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RESEARCH ARTICLE

Inequalities of Hermite-Hadamard Type

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Abstract. Some inequalities of Hermite-Hadamard type for λ-convex functions defined on convex subsets in real or complex linear spaces are given. Applications for norm inequalities are provided as well.

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1. Introduction

We recall here some concepts of convexity that are well known in the literature.

Let I be an interval in $\mathbb{R}$.

Definition 1.1 ([38]). *We say that $f : I \to \mathbb{R}$ is a Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and $t \in (0,1)$ we have*

$$f(tx + (1-t)y) \leq \frac{1}{t} f(x) + \frac{1}{1-t} f(y) . \tag{1}$$

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Some further properties of this class of functions can be found in [28], [29], [31], [44], [47] and [48]. Among others, it has been noted that non-negative monotone and non-negative convex functions belong to this class of functions.

The above concept can be extended for functions $f : C \subseteq X \to [0, \infty)$ where C is a convex subset of the real or complex linear space X and inequality (1) is satisfied for any vectors $x, y \in C$ and $t \in (0, 1)$. If the function $f : C \subseteq X \to \mathbb{R}$ is non-negative and convex, then is of Godunova-Levin type.

Definition 1.2 ([31])**.** *We say that a function $f : I \to \mathbb{R}$ belongs to the class $P(I)$ if it is nonnegative and for all $x, y \in I$ and $t \in [0, 1]$ we have*

$$f(tx + (1-t)y) \le f(x) + f(y). \tag{2}$$

Obviously $Q(I)$ contains $P(I)$ and for applications it is important to note that also $P(I)$ contain all nonnegative monotone, convex and *quasi convex functions,* i. e. nonnegative functions satisfying

$$f(tx + (1-t)y) \le \max\{f(x), f(y)\} \tag{3}$$

for all $x, y \in I$ and $t \in [0, 1]$.

For some results on P-functions see [31] and [45] while for quasi convex functions, the reader can consult [30].

If $f : C \subseteq X \to [0, \infty)$, where C is a convex subset of the real or complex linear space X, then we say that it is of P-type (or quasi-convex) if the inequality (2) (or (3)) holds true for $x, y \in C$ and $t \in [0, 1]$.

Definition 1.3 ([7])**.** *Let s be a real number, $s \in (0, 1]$. A function $f : [0, \infty) \to [0, \infty)$ is said to be s-convex (in the second sense) or Breckner s-convex if*

$$f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in [0, \infty)$ and $t \in [0, 1]$.

For some properties of this class of functions see [1], [2], [7], [8], [26], [27], [39], [41] and [50].

The concept of Breckner s-convexity can be similarly extended for functions defined on convex subsets of linear spaces.

It is well known that if $(X, \|\cdot\|)$ is a normed linear space, then the function $f(x) = \|x\|^p$, $p \ge 1$ is convex on X.

Utilising the elementary inequality $(a+b)^s \le a^s + b^s$ that holds for any $a, b \ge 0$ and $s \in (0, 1]$, we have for the function $g(x) = \|x\|^s$ that

$$\begin{aligned} g(tx + (1-t)y) &= \|tx + (1-t)y\|^s \le (t\|x\| + (1-t)\|y\|)^s \\ &\le (t\|x\|)^s + [(1-t)\|y\|]^s \\ &= t^s g(x) + (1-t)^s g(y) \end{aligned}$$

for any $x, y \in X$ and $t \in [0,1]$, which shows that g is Breckner s-convex on X.

In order to unify the above concepts for functions of real variable, S. Varošanec introduced the concept of h-convex functions as follows.

Assume that I and J are intervals in $\mathbb{R}, (0,1) \subseteq J$ and functions h and f are real non-negative functions defined in J and I, respectively.

Definition 1.4 ([53]). *Let $h : J \to [0,\infty)$ with h not identical to 0. We say that $f : I \to [0,\infty)$ is an h-convex function if for all $x, y \in I$ we have*

$$f(tx + (1-t)y) \le h(t) f(x) + h(1-t) f(y) \tag{4}$$

for all $t \in (0,1)$.

For some results concerning this class of functions see [53], [6], [42], [51], [49] and [52].

This concept can be extended for functions defined on convex subsets of linear spaces in the same way as above by replacing the interval I with the corresponding convex subset C of the linear space X.

We can introduce now another class of functions.

Definition 1.5. *We say that the function $f : C \subseteq X \to [0,\infty)$ is of s-Godunova-Levin type, with $s \in [0,1]$, if*

$$f(tx + (1-t)y) \le \frac{1}{t^s} f(x) + \frac{1}{(1-t)^s} f(y), \tag{5}$$

for all $t \in (0,1)$ and $x, y \in C$.

We observe that for $s = 0$ we obtain the class of P-functions while for $s = 1$ we obtain the class of Godunova-Levin. If we denote by $Q_s(C)$ the class of s-Godunova-Levin functions defined on C, then we obviously have

$$P(C) = Q_0(C) \subseteq Q_{s_1}(C) \subseteq Q_{s_2}(C) \subseteq Q_1(C) = Q(C)$$

for $0 \le s_1 \le s_2 \le 1$.

For different inequalities related to these classes of functions, see [1]-[4], [6], [9]-[37], [40]-[42] and [45]-[52].

A function $h : J \to \mathbb{R}$ is said to be *supermultiplicative* if

$$h(ts) \ge h(t) h(s) \text{ for any } t, s \in J. \tag{6}$$

If inequality (6) is reversed, then h is said to be *submultiplicative*. If the equality holds in (6) then h is said to be a multiplicative function on J.

In [53] it has been noted that if $h : [0,\infty) \to [0,\infty)$ with $h(t) = (x+c)^{p-1}$, then for $c = 0$ the function h is multiplicative. If $c \ge 1$, then for $p \in (0,1)$ the function h is supermultiplicative and for $p > 1$ the function is submultiplicative.

We observe that, if h and g are nonnegative and supermultiplicative, so is their product. In particular, if h is supermultiplicative then its product with a power function $\ell_r(t) = t^r$ is also supermultiplicative.

We can prove now the following generalization of the Hermite-Hadamard inequality for h-convex functions defined on convex subsets of linear spaces.

Theorem 1.1. *Assume that the function $f : C \subseteq X \to [0, \infty)$ is an h-convex function with $h \in L[0,1]$. Let $y, x \in C$ with $y \neq x$ and assume that the mapping $[0,1] \ni t \mapsto f[(1-t)x + ty]$ is Lebesgue integrable on $[0,1]$. Then*

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x + ty]\,dt < [f(x) + f(y)] \int_0^1 h(t)\,dt. \tag{7}$$

Proof. By the h-convexity of f we have

$$f(tx + (1-t)y) \leq h(t) f(x) + h(1-t) f(y) \tag{8}$$

for any $t \in [0,1]$.

Integrating (8) on $[0,1]$ over t, we get

$$\int_0^1 f(tx + (1-t)y)\,dt \leq f(x) \int_0^1 h(t)\,dt + f(y) \int_0^1 h(1-t)\,dt$$

and since $\int_0^1 h(t)\,dt = \int_0^1 h(1-t)\,dt$, we get the second part of (7).

From the h-convexity of f we have

$$f\left(\frac{z+w}{2}\right) \leq h\left(\frac{1}{2}\right) [f(z) + f(w)] \tag{9}$$

for any $z, w \in C$.

If we take in (9) $z = tx + (1-t)y$ and $w = (1-t)x + ty$, then we get

$$f\left(\frac{x+y}{2}\right) \leq h\left(\frac{1}{2}\right) [f(tx + (1-t)y) + f((1-t)x + ty)] \tag{10}$$

for any $t \in [0,1]$.

Integrating (10) on $[0,1]$ over t and taking into account that

$$\int_0^1 f(tx + (1-t)y)\,dt = \int_0^1 f((1-t)x + ty)\,dt$$

we get the first inequality in (7). □

Remark 1.1. *If $f : I \to [0, \infty)$ is an h-convex function on an interval I of real numbers with $h \in L[0,1]$ and $f \in L[a,b]$ with $a, b \in I, a < b$,* *then from (7) we get the Hermite-Hadamard type inequality obtained by Sarikaya et al. in* [49]

$$\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\leq\int_a^b f(u)\,du\leq[f(a)+f(b)]\int_0^1 h(t)\,dt.$$

If we write (7) for $h(t)=t$, then we get the classical Hermite-Hadamard inequality for convex functions.

If we write (7) for the case of P-type functions $f:C\to[0,\infty)$, i.e., $h(t)=1, t\in[0,1]$, then we get the inequality

$$\frac{1}{2}f\left(\frac{x+y}{2}\right)\leq\int_0^1 f[(1-t)x+ty]\,dt\leq f(x)+f(y), \tag{11}$$

that has been obtained for functions of a real variable in [31].

If f is Breckner s-convex on C, for $s\in(0,1)$, then by taking $h(t)=t^s$ in (7) we get

$$2^{s-1}f\left(\frac{x+y}{2}\right)\leq\int_0^1 f[(1-t)x+ty]\,dt\leq\frac{f(x)+f(y)}{s+1}, \tag{12}$$

that was obtained for functions of a real variable in [26].

Since the function $g(x)=\|x\|^s$ is Breckner s-convex on on the normed linear space X, $s\in(0,1)$, then for any $x,y\in X$ we have

$$\frac{1}{2}\|x+y\|^s\leq\int_0^1\|(1-t)x+ty\|^s\,dt\leq\frac{\|x\|^s+\|x\|^s}{s+1}. \tag{13}$$

If $f:C\to[0,\infty)$ is of s-Godunova-Levin type, with $s\in[0,1)$, then

$$\frac{1}{2^{s+1}}f\left(\frac{x+y}{2}\right)\leq\int_0^1 f[(1-t)x+ty]\,dt\leq\frac{f(x)+f(y)}{1-s}. \tag{14}$$

We notice that for $s=1$ the first inequality in (14) still holds, i.e.

$$\frac{1}{4}f\left(\frac{x+y}{2}\right)\leq\int_0^1 f[(1-t)x+ty]\,dt. \tag{15}$$

The case for functions of real variables was obtained for the first time in [31].

2. λ-Convex Functions

We start with the following definition:

Definition 2.1. *Let $\lambda:[0,\infty)\to[0,\infty)$ be a function with the property that $\lambda(t)>0$ for all $t>0$. A mapping $f:C\to\mathbb{R}$ defined on convex subset C of a linear space X is called λ-convex on C if*

$$f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)\leq\frac{\lambda(\alpha)f(x)+\lambda(\beta)f(y)}{\lambda(\alpha+\beta)} \tag{16}$$

for all $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$ and $x, y \in C$.

We observe that if $f : C \to \mathbb{R}$ is λ-convex on C, then f is h-convex on C with $h(t) = \frac{\lambda(t)}{\lambda(1)}$, $t \in [0,1]$.

If $f : C \to [0, \infty)$ is h-convex function with h supermultiplicative on $[0, \infty)$, then f is λ-convex with $\lambda = h$.

Indeed, if $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$ and $x, y \in C$ then

$$\begin{aligned} f\left(\frac{\alpha x + \beta y}{\alpha + \beta}\right) &\leq h\left(\frac{\alpha}{\alpha + \beta}\right) f(x) + h\left(\frac{\beta}{\alpha + \beta}\right) f(y) \\ &\leq \frac{h(\alpha) f(x) + h(\beta) f(y)}{h(\alpha + \beta)}. \end{aligned}$$

The following proposition contain some properties of λ-convex functions.

Proposition 2.1. *Let $f : C \to \mathbb{R}$ be a λ-convex function on C.*

(i) If $\lambda(0) > 0$, then we have $f(x) \geq 0$ for all $x \in C$;

(ii) If there exists $x_0 \in C$ so that $f(x_0) > 0$, then

$$\lambda(\alpha + \beta) \leq \lambda(\alpha) + \lambda(\beta)$$

for all $\alpha, \beta > 0$, i.e. the mapping λ is subadditive on $(0, \infty)$.

(iii) If there exists $x_0, y_0 \in C$ with $f(x_0) > 0$ and $f(y_0) < 0$, then

$$\lambda(\alpha + \beta) = \lambda(\alpha) + \lambda(\beta)$$

for all $\alpha, \beta > 0$, i.e. the mapping λ is additive on $(0, \infty)$.

Proof. (i) For every $\beta > 0$ and $x, y \in C$ we can state

$$f\left(\frac{0x + \beta y}{0 + \beta}\right) \leq \frac{\lambda(0) f(x) + \lambda(\beta) f(y)}{\lambda(\beta)}$$

from where we get

$$f(y) \leq \frac{\lambda(0)}{\lambda(\beta)} f(x) + f(y)$$

and since $\lambda(0) > 0$ we get that $f(x) \geq 0$ for all $x \in C$.

(ii) For all $\alpha, \beta > 0$ we have

$$f\left(\frac{\alpha x_0 + \beta x_0}{\alpha + \beta}\right) \leq \frac{\lambda(\alpha) f(x_0) + \lambda(\beta) f(x_0)}{\lambda(\alpha + \beta)}$$

from where we get

$$f(x_0) \leq \frac{\lambda(\alpha) + \lambda(\beta)}{\lambda(\alpha + \beta)} f(x_0)$$

and since $f(x_0) > 0$, then we get that $\lambda(\alpha + \beta) \leq \lambda(\alpha) + \lambda(\beta)$ for all $\alpha, \beta > 0$.

(iii) If we write the inequality for y_0 we also have

$$f(y_0) \leq \frac{\lambda(\alpha) + \lambda(\beta)}{\lambda(\alpha+\beta)} f(y_0)$$

and since $f(y_0) < 0$ we get that

$$\lambda(\alpha+\beta) \geq \lambda(\alpha) + \lambda(\beta)$$

for all $\alpha, \beta > 0$. □

We have the following result providing many examples of subadditive functions $\lambda : [0,\infty) \to [0,\infty)$.

Theorem 2.1. *Let $h(z) = \sum_{n=0}^{\infty} a_n z^n$ a power series with nonnegative coefficients $a_n \geq 0$ for all $n \in \mathbb{N}$ and convergent on the open disk $D(0,R)$ with $R > 0$ or $R = \infty$. If $r \in (0,R)$ then the function $\lambda_r : [0,\infty) \to [0,\infty)$ given by*

$$\lambda_r(t) := \ln\left[\frac{h(r)}{h(r\exp(-t))}\right] \tag{17}$$

is nonnegative, increasing and subadditive on $[0,\infty)$.

Proof. We use the Čebyšev inequality for synchronous (the same monotonicity) sequences $(c_i)_{i\in\mathbb{N}}, (b_i)_{i\in\mathbb{N}}$ and nonnegative weights $(p_i)_{i\in\mathbb{N}}$, namely

$$\sum_{i=0}^{n} p_i \sum_{i=0}^{n} p_i c_i b_i \geq \sum_{i=0}^{n} p_i c_i \sum_{i=0}^{n} p_i b_i, \tag{18}$$

for any $n \in \mathbb{N}$.

Let $t, s \in (0,1)$ and define the sequences $c_i := t^i$, $b_i := s^i$. These sequences are decreasing and if we apply Čebyšev's inequality for these sequences and the weights $p_i := a_i r^i \geq 0$ we get

$$\sum_{i=0}^{n} a_i r^i \sum_{i=0}^{n} a_i (rts)^i \geq \sum_{i=0}^{n} a_i (rt)^i \sum_{i=0}^{n} a_i (rs)^i \tag{19}$$

for any $n \in \mathbb{N}$.

Since the series

$$\sum_{i=0}^{\infty} a_i r^i, \quad \sum_{i=0}^{\infty} a_i (rts)^i, \quad \sum_{i=0}^{\infty} a_i (rt)^i \text{ and } \sum_{i=0}^{\infty} a_i (rs)^i$$

are convergent, then by letting $n \to \infty$ in (19) we get

$$h(r)\, h(rts) \geq h(rt)\, h(rs)$$

which can be written as

$$\frac{h(r)}{h(rts)} \leq \frac{h(r)}{h(rt)} \cdot \frac{h(r)}{h(rs)}$$

for any $t, s \in (0,1)$.

Let $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$. Then

$$\lambda_r(\alpha+\beta) = \ln\left[\frac{h(r)}{h(r\exp(-\alpha-\beta))}\right] = \ln\left[\frac{h(r)}{h(r\exp(-\alpha)\exp(-\beta))}\right] \tag{20}$$
$$= \ln\left[\frac{h(r)}{h(r\exp(-\alpha))}\cdot\frac{h(r)}{h(r\exp(-\beta))}\right]$$
$$= \ln\left[\frac{h(r)}{h(r\exp(-\alpha))}\right] + \ln\left[\frac{h(r)}{h(r\exp(-\beta))}\right]$$
$$= \lambda_r(\alpha) + \lambda_r(\beta).$$

Since $h(r) \geq h(r\exp(-t))$ for any $t \in [0,\infty)$ we deduce that λ_r is nonnegative and subadditive on $[0,\infty)$.

Now, observe that λ_r is differentiable on $(0,\infty)$ and

$$\begin{aligned} \lambda_r'(t) &: = -(\ln[h(r\exp(-t))])' \\ &= -\frac{h'(r\exp(-t))(r\exp(-t))'}{h(r\exp(-t))} \\ &= \frac{r\exp(-t)h'(r\exp(-t))}{h(r\exp(-t))} \geq 0 \end{aligned} \tag{21}$$

for $t \in (0,\infty)$, where

$$h'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

This proves the monotonicity of λ_r. $\square$

We have the following fundamental examples of power series with positive coefficients

$$h(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z},\ z \in D(0,1) \tag{22}$$
$$h(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n = \exp(z) \qquad z \in \mathbb{C},$$
$$h(z) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n} = \cosh z,\ z \in \mathbb{C};$$
$$h(z) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1} = \sinh z,\ z \in \mathbb{C};$$
$$h(z) = \sum_{n=1}^{\infty} \frac{1}{n} z^n = \ln\frac{1}{1-z},\ z \in D(0,1).$$

Other important examples of functions as power series representations with positive coefficients are:

$$h(z)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1}=\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right),\qquad z\in D(0,1); \tag{23}$$

$$h(z)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}z^{2n+1}=\sin^{-1}(z),\qquad z\in D(0,1);$$

$$h(z)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1}=\tanh^{-1}(z),\qquad z\in D(0,1);$$

$$h(z)={}_2F_1(\alpha,\beta,\gamma,z)=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}z^{n},\alpha,\beta,\gamma>0,$$
$$z\in D(0,1);$$

where Γ is *Gamma function*.

Remark 2.1. *Now, if we take* $h(z)=\frac{1}{1-z}$, $z\in D(0,1)$, *then*

$$\lambda_r(t)=\ln\left[\frac{1-r\exp(-t)}{1-r}\right] \tag{24}$$

is nonnegative, increasing and subadditive on $[0,\infty)$ *for any* $r\in(0,1)$.

If we take $h(z)=\exp(z)$, $z\in\mathbb{C}$, *then*

$$\lambda_r(t)=r[1-\exp(-t)] \tag{25}$$

is nonnegative, increasing and subadditive on $[0,\infty)$ *for any* $r>0$.

Corollary 2.1. *Let* $h(z)=\sum_{n=0}^{\infty}a_nz^n$ *a power series with nonnegative coefficients* $a_n\geq0$ *for all* $n\in\mathbb{N}$ *and convergent on the open disk* $D(0,R)$ *with* $R>0$ *or* $R=\infty$ *and* $r\in(0,R)$. *For a mapping* $f:C\to\mathbb{R}$ *defined on convex subset* C *of a linear space* X, *the following statements are equivalent:*

(i) The function f *is* λ_r*-convex with* $\lambda_r:[0,\infty)\to[0,\infty)$,

$$\lambda_r(t):=\ln\left[\frac{h(r)}{h(r\exp(-t))}\right];$$

(ii) We have the inequality

$$\left[\frac{h(r)}{h(r\exp(-\alpha-\beta))}\right]^{f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)} \tag{26}$$
$$\leq\left[\frac{h(r)}{h(r\exp(-\alpha))}\right]^{f(x)}\left[\frac{h(r)}{h(r\exp(-\beta))}\right]^{f(y)}$$

for any $\alpha,\beta\geq0$ *with* $\alpha+\beta>0$ *and* $x,y\in C$.

(iii) We have the inequality

$$\frac{[h(r\exp(-\alpha))]^{f(x)}[h(r\exp(-\beta))]^{f(y)}}{[h(r\exp(-\alpha-\beta))]^{f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)}} \leq [h(r)]^{f(x)+f(y)-f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)} \tag{27}$$

for any $\alpha, \beta \geq 0$ with $\alpha+\beta>0$ and $x, y \in C$.

Proof. We have

$$f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)\lambda_r(\alpha+\beta) \leq \lambda_r(\alpha) f(x) + \lambda_r(\beta) f(y)$$

for any $\alpha, \beta \geq 0$ with $\alpha+\beta>0$ and $x, y \in C$, is equivalent to

$$\begin{aligned} &\ln\left[\frac{h(r)}{h(r\exp(-\alpha-\beta))}\right]^{f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)} \\ &\leq \ln\left[\frac{h(r)}{h(r\exp(-\alpha))}\right]^{f(x)} + \ln\left[\frac{h(r)}{h(r\exp(-\beta))}\right]^{f(y)} \\ &= \ln\left\{\left[\frac{h(r)}{h(r\exp(-\alpha))}\right]^{f(x)}\left[\frac{h(r)}{h(r\exp(-\beta))}\right]^{f(y)}\right\} \end{aligned} \tag{28}$$

for any $\alpha, \beta \geq 0$ with $\alpha+\beta>0$ and $x, y \in C$.

The inequality (28) is equivalent to (26) and the proof of the equivalence " $(i) \Leftrightarrow (ii)$" is concluded. The last part is obvious. □

Remark 2.2. *We observe that, in the case when*

$$\lambda_r(t) = r[1-\exp(-t)],\ t \geq 0,$$

then the function f is λ_r-convex on convex subset C of a linear space X iff

$$f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right) \leq \frac{[1-\exp(-\alpha)]f(x)+[1-\exp(-\beta)]f(y)}{1-\exp(-\alpha-\beta)} \tag{29}$$

for any $\alpha, \beta \geq 0$ with $\alpha+\beta>0$ and $x, y \in C$.

We observe that this definition is independent of $r>0$.

The inequality (29) is equivalent to

$$f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right) \leq \frac{\exp(\beta)[\exp(\alpha)-1]f(x)+\exp(\alpha)[\exp(\beta)-1]f(y)}{\exp(\alpha+\beta)-1} \tag{30}$$

for any $\alpha, \beta \geq 0$ with $\alpha+\beta>0$ and $x, y \in C$.

3. Hermite-Hadamard Type Inequalities

For an arbitrary mapping $f : C \subset X \to \mathbb{R}$ where C is a convex subset of the linear space X, we can define the mapping

$$g_{x,y} : [0,1] \to \mathbb{R},\ g_{x,y}(t) := f(tx + (1-t)y),$$

where x, y are two distinct fixed elements in C.

Proposition 3.1. *With the above assumptions, the following statements are equivalent:*
(i) f is λ-convex on C;
(ii) For every $x, y \in C$, the mapping $g_{x,y}$ is λ-convex on $[0,1]$.

Proof. "(i) $\Rightarrow$ (ii)". Let $t_1, t_2 \in [0,1]$ and $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$. Then we have

$$\begin{aligned} & g_{x,y}\left(\frac{\alpha t_1 + \beta t_2}{\alpha + \beta}\right) \qquad (31) \\ &= f\left[\left(\frac{\alpha t_1 + \beta t_2}{\alpha + \beta}\right)x + \left(1 - \frac{\alpha t_1 + \beta t_2}{\alpha + \beta}\right)y\right] \\ &= f\left[\frac{\alpha(t_1 x + (1-t_1)y) + \beta(t_2 x + (1-t_2)y)}{\alpha + \beta}\right] \\ &\leq \frac{\lambda(\alpha) f(t_1 x + (1-t_1)y) + \lambda(\beta) f(t_2 x + (1-t_2)y)}{\lambda(\alpha + \beta)} \\ &= \frac{\lambda(\alpha) g_{x,y}(t_1) + \lambda(\beta) g_{x,y}(t_2)}{\lambda(\alpha + \beta)} \end{aligned}$$

and the implication is proved.

"(ii) $\Rightarrow$ (i)". Let $x, y \in C$ and $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$. Then we have

$$\begin{aligned} f\left(\frac{\alpha x + \beta y}{\alpha + \beta}\right) &= g_{x,y}\left(\frac{\alpha}{\alpha + \beta}\right) = g_{x,y}\left(\frac{\alpha \cdot 1 + \beta \cdot 0}{\alpha + \beta}\right) \\ &\leq \frac{\lambda(\alpha) g_{x,y}(1) + \lambda(\beta) g_{x,y}(0)}{\lambda(\alpha + \beta)} \\ &= \frac{\lambda(\alpha) f(x) + \lambda(\beta) f(y)}{\lambda(\alpha + \beta)} \end{aligned}$$

and the implication is thus proved. □

We can introduce the following mapping $k_{x,y} : [0,1] \to \mathbb{R}$

$$k_{x,y}(t) := \frac{1}{2}\left[f(tx + (1-t)y) + f((1-t)x + ty)\right]$$

for $x, y \in C$, $x \neq y$.

Theorem 3.1. *Let $f : C \to [0, \infty)$ be a λ-convex function on C. Assume that $x, y \in C$ with $x \neq y$.*

(i) We have the equality

$$k_{x,y}(1-t) = k_{x,y}(t) \text{ for all } t \in [0,1];$$

(ii) The mapping $k_{x,y}$ is λ-convex on $[0,1]$;

(iii) One has the inequalities

$$k_{x,y}(t) \leq \frac{\lambda(t) + \lambda(1-t)}{\lambda(1)} \cdot \frac{f(x) + f(y)}{2} \tag{32}$$

and

$$\frac{\lambda(2\alpha)}{2\lambda(\alpha)} f\left(\frac{x+y}{2}\right) \leq k_{x,y}(t) \tag{33}$$

for all $t \in [0,1]$ and $\alpha > 0$.

(iv) Let $y, x \in C$ with $y \neq x$ and assume that the mappings $[0,1] \ni t \mapsto f[(1-t)x + ty]$ and λ are Lebesgue integrable on $[0,1]$, then we have the Hermite-Hadamard type inequalities

$$\frac{\lambda(2\alpha)}{2\lambda(\alpha)} f\left(\frac{x+y}{2}\right) \leq \int_0^1 f((1-t)x + ty)\,dt < \frac{f(x) + f(y)}{\lambda(1)} \int_0^1 \lambda(t)\,dt \tag{34}$$

for any $\alpha > 0$.

Proof. The statements (i) and (ii) are obvious.

(iii). By the λ-convexity of f we have:

$$f(tx + (1-t)y) \leq \frac{\lambda(t) f(x) + \lambda(1-t) f(y)}{\lambda(1)}$$

and

$$f((1-t)x + ty) \leq \frac{\lambda(1-t) f(x) + \lambda(t) f(y)}{\lambda(1)},$$

which gives by addition inequality (32).

We also have

$$\frac{\lambda(\alpha) f(z) + \lambda(\alpha) f(u)}{\lambda(2\alpha)} \geq f\left(\frac{\alpha z + \alpha u}{\alpha + \alpha}\right) = f\left(\frac{z+u}{2}\right)$$

i.e.,

$$\frac{\lambda(\alpha)}{\lambda(2\alpha)} [f(z) + f(u)] \geq f\left(\frac{z+u}{2}\right)$$

for all $z, u \in C$.

If we write this inequality for $z = tx + (1-t)y$ and $u = (1-t)x + ty$ we get

$$\frac{\lambda(\alpha)}{\lambda(2\alpha)} [f(tx + (1-t)y) + f((1-t)x + ty)] \geq f\left(\frac{x+y}{2}\right),$$

which is equivalent to (33).

Integrating (33) and (34) over t on $[0,1]$ we get

$$\frac{2\lambda(\alpha)}{\lambda(2\alpha)}\cdot f\left(\frac{x+y}{2}\right) \le \frac{1}{2}\int_0^1 [f(tx+(1-t)y)+f((1-t)x+ty)]\,dt \tag{35}$$
$$\le \frac{f(x)+f(y)}{2}\int_0^1 \frac{\lambda(t)+\lambda(1-t)}{\lambda(1)}dt.$$

Since

$$\int_0^1 f(tx+(1-t)y)\,dt = \int_0^1 f((1-t)x+ty)\,dt$$

and

$$\int_0^1 \lambda(t)\,dt = \int_0^1 \lambda(1-t)\,dt$$

then by (35) we get the desired result (34). □

Remark 3.1. *Since λ is subadditive, then*

$$\frac{\lambda(2\alpha)}{2\lambda(\alpha)} \le 1 \text{ for any } \alpha > 0.$$

From (34) we have the best inequality

$$\sup_{\alpha>0}\left\{\frac{\lambda(2\alpha)}{2\lambda(\alpha)}\right\} f\left(\frac{x+y}{2}\right) \le \int_0^1 f((1-t)x+ty)\,dt \tag{36}$$
$$\le \frac{f(x)+f(y)}{\lambda(1)}\int_0^1 \lambda(t)\,dt.$$

If the right limit

$$k = \lim_{s\to 0+}\frac{\lambda(s)}{s}$$

exists and is finite with $k>0$, then

$$\lim_{\alpha\to 0+}\frac{\lambda(2\alpha)}{2\lambda(\alpha)} = \lim_{\alpha\to 0+}\frac{\left(\frac{\lambda(2\alpha)}{2\alpha}\right)}{\left(\frac{\lambda(\alpha)}{\alpha}\right)} = \frac{\lim_{\alpha\to 0+}\left(\frac{\lambda(2\alpha)}{2\alpha}\right)}{\lim_{\alpha\to 0+}\left(\frac{\lambda(\alpha)}{\alpha}\right)} = \frac{k}{k} = 1$$

and by (34) we get

$$f\left(\frac{x+y}{2}\right) \le \int_0^1 f((1-t)x+ty)\,dt \le \frac{f(x)+f(y)}{\lambda(1)}\int_0^1 \lambda(t)\,dt. \tag{37}$$

Corollary 3.1. *Assume that the function $f : C \to [0,\infty)$ is λ_r-convex with $\lambda_r : [0,\infty) \to [0,\infty)$,*

$$\lambda_r(t) := \ln\left[\frac{h(r)}{h(r\exp(-t))}\right]$$

and h is as in Corollary 2.1.

If $y, x \in C$ with $y \neq x$ and the mapping $[0,1] \ni t \mapsto f[(1-t)x+ty]$ is Lebesgue integrable on $[0,1]$, then we have the Hermite-Hadamard type inequalities

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f((1-t)x+ty)\,dt \tag{38}$$
$$\leq \frac{f(x)+f(y)}{\ln\left[\frac{h(r)}{h(re^{-1})}\right]} \int_0^1 \ln\left[\frac{h(r)}{h(r\exp(-t))}\right] dt.$$

Proof. We know that λ_r is differentiable on $(0,\infty)$ and

$$\lambda_r'(t) := \frac{r\exp(-t)\,h'(r\exp(-t))}{h(r\exp(-t))}$$

for $t \in (0,\infty)$, where

$$h'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

Since $\lambda_r(0) = 0$, then

$$k = \lim_{s\to 0+} \frac{\lambda(s)}{s} = \lambda_+'(0) = \frac{rh'(r)}{h(r)} > 0 \text{ for } r \in (0,R)$$

and by (37) we get (38). □

Furthermore, we observe that the following elementary inequality holds:

$$(\alpha+\beta)^p \geq (\leq)\, \alpha^p + \beta^p \tag{39}$$

for any $\alpha, \beta \geq 0$ and $p \geq 1$ $(0 < p < 1)$.

Indeed, if we consider the function $f_p : [0,\infty) \to \mathbb{R}$, $f_p(t) = (t+1)^p - t^p$ we have $f_p'(t) = p\left[(t+1)^{p-1} - t^{p-1}\right]$. Observe that for $p > 1$ and $t > 0$ we have that $f_p'(t) > 0$ showing that f_p is strictly increasing on the interval $[0,\infty)$. Now for $t = \frac{\alpha}{\beta}$ $(\beta > 0, \alpha \geq 0)$ we have $f_p(t) > f_p(0)$ giving that $\left(\frac{\alpha}{\beta}+1\right)^p - \left(\frac{\alpha}{\beta}\right)^p > 1$, i.e., the desired inequality (39).

For $p \in (0,1)$ we have f_p strictly decreasing on $[0,\infty)$ which proves the second case in (39).

If we consider the power function $\hat{\lambda}_q(t) = t^q$ with $q \in (0,1)$, then $\hat{\lambda}_q$ is subadditive and by (34) we have

$$\frac{1}{2^{1-q}} f\left(\frac{x+y}{2}\right) \leq \int_0^1 f((1-t)x+ty)\,dt \leq \frac{f(x)+f(y)}{q+1}, \tag{40}$$

therefore we recapture the inequality (12) that was obtained from (7).

For $q \geq 1$ and if we consider the function $\check{\lambda}_q(t) = \frac{1}{t^q}$, then for any $t, s > 0$ we have

$$\check{\lambda}_q(t+s) = \frac{1}{(t+s)^q} \leq \frac{1}{t^s + s^q} \leq \frac{1}{t^s} + \frac{1}{s^q} = \check{\lambda}_q(t) + \check{\lambda}_q(s)$$

which shows that $\check{\lambda}_q$ is subadditive.

If $f : C \to [0, \infty)$ is a $\check{\lambda}_q$-convex function on C, i.e.

$$f\left(\frac{\alpha x + \beta y}{\alpha + \beta}\right) \leq \frac{\alpha^{-q} f(x) + \beta^{-q} f(y)}{(\alpha + \beta)^{-q}} \tag{41}$$

for all $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$ and $x, y \in C$, where $q \geq 1$, then we observe that the inequality (41) is equivalent to

$$f\left(\frac{\alpha x + \beta y}{\alpha + \beta}\right) \leq \left(\frac{\alpha + \beta}{\alpha\beta}\right)^q [\beta^q f(x) + \alpha^q f(y)] \tag{42}$$

for all $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$ and $x, y \in C$, where $q \geq 1$.

Since $\check{\lambda}_q$ is not integrable on $[0, 1]$ we cannot apply the second inequality from (34). However, from the first inequality we get

$$\frac{1}{2^{q+1}} f\left(\frac{x+y}{2}\right) \leq \int_0^1 f((1-t)x + ty)\, dt \tag{43}$$

provided that f is $\check{\lambda}_q$-convex and the integral $\int_0^1 f((1-t)x + ty)\, dt$ exists for some $x, y \in C$.

Moreover, if we assume that $f : C \to [0, \infty)$ is a λ-convex function on C with $\lambda(t) = 1 - \exp(-t)$, $t \geq 0$, i.e.

$$f\left(\frac{\alpha x + \beta y}{\alpha + \beta}\right) \leq \frac{\exp(\beta)[\exp(\alpha) - 1] f(x) + \exp(\alpha)[\exp(\beta) - 1] f(y)}{\exp(\alpha + \beta) - 1} \tag{44}$$

for any $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$ and $x, y \in C$, then by (37) we have

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f((1-t)x + ty)\, dt \leq \frac{f(x) + f(y)}{1 - e^{-1}} \int_0^1 [1 - \exp(-t)]\, dt,$$

that is equivalent to

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f((1-t)x + ty)\, dt \leq \frac{f(x) + f(y)}{e - 1}, \tag{45}$$

provided the integral $\int_0^1 f((1-t)x + ty)\, dt$ exists for some $x, y \in C$.

4. Inequalities for Double Integrals

We have the following result:

Theorem 4.1. *Let $f : C \to [0, \infty)$ be a λ-convex function on C. Let $y, x \in C$ with $y \neq x$ and assume that the mappings $[0,1] \ni t \mapsto f[(1-t)x + ty]$ and λ are Lebesgue integrable on $[0,1]$, then for $0 \leq a < b$ we have the Hermite-Hadamard type inequalities*

$$\begin{aligned}
&\frac{\lambda(2\eta)}{2\lambda(\eta)} f\left(\frac{x+y}{2}\right)(b-a)^2 \qquad (46)\\
&\leq \frac{1}{2}\int_a^b\int_a^b \left[f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)d\alpha d\beta + f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right)\right]d\alpha d\beta\\
&\leq [f(x)+f(y)]\int_a^b\int_a^b \frac{\lambda(\alpha)}{\lambda(\alpha+\beta)}d\alpha d\beta
\end{aligned}$$

for any $\eta > 0$.

Proof. By the λ-convexity of f we have

$$f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right) \leq \frac{\lambda(\alpha)f(x)+\lambda(\beta)f(y)}{\lambda(\alpha+\beta)}$$

and

$$f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right) \leq \frac{\lambda(\beta)f(x)+\lambda(\alpha)f(y)}{\lambda(\alpha+\beta)}$$

for all $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$.

By adding these inequalities we obtain

$$f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right) + f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right) \leq \frac{\lambda(\alpha)+\lambda(\beta)}{\lambda(\alpha+\beta)}[f(x)+f(y)] \qquad (47)$$

for all $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$.

Since the mappings $[0,1] \ni t \mapsto f[(1-t)x + ty]$ and λ are Lebesgue integrable on $[0,1]$, then the integrals

$$\int_a^b\int_a^b f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)d\alpha d\beta \text{ and } \int_a^b\int_a^b f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right)d\alpha d\beta$$

exist and by integrating the inequality (47) on the square $[a,b]^2$ we get

$$\begin{aligned}
&\int_a^b\int_a^b f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)d\alpha d\beta + \int_a^b\int_a^b f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right)d\alpha d\beta\\
&\leq [f(x)+f(y)]\int_a^b\int_a^b \frac{\lambda(\alpha)+\lambda(\beta)}{\lambda(\alpha+\beta)}d\alpha d\beta\\
&= 2[f(x)+f(y)]\int_a^b\int_a^b \frac{\lambda(\alpha)}{\lambda(\alpha+\beta)}d\alpha d\beta
\end{aligned}$$

and the second inequality in (46) is proved.

We know from the proof of Theorem 3.1 that

$$\frac{\lambda(\eta)}{\lambda(2\eta)}\left[f(z)+f(u)\right]\geq f\left(\frac{z+u}{2}\right)$$

for all $z,u\in C$ and $\eta>0$.

Taking

$$z=\frac{\alpha x+\beta y}{\alpha+\beta}\text{ and }u=\frac{\beta x+\alpha y}{\alpha+\beta}$$

we get

$$\frac{\lambda(\eta)}{\lambda(2\eta)}\left[f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)+f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right)\right]\geq f\left(\frac{x+y}{2}\right)\tag{48}$$

for all $\alpha,\beta\geq 0$ with $\alpha+\beta>0$ and $\eta>0$.

Integrating inequality (48) on the square $[a,b]^2$ we get the first part of (46). □

Remark 4.1. *If we write inequality (46) for $f:C\to[0,\infty)$ a $\check{\lambda}_q$-convex function on C, then we get the inequality*

$$\begin{aligned}&\frac{1}{2^{q+1}}f\left(\frac{x+y}{2}\right)(b-a)^2 \qquad (49)\\&\leq\frac{1}{2}\int_a^b\int_a^b\left[f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)d\alpha d\beta+f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right)\right]d\alpha d\beta\\&\leq[f(x)+f(y)]\int_a^b\int_a^b\left(\frac{\alpha+\beta}{\alpha}\right)^q d\alpha d\beta,\end{aligned}$$

provided that the mapping $[0,1]\ni t\mapsto f[(1-t)x+ty]$ is Lebesgue integrable on $[0,1]$.

For $q=1$ we have

$$\begin{aligned}\int_a^b\int_a^b\frac{\alpha+\beta}{\alpha}d\beta d\alpha&=\int_a^b\int_a^b\left(1+\frac{\beta}{\alpha}\right)d\beta d\alpha\\&=(b-a)^2+(\ln b-\ln a)\frac{b^2-a^2}{2}\\&=(b-a)^2\left(1+\frac{\ln b-\ln a}{b-a}\cdot\frac{a+b}{2}\right)\\&=(b-a)^2\left[1+\frac{A(a,b)}{L(a,b)}\right]\end{aligned}$$

where

$$L(a,b):=\frac{b-a}{\ln b-\ln a}$$

is the logarithmic mean.

Then from (49) we get

$$\frac{1}{4}f\left(\frac{x+y}{2}\right) \tag{50}$$
$$\leq \frac{1}{2(b-a)^2}\int_a^b\int_a^b\left[f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)d\alpha d\beta + f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right)\right]d\alpha d\beta$$
$$\leq [f(x)+f(y)]\left[1+\frac{A(a,b)}{L(a,b)}\right],$$

provided that $f: C \to [0,\infty)$ *is a* $\check{\lambda}_1$*-convex function on* C *and the mapping* $[0,1] \ni t \mapsto f[(1-t)x+ty]$ *is Lebesgue integrable on* $[0,1]$.

For $q=2$ *we have*

$$\begin{aligned}\int_a^b\int_a^b\left(\frac{\alpha+\beta}{\alpha}\right)^2 d\beta d\alpha &= \int_a^b\int_a^b\left(1+\frac{\beta}{\alpha}\right)^2 d\beta d\alpha\\ &= \int_a^b\int_a^b\left(1+\frac{2\beta}{\alpha}+\frac{\beta^2}{\alpha^2}\right)d\beta d\alpha\\ &= (b-a)^2\left(1+2\frac{\ln b-\ln a}{b-a}\cdot\frac{a+b}{2}+\frac{a^2+ab+b^2}{3ab}\right)\\ &= \left(2\frac{\ln b-\ln a}{b-a}\cdot\frac{a+b}{2}+\frac{a^2+4ab+b^2}{3ab}\right)\\ &= 2(b-a)^2\left[\frac{1}{3}+\frac{2}{3}\cdot\frac{A(a,b)}{G(a,b)}+\frac{A(a,b)}{L(a,b)}\right],\end{aligned}$$

where $G(a,b) := \sqrt{ab}$ *is the geometric mean.*

Then from (49) we get

$$\frac{1}{8}f\left(\frac{x+y}{2}\right) \tag{51}$$
$$\leq \frac{1}{2(b-a)^2}\int_a^b\int_a^b\left[f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)d\alpha d\beta + f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right)\right]d\alpha d\beta$$
$$\leq 2[f(x)+f(y)]\left[\frac{1}{3}+\frac{2}{3}\cdot\frac{A(a,b)}{G(a,b)}+\frac{A(a,b)}{L(a,b)}\right],$$

provided that $f: C \to [0,\infty)$ *is a* $\check{\lambda}_2$*-convex function on* C *and the mapping* $[0,1] \ni t \mapsto f[(1-t)x+ty]$ *is Lebesgue integrable on* $[0,1]$.

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