteq \mathbb{C}$. It is known that, in general, the equality $\sigma_{\rm a}(T) = \sigma_{\rm a}(T+K)$, where K is a finite rank operator which commutes with T, does not hold. Actually, we have

$$\operatorname{acc}\sigma_{\rm a}(T) = \operatorname{acc}\sigma_{\rm a}(T+K),$$

while the isolated points of $\sigma_{\rm a}(T)$ and $\sigma_{\rm a}(T+K)$ may be different.

Lemma 3.14. *Suppose that $T, K \in L(X)$ commute and that K^n is a finite rank operator for some $n \in \mathbb{N}$. If $\operatorname{iso}\sigma_{\rm a}(T) = \operatorname{iso}\sigma_{\rm a}(T+K)$ then $\sigma_{\rm a}(T) = \sigma_{\rm a}(T+K)$ and $\sigma(T) = \sigma(T+K)$.*

Proof. By [36, Theorem 2.2] we have

$$\operatorname{acc}\sigma_{\rm a}(T) = \operatorname{acc}\sigma_{\rm a}(T+K)$$

and

$$\operatorname{acc}\sigma(T) = \operatorname{acc}\sigma(T+K).$$

Hence

$$\sigma_{\rm a}(T+K) = \operatorname{iso}\sigma_{\rm a}(T+K) \cup \operatorname{acc}\sigma_{\rm a}(T+K) = \operatorname{iso}\sigma_{\rm a}(T) \cup \operatorname{acc}\sigma_{\rm a}(T) = \sigma_{\rm a}(T).$$

To show that $\sigma(T) = \sigma(T+K)$, observe first that if $\lambda \in \operatorname{iso}\sigma(T)$, then $\lambda \in \sigma_{\mathrm{a}}(T)$ and hence, $\lambda \in \operatorname{iso}\sigma_{\mathrm{a}}(T) = \operatorname{iso}\sigma_{\mathrm{a}}(T+K)$. Therefore

$$\begin{aligned} \sigma(T) &= \operatorname{iso}\sigma(T) \cup \operatorname{acc}\sigma(T) \subseteq \operatorname{iso}\sigma_{\mathrm{a}}(T) \cup \operatorname{acc}\sigma(T) \\ &= \operatorname{iso}\sigma_{\mathrm{a}}(T+K) \cup \operatorname{acc}\sigma(T+K) \subseteq \sigma_{\mathrm{a}}(T+K) \cup \operatorname{acc}\sigma(T+K) \\ &\subseteq \sigma(T+K) \end{aligned}$$

Since K commutes with $T+K$, a symmetric argument shows $\sigma(T+K) \subseteq \sigma((T+K)-K) = \sigma(T)$. Therefore, $\sigma(T) = \sigma(T+K)$. ∎

Theorem 3.15. *Suppose that $T, K \in L(X)$ commute and that K^n is a finite rank operator for some $n \in \mathbb{N}$. Furthermore, assume that $\operatorname{iso}\sigma_{\mathrm{a}}(T) = \operatorname{iso}\sigma_{\mathrm{a}}(T+K)$. If T has property (gab) then also $T+K$ has property (gab).*

Proof. We have $\sigma_{\mathrm{d}}(T) = \sigma_{\mathrm{d}}(T+K)$ and $\sigma_{\mathrm{ld}}(T) = \sigma_{\mathrm{ld}}(T+K)$, see [35, Theorem 2.11], or [13, Theorem 2.8]. By Lemma 3.14 we then obtain $\Pi(T) = \Pi(T+K)$ and $\Pi_a(T) = \Pi_a(T+K)$.

Now, assume that T has property (gab). Then Browder's theorem holds for T and it is well known that the Weyl spectrum and the Browder spectrum are stable under Riesz commuting perturbations, so Browder's theorem holds for $T+K$, since K is a Riesz operator. Property (gab) for T entails, by Theorem 3.5, that $\Pi(T) = \Pi_a(T)$, and hence $\Pi(T) = \Pi(T+K) = \Pi_a(T) = \Pi_a(T+K)$, so, again by Theorem 3.5, $T+K$ has property (gab). ∎

Clearly, if N is a nilpotent operator which commutes with T, then $\sigma_{\mathrm{a}}(T) = \sigma_{\mathrm{a}}(T+N)$, so Theorem 3.15 applies to nilpotent commuting perturbations. The result of Theorem 3.15 cannot applied to a commuting quasi-nilpotent perturbation Q, although $\sigma_{\mathrm{a}}(T) = \sigma_{\mathrm{a}}(T+Q)$. Under the very special condition $\operatorname{iso}\sigma_{\mathrm{ub}}(T) = \emptyset$ we have $\Pi_a(T+Q) = \Pi_a(T)$ and $\sigma_{\mathrm{ubw}}(T+Q) = \sigma_{\mathrm{ubw}}(T)$, see [25, Corollary2.8 and Corollary 2.1], from which it follows that, under this condition, the property (gab) for T entails property (gab) for $T+Q$.

Theorem 3.16. *Suppose that $T, K \in L(X)$ commute and that K^n is a finite rank operator for some $n \in \mathbb{N}$. If $\operatorname{iso}\sigma_{\mathrm{a}}(T) = \emptyset$ and T has property (gab) then also $T+K$ has property (gab).*

Proof. The condition $\operatorname{iso}\sigma_{\mathrm{a}}(T) = \emptyset$ implies that also $\operatorname{iso}\sigma_{\mathrm{a}}(T+K) = \emptyset$ (the proof of this is exactly the same of [3, Lemma 2.6], where was considered a finite rank operator). Thus, we are in the situation of Theorem 3.15, hence T transfers property (gab) to $T+K$. ∎

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